The Geometry of Relevant Implication

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Abstract

This paper is a continuation of earlier work by the author on the connection between the logic **KR** and projective geometry. It contains a simplified construction of **KR** model structures; as a consequence, it extends the previous results to a much more extensive class of projective spaces and the corresponding modular lattices.

1 The Logic KR

The logic **KR** occupies a rather unusual place in the family of relevant logics. In fact, it is questionable whether it should even be classified as a relevant logic, since it is the result of adding to **R** the axiom *ex falso quodlibet*, that is to say, $(A \wedge \neg A) \rightarrow B$. This is of course one of the paradoxes of material implication that relevant logics were devised specifically to avoid, a paradox of consistency. The other type of paradox is a paradox of relevance, of which the paradigm case is the weakening axiom $A \rightarrow (B \rightarrow A)$. The surprising thing about **KR** is that although it contains the first type of paradox, it avoids the second, contrary to what we might at first suspect. In fact, it is a complex and highly non-trivial system. The credit for its initial investigation belongs to Adrian Abraham, Robert K. Meyer and Richard Routley [12].

The model theory for **KR** is elegantly simple. The usual ternary relational semantics for **R** includes an operation $*$ designed to deal with the truth condition for negation

$$x \models \neg A \Leftrightarrow x^* \not\models A.$$

The effect of adding *ex falso quodlibet* to **R** is to identify x and x^*; this in turn has a notable effect on the ternary accessibility relation. The postulates for an **R** model structure include the following implication:

$$Rxyz \Rightarrow (Ryxz \;\&\; Rxz^*y^*).$$

The result of the identification of x and x^* is that the ternary relation in a **KR** model structure (KRms) is *totally symmetric*. In detail, a KRms $\mathcal{K} = \langle S, R, 0\rangle$ is a 3-place relation R on a set containing a distinguished element 0, and satisfying the postulates:

1. $R0ab \Leftrightarrow a = b$;
2. $Raaa$;
3. $Rabc \Rightarrow (Rbac \,\&\, Racb)$ (total symmetry);
4. $(Rabc \,\&\, Rcde) \Rightarrow \exists f(Radf \,\&\, Rfbe)$ (Pasch's postulate).

The result of adding the weakening axiom $A \to (B \to A)$ to **R** is a collapse into classical logic. The addition of $(A \wedge \neg A) \to B$ does not result in such a collapse – but is the result a trivial or uninteresting system? This is very far from the case, as we shall see in the next section.

2 KR and modular lattices

Given a **KR** model structure $\mathcal{K} = \langle S, R, 0\rangle$, we can define an algebra $\mathfrak{A}(\mathcal{K})$ as follows:

Definition 2.1. *The algebra $\mathfrak{A}(\mathcal{K}) = \langle \mathcal{P}(S), \cap, \cup, \neg, \top, \bot, t, \circ\rangle$ is defined on the Boolean algebra $\langle \mathcal{P}(S), \cap, \cup, \neg, \top, \bot\rangle$ of all subsets of S, where $\top = S, \bot = \emptyset, t = \{0\}$, and the operator $A \circ B$ is defined by*

$$A \circ B = \{c \mid \exists a \in A, b \in B(Rabc)\}.$$

The algebra $\mathfrak{A}(\mathcal{K})$ is a De Morgan monoid [1],[3] in which $A \cap \neg A = \bot$; we shall call any such algebra a **KR**-*algebra*. Hence the fusion operator $A \circ B$ is associative, commutative, and monotone. In addition, it satisfies the square-increasing property, and t is the monoid identity:

$$A \circ (B \circ C) = (A \circ B) \circ C,$$

$$A \circ B = B \circ A,$$

$$(A \subseteq B \wedge C \subseteq D) \Rightarrow A \circ C \subseteq B \circ D,$$

$$A \subseteq A \circ A,$$

$$A \circ t = A.$$

In what follows, we shall assume basic results from the theory of De Morgan monoids, referring the reader to the expositions in Anderson and Belnap [1] and Dunn and Restall [3] for more background.

In a **KR**-algebra, we can single out a subset of the elements that form a lattice; this lattice plays a key role in the analysis of the logic **KR**.

