

## Boolean algebras [edit](#)

### Abbreviation: BA

**Definition 1.** A *Boolean algebra* is a structure  $\mathbf{A} = \langle A, \vee, 0, \wedge, 1, - \rangle$  of type  $\langle 2, 0, 2, 0, 1 \rangle$  such that

$0, 1$  are identities for  $\vee, \wedge$ :  $x \vee 0 = x, x \wedge 1 = x$

$-$  gives a complement:  $x \wedge -x = 0, x \vee -x = 1$

$\vee, \wedge$  are associative:  $x \vee (y \vee z) = (x \vee y) \vee z, x \wedge (y \wedge z) = (x \wedge y) \wedge z$

$\vee, \wedge$  are commutative:  $x \vee y = y \vee x, x \wedge y = y \wedge x$

$\vee, \wedge$  are mutually distributive:  $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

**Definition 2.** A *Boolean algebra* is a structure  $\mathbf{A} = \langle A, \vee, 0, \wedge, 1, - \rangle$  of type  $\langle 2, 0, 2, 0, 1 \rangle$  such that

$\langle A, \vee, 0, \wedge, 1 \rangle$  is a bounded distributive lattice

$-$  gives a complement:  $x \wedge -x = 0, x \vee -x = 1$

**Morphisms.** Let  $\mathbf{A}$  and  $\mathbf{B}$  be Boolean algebras. A morphism from  $\mathbf{A}$  to  $\mathbf{B}$  is a function  $h : A \rightarrow B$  that is a homomorphism:

$$h(x \vee y) = h(x) \vee h(y), h(-x) = -h(x)$$

It follows that  $h(x \wedge y) = h(x) \wedge h(y), h(0) = 0, h(1) = 1$ .

**Definition 3.** A *Boolean ring* is a structure  $\mathbf{A} = \langle A, +, 0, \cdot, 1 \rangle$  of type  $\langle 2, 0, 2, 0 \rangle$  such that

$\langle A, +, 0, \cdot, 1 \rangle$  is a commutative ring with unit

$\cdot$  is idempotent:  $x \cdot x = x$

Remark: The term-equivalence with Boolean algebras is given by  $x \wedge y = x \cdot y, -x = x + 1, x \vee y = -(x \wedge -y)$  and  $x + y = (x \vee y) \wedge -(x \wedge y)$ .

**Definition 4.** A *Boolean algebra* is a Heyting algebra  $\mathbf{A} = \langle A, \vee, 0, \wedge, 1, \rightarrow \rangle$  such that

$x \rightarrow 0$  is an involution:  $(x \rightarrow 0) \rightarrow 0 = x$

### Basic Results.

#### Examples.

1.  $\langle \mathcal{P}(S), \cup, \emptyset, \cap, S, - \rangle$ , the collection of subsets of a sets  $S$ , with union, intersection, and setcomplementation.

**Finite Members.** Number of algebras =  $\begin{cases} 1 & \text{if size} = 2^n \\ 0 & \text{otherwise} \end{cases}$

**Properties.** (description)

Class type	variety
Equational theory	decidable in NPTIME
Quasiequational theory	decidable
First-order theory	decidable
Congruence distributive	yes
Congruence modular	yes
Congruence $n$ -permutable	yes, $n = 2$
Congruence regular	yes
Congruence uniform	yes
Congruence extension property	yes
Definable principal congruences	yes
Equationally def. pr. cong.	yes
Amalgamation property	yes
Strong amalgamation property	yes
Epimorphisms are surjective	yes
Locally finite	yes
Residual size	2

**Subclasses.**

One-element algebras

Complete Boolean algebras

**Superclasses.**

Bounded distributive lattices

Generalized Boolean algebras

Heyting algebras

## REFERENCES

[1]