Distributive $\ell$-Pregroups

R. Ball, N. Galatos, and P. Jipsen

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A lattice-ordered pregroup, or just $\ell$-pregroup, is a structure of the form $\langle L, \cdot, 1, l, r, \lor, \land, \rangle$, where

- $\langle L, \cdot, 1 \rangle$ is a monoid,
- $\langle L, \lor, \land \rangle$ is a lattice,
- multiplication on either side preserves order,
- and $x^l x \leq 1 \leq xx^l$ and $xx^r \leq 1 \leq x^r x$.

Alternatively, $L$ is a residuated lattice such that $x^{lr} = x = x^{rl}$ and $(xy)^l = y^l x^l$.

An $\ell$-pregroup is distributive if it is distributive as a lattice.

The variety of distributive $\ell$-pregroups has the variety of $\ell$-groups as an important subvariety. It is picked out by the equation $x^l = x^r$.

The elements which satisfy the foregoing equation form an $\ell$-group inside any $\ell$-pregroup.
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The fat question: are all $\ell$-pregroups distributive?

- A modular $\ell$-pregroup is distributive. This fact first came to light as the result of a two-month run on an automated theorem prover. Peter has reduced this proof to a single page. Nevertheless, the proof remains opaque.

- Is an $\ell$-pregroup modular?

**Theorem**

*If a pregroup contains a pentagon then the pivot element cannot be invertible.*

**Proof.**

It suffices to prove this for pivot element 1.

- $da = (1 \land b) a = a \land ba \geq b$
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An ℓ-semigroup with right identity is distributive iff it can be embedding into End(Ω), the ℓ-monoid of order-preserving endomorphisms of some chain Ω.

- The question becomes which \( f \in \text{End}(Ω) \) have residuals \( f^l \) and \( f^r \)? Which have residuals of all orders?
- Note that \( f \) and \( f^l \) form a Galois pair, as do \( f \) and \( f^r \). It follows that if both \( f^l \) and \( f^r \) exist then \( f \) must preserve all existing joins and meets in \( Ω \).
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Which endomorphisms have residuals?

- **Theorem**
  
  An endomorphism $f \in \text{End}(\Omega)$ has a left residual $f^l$ iff, for each $\alpha \in \Omega$, $\{\beta : \beta f \leq \alpha\}$ contains a greatest element. And in that case
  
  $$\alpha f^l = \bigvee_{\beta f \leq \alpha} \beta$$
  
  And dually.

- **Proof.**
  
  - We claim that $\beta f \leq \alpha$ iff $\beta \leq \alpha f^l$.
  - Recall that $f^l f \leq 1 \leq ff^l$. Therefore
  - $\beta f \leq \alpha$ implies (apply $f^l$ to both sides)
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  - The argument for the converse is similar.
  - The claim proves the theorem.
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Two violations of the theorem

- To intervals of constancy
- To lacunas in the range
Endomorphisms with residuals must have coterminial range

- In order for an endomorphism $f \in \text{End}(\Omega)$ to have a left residual $f^l$, its range $[\Omega]f$ must be co-initial in $\Omega$, i.e., for all $\alpha \in \Omega$ there must be some $\beta \in \Omega$ such that $\beta f \leq \alpha$.

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- We say that the range of $f$ is \textit{coterminal in $\Omega$} if it is both co-initial and cofinal.
Intervals of constancy

Definition
Elements $\alpha, \beta \in \Omega$ form a covering pair if $\alpha < \beta$ and, for all $\gamma$, $\alpha \leq \gamma \leq \beta$ implies $\gamma = \alpha$ or $\gamma = \beta$. We write $\alpha \prec \beta$, and we say that $\alpha$ is covered by $\beta$. We denote $\beta$ by $\alpha + 1$ and to $\alpha$ as $\beta - 1$.

Definition
An interval of constancy of an endomorphism $f$ is a convex subset $\Lambda \subseteq \Omega$ of cardinality at least 2 such that $\alpha f = \beta f$ for all $\alpha, \beta \in \Lambda$. Such an interval is said to be maximal if it is contained in no strictly larger interval of constancy.

Lemma
Let $f$ be an endomorphism for which both left and right residuals exist. Then every interval of constancy of $f$ is contained in a maximal such interval, and every maximal interval $\Lambda$ is of the form $[\gamma f^r, \gamma f^l]$ for $[\Lambda]f = \{\gamma\}$. 
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A lacuna in the range of $f$ is a nonempty convex subset $\Lambda \subseteq \Omega$ which is disjoint from the range of $f$. Such an interval is said to be maximal if it is contained in no strictly larger lacuna in the range of $f$.

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What else can we say about intervals of constancy?

Lemma

Suppose $f$ is an endomorphism whose second order residuals exist. Suppose also that $[\alpha, \beta]$ is a maximal interval of constancy of $f$. Then beta is covered and $\alpha$ is a cover.

Proof.

(1) Suppose $[\alpha, \beta] \equiv \Lambda$ is a maximal interval of constancy of $f$, say $[\Lambda]f = \{\gamma\}$, and for argument’s sake suppose $\alpha$ is not a cover, i.e., so that $\alpha = \bigvee \Delta$ for $\Delta \equiv \{\delta : \delta < \alpha\}$. Since both $f$ and $f^l$ preserve order, we have $\bigvee \Delta \delta ff^l = \alpha ff^l = \gamma f^l = \beta$.