Definition 2.2. *Let $\mathfrak{A}$ be a **KR**-algebra. The family $\mathcal{L}(\mathfrak{A})$ is defined to be the elements of $\mathfrak{A}$ that are $\geq t$ and idempotent, that is to say, $a \in \mathcal{L}(\mathfrak{A})$ if and only if $a \circ a = a$ and $t \leq a$. If $\mathcal{K}$ is a **KR** model structure, then we define $\mathcal{L}(\mathcal{K})$ to be $\mathcal{L}(\mathfrak{A}(\mathcal{K}))$.*

The following lemma provides a useful characterization of the elements of $\mathcal{L}(\mathfrak{A})$; it is based on some old observations of Bob Meyer.

Lemma 2.3. *Let $\mathfrak{A}$ be a **KR**-algebra. Then the following conditions are equivalent:*

1. $a \in \mathcal{L}(\mathfrak{A})$;
2. $a = (a \to a)$;
3. $\exists b[a = (b \to b)]$.

Proof. ($\mathbf{1} \Rightarrow \mathbf{2} \Rightarrow \mathbf{3}$): Since $t \leq a$, we have $t \leq (a \to a) \to a$, $t \circ (a \to a) \leq a$, hence $(a \to a) \leq a$. Since $a \circ a \leq a$, $a \leq (a \to a)$, so $a = (a \to a)$, proving the second and hence the third condition.

($\mathbf{3} \Rightarrow \mathbf{1}$): First, we have $t \leq (b \to b) = a$. Second, $(b \to b) \leq (b \to b) \to (b \to b)$, so $(b \to b) \circ (b \to b) \leq (b \to b)$, that is to say, $a \circ a \leq a$, so $a \circ a = a$. $\square$

If $\mathcal{K} = \langle S, R, 0 \rangle$ is a **KR** model structure, then a subset A of S is a *linear subspace* if it satisfies the condition

$$(a, b \in A \wedge Rabc) \Rightarrow c \in A.$$

A lattice is *modular* if it satisfies the implication

$$x \geq z \Rightarrow x \wedge (y \vee z) = (x \wedge y) \vee z.$$

For background on modular lattice theory, the reader can consult the texts of Birkhoff [2] or Grätzer [6].

We require a few basic lattice-theoretic definitions here. A *chain* in a lattice L is a totally ordered subset of L; the length of a finite chain C is $|C| - 1$. A chain C in a lattice L is *maximal* if for any chain D in L, if $C \subseteq D$ then $C = D$. If L is a lattice, $a, b \in L$ and $a \leq b$, then the *interval* $[a, b]$ is defined to be the sublattice $\{c : a \leq c \leq b\}$.

Let L be a lattice with least element 0. We define the *height* function: for $a \in L$, let $h(a)$ denote the length of a longest maximal chain in $[0, a]$ if there is a finite longest maximal chain; otherwise put $h(a) = \infty$. If L has a largest element 1, and $h(1) < \infty$, then L has *finite height.*

Let L be a modular lattice with 0 of finite height. Then for $a \in L$, $h(a)$ is the length of *any* maximal chain in $[0, a]$. In addition, the height function in L satisfies the condition

$$h(a) + h(b) = h(a \wedge b) + h(a \vee b),$$

for all $a, b \in L$. For a lattice of finite height, this last condition is equivalent to modularity. These results are proved in the text of Grätzer [6, Chapter IV, §2].

Lemma 2.4. *If $\mathcal{K}$ is a* **KR** *model structure, then the elements of $\mathcal{L}(\mathcal{K})$ are exactly the non-empty linear subspaces of $\mathcal{K}$.*

Proof. The lemma follows from the definition of $A \circ B$ and the fact that $Raa0$ and $Raaa$ hold in any **KR** model structure. □

Theorem 2.5. *If $\mathfrak{A}$ is a* **KR***-algebra, then $\mathcal{L}(\mathcal{K})$, ordered by containment, forms a modular lattice, with least element t, and the lattice operations of join and meet defined by $a \wedge b$ and $a \circ b$.*

Proof. The fact that $\mathcal{L}(\mathcal{K})$ forms a lattice, with $\wedge$ as the lattice meet and $\circ$ as the lattice join, can be proved from the basic properties of De Morgan lattices.

