Abelian lattice-ordered groups edit

Abbreviation: AbLGrp

Definition 1. An *abelian lattice-ordered group* (or abelian *ℓ-group*) is a lattice-ordered group $\mathbf{L} = \langle L, \vee, \wedge, \cdot, ^{-1}, e\rangle$ such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$

Morphisms. Let $\mathbf{L}$ and $\mathbf{M}$ be ℓ-groups. A morphism from $\mathbf{L}$ to $\mathbf{M}$ is a function $f : L \to M$ that is a homomorphism: $f(x \vee y) = f(x) \vee f(y)$ and $f(x \cdot y) = f(x) \cdot f(y)$.

Remark: It follows that $f(x \wedge y) = f(x) \wedge f(y)$, $f(x^{-1}) = f(x)^{-1}$, and $f(e) = e$

Definition 2. An *abelian lattice-ordered group* (or *abelian ℓ-group*) is a commutative residuated lattice $\mathbf{L} = \langle L, \vee, \wedge, \cdot, \to, e\rangle$ that satisfies the identity $x \cdot (x \to e) = e$.

Remark: $x^{-1} = x \to e$ and $x \to y = x^{-1}y$

Examples. $\langle \mathbb{Z}, \max, \min, +, -, 0\rangle$, the integers with maximum, minimum, addition, unary subtraction and zero. The variety of abelian ℓ-groups is generated by this algebra.

Basic Results. The lattice reducts of (abelian) ℓ-groups are distributive lattices.

Properties. (description)

Classtype	variety
Equational theory	decidable
Quasiequational theory	decidable
First-order theory	hereditarily undecidable [1] [2]
Locally finite	no
Residual size	
Congruence distributive	yes (see lattices)
Congruence modular	yes
Congruence n-permutable	yes, $n = 2$ (see groups)
Congruence regular	yes, (see groups)
Congruence uniform	yes, (see groups)
Congruence extension property	
Definable principal congruences	
Equationally def. pr. cong.	
Amalgamation property	yes
Strong amalgamation property	
Epimorphisms are surjective	

Finite Members. $f(n) =$ number of members of size n.

None

Subclasses.
Totally ordered abelian groups

Superclasses.
Representable lattice-ordered groups

References

[1] Yuri Gurevic, *Hereditary undecidability of a class of lattice-ordered Abelian groups*, Algebra i Logika Sem., **6**, 1967, 45–62 MRreview

[2] Stanley Burris, *A simple proof of the hereditary undecidability of the theory of lattice-ordered abelian groups*, Algebra Universalis, **20**, 1985, 400–401 MRreview