We claim that $[\Delta]f$ has no greatest element. For if so, say $\delta f = \delta_1 f$ for some $\delta_1 < \alpha$ and all $\delta_1 < \delta < \alpha$, then $f$ has another interval of constancy which includes $[\delta_1, \alpha)$ but is disjoint from $[\alpha, \beta]$. This contradicts the closure of maximal intervals of constancy and proves the claim.

The claim implies that each $\delta ff^l$ is bounded above by $\alpha$, i.e., $\bigvee \Delta ff^l = \alpha$, contrary to the conclusion above.
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What else can we say about lacunas in the range?

Lemma

Suppose $f$ is an endomorphism whose second order residuals exist. Suppose also that $(\alpha f, \beta f)$, $\alpha \prec \beta$, is a maximal lacuna in the range of $f$. Then $\alpha f$ is covered and $\beta f$ is a cover.
Intervals of constancy correspond to lacunas in the range

Lemma
Let \([\alpha, \beta] \equiv \Lambda\) be a maximal interval of constancy for an endomorphism \(f\) having all its second residuals.

\begin{itemize}
  \item \((\alpha - 1, \beta)\) is a maximal lacuna in the range of \(f^l\), and every such lacuna arises in this fashion.
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Let \((\alpha, \beta) \equiv \Lambda\) be a maximal lacuna in the range of an endomorphism \(f\) having all its second residuals.

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A point \( \alpha \in \Omega \) is called integral if \( \alpha + n \) exists in \( \Omega \) for all \( n \in \mathbb{Z} \).

Theorem
If an endomorphism \( f \) has residuals of all orders then the endpoints of its maximal intervals of constancy, along with the endpoints of the maximal lacunas in its support, are all integral points.
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▶ Theorem
An endomorphism \( f \in \text{End}(\Omega) \) has residuals of all orders iff it has these properties.
- The range of \( f \) is coterminal in \( \Omega \).
- For each \( \alpha \in \Omega \), the set \( \{ \beta : \beta f \leq \alpha \} \) has a greatest element, and dually.
- Each maximal interval of constancy of \( f \) has the form \([\alpha, \beta]\), where \( \alpha \) and \( \beta \) are integral points.
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The family of endomorphisms which satisfy these conditions, call it \( E(\Omega) \), forms a distributive \( \ell \)-pregroup. It is the unique largest \( \ell \)-pregroup contained in \( \text{End}(\Omega) \).
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An endomorphism \( f \in \text{End}(\Omega) \) has residuals of all orders iff it has these properties.

- The range of \( f \) is coterminal in \( \Omega \).
- For each \( \alpha \in \Omega \), the set \( \{ \beta : \beta f \leq \alpha \} \) has a greatest element, and dually.
- Each maximal interval of constancy of \( f \) has the form \([\alpha, \beta]\), where \( \alpha \) and \( \beta \) are integral points.
- Each maximal lacuna in the range of \( f \) has the form \((\alpha, \beta)\) for integral points \( \alpha \) and \( \beta \).

▶ **Theorem**

The family of endomorphisms which satisfy these conditions, call it \( E(\Omega) \), forms a distributive \( \ell \)-pregroup. It is the unique largest \( \ell \)-pregroup contained in \( \text{End}(\Omega) \).
A Holland-style representation for distributive $\ell$-pregroups

Theorem

Every $\ell$-pregroup is isomorphic to a sub-$\ell$-pregroup of $E(\Omega)$ for some chain $\Omega$.

- If $\Omega$ has no covering pairs then $\text{End}(\Omega) = \text{Aut}(\Omega)$. In fact, if $\Omega$ has no integral points then $\text{End}(\Omega) = \text{Aut}(\Omega)$.
- Every automorphism of $E(\Omega)$ must take integral points to integral points.
- A sub-$\ell$-pregroup $G \subseteq E(\Omega)$ is called quasitransitive if it has a point $\alpha_0 \in \Omega$, called the source, such that for all $\beta \in \Omega$ there is some $g \in G$ for which $\alpha_0 g = \beta$.
- The quasitransitive sub-$\ell$-pregroups of $E(\Omega)$ are the building blocks of a structure theory.
- The theory of $\ell$-permutation groups is well-developed and deep. The theory of $\ell$-pregroups which are not $\ell$-groups should be simpler.
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A Holland-style representation for distributive \(\ell\)-pregroups

- **Theorem**

  *Every \(\ell\)-pregroup is isomorphic to a sub-\(\ell\)-pregroup of \(E(\Omega)\) for some chain \(\Omega\).*

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  - A sub-\(\ell\)-pregroup \(G \subseteq E(\Omega)\) is called *quasitransitive* if it has a point \(\alpha_0 \in \Omega\), called the source, such that for all \(\beta \in \Omega\) there is some \(g \in G\) for which \(\alpha_0 g = \beta\).

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An example

$$\Omega \equiv \mathbb{Z} \rightarrow \mathbb{Z}$$

$$(m, n) f \equiv \begin{cases} (2n, n) & \text{if } n \geq 1 \\ (2n - 1, n) & \text{if } n \leq 0 \end{cases}$$

$$(k, l) f' \equiv \begin{cases} \left(\frac{k}{2}, l\right) & \text{if } k \text{ is even and } l \geq 1 \\ \left(\frac{k}{2}, 0\right) & \text{if } k \text{ is even and } l \leq 0 \\ \left(\frac{k+1}{2}, l\right) & \text{if } k \text{ is odd and } l \leq 0 \\ \left(\frac{k+1}{2}, 0\right) & \text{if } k \text{ is odd and } l \geq 1 \end{cases}$$

$f$ has no intervals of constancy but infinitely many lacunas in its range.

$f'$ has infinitely many intervals of constancy and no lacunas in its range.
Thank you!