We now prove modularity; in the following computation, we use juxtaposition ab for meet $a \wedge b$, and $\overline{a}$ for the Boolean complement. Note that $a \vee b$ is the extensional (Boolean) join, not the lattice join in $\mathcal{L}(\mathcal{K})$. If $a \geq c$, then

$$\begin{array}{rcl} a(b \circ c) & = & a[(ba \vee b\overline{a}) \circ c] \\ & = & a[(ba \circ c) \vee (b\overline{a} \circ c)] \\ & \leq & a[(ba \circ c) \vee (\overline{a} \circ a)] \\ & = & a(ba \circ c) \vee a\overline{a} \\ & \leq & ab \circ c. \end{array}$$

The opposite inequality $ab \circ c \leq a(b \circ c)$ follows from the lattice properties of $\mathcal{L}(\mathcal{K})$, so $a(b \circ c) = ab \circ c$. In the fourth line above, the equation $\overline{a} \circ a = \overline{a}$ follows from Lemma 2.3, since for $a \in \mathcal{L}(\mathcal{K})$, $a = a \to a$, so $\overline{a} = \overline{a \to a} = a \circ \overline{a}$. □

The preceding theorem shows that there is a modular lattice canonically associated with any **KR**-algebra. It is natural to ask the question: how general is this

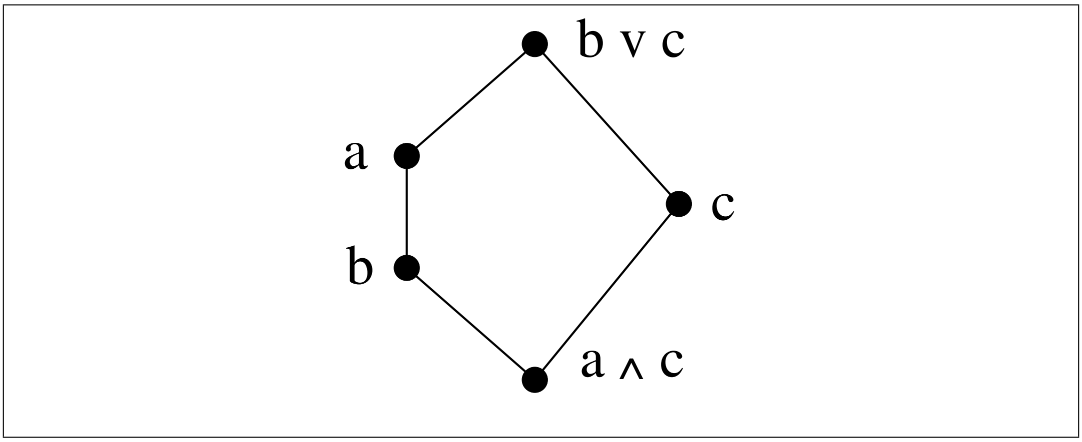

Figure 1: $\mathfrak{N}_5$: the five-element non-modular lattice

construction? That is to say, which modular lattices arise in this way? In earlier papers [13],[14],[15] I provided a partial answer to this question by showing that a very large family of modular lattices, closely connected with classical projective geometries, can be represented as the lattices $\mathcal{L}(\mathcal{K})$ associated with **KR** model structures. This construction made possible the solution of some long-standing problems in the area of relevance logic, particularly those of decidability and interpolability.

The lattices arising from projective spaces, however, are of a rather special type, and the construction given in my earlier work does not make clear whether more general modular lattices can be represented. In this section, I give a very simple construction for **KR** model structures showing that *any* modular lattice can be represented as a sublattice of a lattice $\mathcal{L}(\mathcal{K})$. The earlier representation of geometric lattices can be obtained as a direct corollary of this construction, as is shown in Section 4.

Definition 2.6. *Let L be a lattice with least element 0. Define a ternary relation R on the elements of L by:*

$$Rabc \Leftrightarrow a \vee b = b \vee c = a \vee c,$$

and let $\mathcal{K}(L)$ be $\langle L, R, 0\rangle$.

Theorem 2.7. *$\mathcal{K}(L)$ is a* **KR** *model structure if and only if L is modular.*

Proof. The first three postulates for a **KR** model structure follow immediately from the definition of R, using only the fact that L is a lattice. Now assume that L is modular; we need to verify the last postulate (the Pasch postulate). Assume that $Rabc$ and $Rcde$, that is to say, $a \vee b = b \vee c = a \vee c$ and $c \vee d = c \vee e = d \vee e$. Define $f = (a \vee d) \wedge (b \vee e)$. We need to show that $Radf$ and $Rfbe$, that is to say, $a \vee f = a \vee d = d \vee f$ and $b \vee f = b \vee e = e \vee f$. We compute

$$\begin{aligned} a \vee f &= a \vee [(a \vee d) \wedge (b \vee e)] && (1)\\ &= (a \vee d) \wedge (a \vee b \vee e) && (2)\\ &= (a \vee d) \wedge (a \vee c \vee e) && (3)\\ &= (a \vee d) \wedge (a \vee c \vee d) && (4)\\ &= a \vee d, && (5) \end{aligned}$$

where the equality (2) follows by modularity. The remaining three equalities follow by an exactly symmetrical argument.

For the converse implication, assume that the Pasch postulate holds, but L is not modular. Then L has a sublattice isomorphic to $\mathfrak{N}_5$, the five-element nonmodular lattice (see Figure 1). In $\mathfrak{N}_5$, we have $R(a, c, a \vee c)$ and $(b \vee c, b, c)$. Since $a \vee c = b \vee c$, it follows by the Pasch postulate that there is an f so that $R(a, b, f)$ and $R(f, c, c)$. Then $f \leq a \vee f = a \vee b = a$, and $f \leq f \vee c = c \vee c = c$, so $f \leq a \wedge c$. Thus $b \vee f \leq b \vee (a \wedge c)$; hence $a \leq a \vee b = b \vee f \leq b \vee (a \wedge c) = b$, contradicting $a > b$. $\square$

Definition 2.8. *If L is a lattice, then an* ideal *of L is a non-empty subset I of L such that*

1. *If $a, b \in I$ then $a \vee b \in I$;*
2. *If $b \in I$ and $a \leq b$, then $a \in I$.*

The family of ideals of a lattice L, ordered by containment, forms a complete lattice $I(L)$. The original lattice L is embedded in $I(L)$ by mapping an element $a \in L$ into the *principal ideal* containing a, $(a] = \{b \mid b \leq a\}$. It is easy to verify that the mapping $a \longmapsto (a]$ is a lattice isomorphism between L and a sublattice of $I(L)$.

Theorem 2.9. *Let L be a modular lattice with least element 0, and $\mathcal{K}(L) = \langle L, R, 0\rangle$ the* **KR** *model structure constructed from L. Then $\mathcal{L}(\mathcal{K}(L))$ is identical with the lattice of ideals of L.*

Proof. We need to show that the non-empty linear subspaces of $\mathcal{K}(L)$ are exactly the ideals of L. Let $S \subseteq L$ be a non-empty linear subspace of L. If $a, b \in S$, and $a \vee b = c$, then $Rabc$, so $c \in S$. If $a \leq b$ and $b \in S$, then $Rbba$, so that $a \in S$, showing that S is an ideal. Conversely, assume that S is an ideal of L. By definition, S is non-empty. If $a, b \in S$ and $Rabc$, then $a \vee b = a \vee c \in S$, so $c \in S$, since $c \leq a \vee c$. $\square$

Corollary 2.10. *Any modular lattice of finite height (hence any finite modular lattice) is representable as $\mathcal{L}(\mathcal{K})$ for some* **KR** *model structure $\mathcal{K}$. In addition, any modular lattice is representable as a sublattice of $\mathcal{L}(\mathcal{K})$ for some* **KR** *model structure $\mathcal{K}$.*

The preceding theorem and corollary constitute a general representation theory for modular lattices. Faigle and Herrmann [4] provided a related representation theorem for modular lattices of finite height. They define a set of axioms for a projective geometry as an incidence structure on partially ordered sets of "points" and "lines," and show that every modular lattice of finite length is isomorphic to the lattice of linear subsets of some finite-dimensional projective geometry.

3 Anticipations of the main construction

The construction of Definition 2.6 is very simple and natural, and it is not surprising that it has occurred earlier in the mathematical literature. In a paper of 1959 [8, p. 463], Bjarni Jónsson asked whether every modular lattic is isomorphic to a lattice of commuting equivalence elements of some relation algebra. His question was answered affirmatively by Roger Maddux in a paper published in 1981 [10]. The construction that he used to answer Jónsson's question is the same as that of Definition 2.6; his paper also contains a version of Theorem 2.9. Maddux's monograph on relation algebras also describes the construction [11, pp. 501-502].

A surprising anticipation of Maddux's construction can be found in a paper by D.K. Harrison [7] published in 1979. Harrison defines a *Pasch geometry* (also known as a *multigroup*) to be a set A with a distinguished element e and a ternary relation Δ defined on A satisfying four postulates. His first postulate is:

> For each $a \in A$ there exists a unique $b \in A$ with $(a, b, e) \in \Delta$; denote b by $a^{\#}$.

In Harrison's terminology, a **KR** model structure is a Pasch geometry in which $a^{\#} = a$ for all $a \in A$. Proposition 8 of his paper shows that if the construction of Definition 2.6 is applied to a lattice L with least element e, then the resulting structure (L, Δ, e) is a Pasch geometry if and only if L is modular. The second part of Theorem 2.7 above is adapted from Harrison's proof of his Proposition 8.

4 KR and projective spaces

In an earlier paper [13], I showed that there is a close connection between **KR** and projective geometry. More precisely, I proved that every lattice arising from

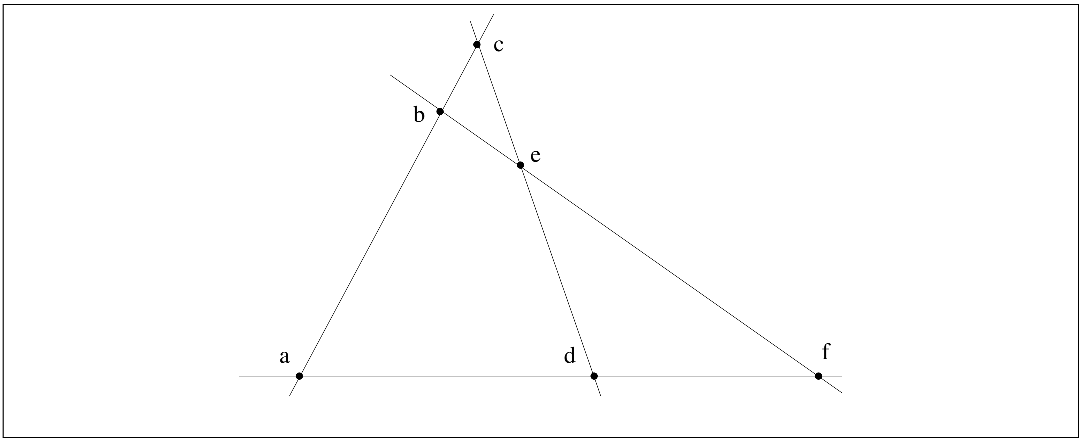

Figure 2: The Pasch Postulate

a broad class of projective spaces can be represented as $\mathcal{L}(\mathcal{K})$ for some **KR** model structure $\mathcal{K}$. The proof proceeded by a direct construction of a model structure from a projective space; the construction is essentially the same as that given earlier by Roger C. Lyndon [9] to produce examples of non-representable relation algebras. In the 1983 paper [13], the construction is only sketched; my paper on interpolation from 1993 [15] contains a full exposition.

The present section gives a new proof of the earlier results, based on Theorems 2.7 and 2.9. Before giving the proof, we need some definitions and results relating to projective spaces and the lattices that arise from them; they are adapted from the text of Grätzer [6, Chapter IV, §5].

Definition 4.1. *Let A be a set and L a collection of subsets of A. The pair $\langle A, L\rangle$ is a* projective space *iff the following properties hold:*

1. *Every $l \in L$ has at least two elements;*

2. *For any two distinct $p, q \in A$, there is exactly one $l \in L$ so that $p, q \in l$;*

3. *Pasch Postulate: For $a, b, c, d, e \in A$ and $l_1, l_2 \in L$ satisfying $a, b, c \in l_1$ and $c, d, e \in l_2$, there exist $f \in A$ and $l_3, l_4 \in L$ satisfying $a, d, f \in l_3$ and $b, e, f \in l_4$.*

We call the members of A *points* and those of L *lines*. For $p, q \in A$, $p \neq q$, let $p + q$ denote the unique line containing p and q; if $p = q$, set $p + q = \{p\}$. Apart from degenerate cases, the Pasch Postulate states that if a line $b + e$ intersects two sides, $a + c$ and $c + d$ of the triangle $\{a, c, d\}$, then it intersects the third side, $a + d$; see Figure 2.

If L is a lattice with least element 0, then $a \in L$ is an *atom* if $h(a) = 1$. An element a of a complete lattice L is *compact* if and only if $a \leq \bigvee X$ for some $X \subseteq L$ implies that $a \leq \bigvee Y$ for some finite $Y \subseteq X$.

Definition 4.2. *A lattice L is a* modular geometric lattice *iff L is complete, every element of L is a join of atoms, all atoms are compact, and L is modular.*

A subset X of the set of atoms of a projective space is a *linear subspace* iff $p + q \subseteq X$ whenever $p, q \in X$.

Theorem 4.3. *The linear subspaces of a projective space form a modular geometric lattice, where $A \wedge B = A \cap B$ and*

$$A \vee B = \bigcup \{a + b \mid a \in A, b \in B\}.$$

Proof. See Grätzer [6, Chapter IV, §5, Theorem 5]. □

The construction of a modular geometric lattice from a projective space can be reversed. Given such a lattice L, define a geometry $G(L)$ by defining the points to be the set of atoms of L, while the lines are the elements of L with height 2.

Theorem 4.4. *If L is a modular geometric lattice, then $G(L)$ is a projective space, and L is isomorphic to the lattice of linear subspaces of $G(L)$.*

Proof. See Grätzer [6, Chapter IV, §5]. □

The two preceding theorems show that there is an exact correspondence between projective spaces and modular geometric lattices.

Lemma 4.5. *Let L be a modular geometric lattice. Then the set F of elements of L of finite height is an ideal of L, and every element of F is a finite join of atoms. L is isomorphic to $I(F)$, the lattice of all ideals of F.*

Proof. See Grätzer [6, Corollary 2, p. 179]. □

Theorem 4.6. *Let L be a modular geometric lattice. Then L is isomorphic to $\mathcal{L}(\mathcal{K})$, for some* **KR** *model structure $\mathcal{K}$.*

Proof. Let F be the family of elements of L of finite height. Then F forms a modular lattice, so we can construct a **KR** model structure $\mathcal{K}(F)$ by Definition 2.6. By Theorem 2.9 and Lemma 4.5, L is isomorphic to $\mathcal{L}(\mathcal{K}(F))$. □

The preceding theorem includes the results of [13], but in fact goes further, because the earlier results omitted certain projective spaces and the corresponding

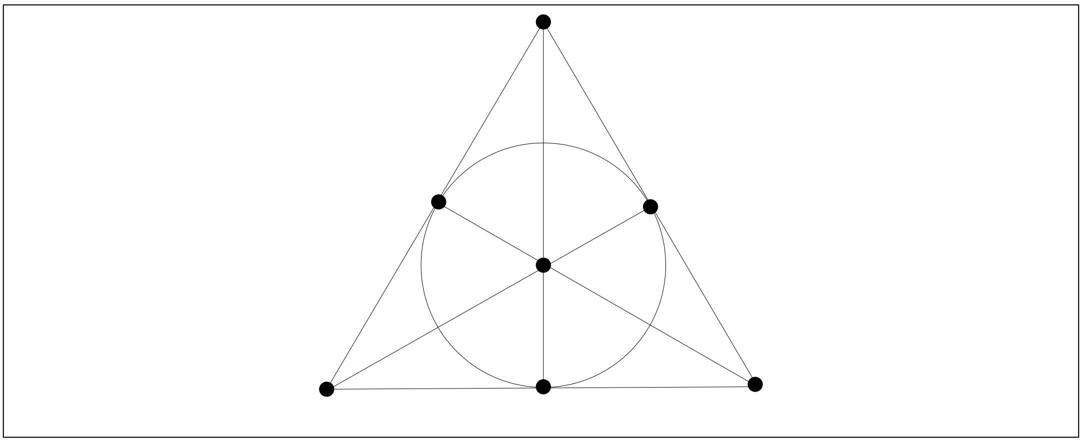

Figure 3: The Fano Plane

geometries. In particular, the construction of **KR** model structures in my 1984 paper required that the underlying projective spaces have at least four points on each line (the construction of Lyndon [9] has the same restriction). This restriction means that the important special case of geometries constructed from the two-element field are not represented. In particular, the best known example of a finite geometry, the Fano plane (Figure 3), is not included in the family of lattices represented in the construction of [13]. The present construction is not only much simpler, but includes these geometries in its scope.

Roger Maddux has reminded me of the fact that Lyndon does treat the case of geometries over the two-element field, though only as an aside [9, p. 24]. The difficulty in the case of the two-element field arises from contraction. If we assume the second postulate in the definition of a **KR** model structure, *Raaa*, then there are not enough points on a line to validate the Pasch postulate. If we omit this postulate, though, we can construct models for contraction-free logics, following Lyndon's method.

5 An application, a problem and acknowledgments

The simple construction of this paper indicates that further results about **KR** and other relevant logics can very likely be obtained by adapting ideas from the well developed and deep theory of modular lattices. As a minor application illustrating these possibilities, we show that if $\mathfrak{A}(G)$ is a **KR**-algebra freely generated by G, then there is a set $G^* \subseteq \mathcal{L}(\mathfrak{A}(G))$ so that G^* freely generates a sublattice of $\mathcal{L}(\mathfrak{A}(G))$. No doubt other such applications can be found, and we include as an open problem another possible use for the construction.

Theorem 5.1. *Let $\mathfrak{A}$ be a* **KR***-algebra, and G a subset that freely generates $\mathfrak{A}$. If $G^* = \{a \to a : a \in G\}$, then G^* freely generates a sublattice of $\mathcal{L}(\mathfrak{A})$.*

Proof. Let L be the sublattice of $\mathcal{L}(\mathfrak{A})$ generated by G^*. If M is a modular lattice with least element 0, and $f : G^* \longmapsto M$ a function from G^* to M, then we need to show that f can be extended to a lattice homomorphism from L to M.

Using Definition 2.6, we can define the **KR** model structure $\mathcal{K}(M)$, and hence by Definition 2.1, the **KR**-algebra $\mathfrak{B} = \mathfrak{A}(\mathcal{K}(M))$. For $a \in G$, define $g(a) = f(a \to a)$. Since G freely generates $\mathfrak{A}$, g can be extended to a homomorphism h from $\mathfrak{A}$ to $\mathfrak{B}$. By Theorem 2.9, $\mathcal{L}(\mathfrak{B})$ is identical with the lattice of ideals of M, so that we can identify M with a sublattice of $\mathcal{L}(\mathfrak{B})$ by the embedding $a \rightleftharpoons (a]$ that maps an element $a \in M$ into the principal ideal generated by a.

For $a \in G^*$, let $a = b \to b$, for $b \in G$. Then

$$h(a) = h(b \to b) = h(b) \to h(b) = g(b) \to g(b)$$

$$= f(b \to b) \to f(b \to b) = f(a) \to f(a) = f(a).$$

Thus, h restricted to L is a lattice homomorphism from L to M extending f, showing that G^* freely generates L. □

Corollary 5.2. *In the logic* **KR***, there are infinitely many distinct formulas built from the formulas $p \to p, q \to q, r \to r$ and $s \to s$ using only the connectives $\wedge$ and $\circ$.*

Proof. Theorem 5.1 shows that the formulas $p \to p, q \to q, r \to r$ and $s \to s$ generate an algebra of formulas isomorphic to the free modular lattice on four generators. This algebra is infinite [2, p. 64]. □

Beth's theorem equating implicit and explicit definability is known to fail in many of the well known relevant logics such as **R**. However, the proof of this result [16] depends on the fact that classical Boolean negation is missing from these logics, and so does not apply to **KR**.

Problem 5.3. *Does Beth's definability theorem hold in the logic* **KR***?*

The construction of this paper suggests a way to attack this problem. The algebraic counterpart of the Beth definability theorem in a variety of algebras is the property that epimorphisms are surjective. Ralph Freese [5, Theorem 3.3] has shown that this property fails in the category of modular lattices and lattice homomorphisms. Consequently, a possible strategy to attack this problem would be to adapt Freese's proof to the algebra of **KR**.

This paper was presented at a special session on algebraic logic (organized by Nick Galatos and Peter Jipsen) at the regional meeting of the ASM in Denver, October 2016. At my talk, Roger Maddux told me of his earlier work on the construction of §2; I am indebted to him for his comments on this paper, and for providing the list of references in §3.

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