Unification on Subvarieties of Pseudocomplemented lattices

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Introduction

Algebraic Unification

Introduction
Algebraic Unification

[1] S. Ghilardi,
Unification through projectivity,

Unification Problem: Finitely presented algebra $A$
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Unification Problem: Finitely presented algebra $A$

Solution (Unifier): $h: A \rightarrow P$

$P$ is projective

Unification Problem: Finitely presented algebra $A$

Solution (Unifier): $h: A \rightarrow P$

$P$ is projective

Pre-order:

\[
\begin{array}{c}
A \\ \downarrow h' \\
\downarrow h \\
P \\ \downarrow f \\
P'
\end{array}
\]
Introduction

Algebraic Unification

Let $A \in V$ a finitely presented algebra of a variety $V$ and $\mathcal{U}_V(A)$ the pre-order of its unifiers. Then $A$ is said to have unification type:
Introduction

Algebraic Unification

Let $A \in \mathcal{V}$ a finitely presented algebra of a variety $\mathcal{V}$ and $\mathcal{U}_\mathcal{V}(A)$ the pre-order of its unifiers. Then $A$ is said to have unification type:

$$\mathcal{U}_\mathcal{V}(A)$$
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Algebraic Unification

Let \( A \in V \) a finitely presented algebra of a variety \( V \) and \( \mathcal{U}_V(A) \) the pre-order of its unifiers. Then \( A \) is said to have unification type:

\[ \mathcal{U}_V(A) \]

\[ \begin{array}{c}
1 \\
n
\end{array} \]
Introduction

Algebraic Unification

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Introduction
Algebraic Unification

A variety $\mathcal{V}$ is said to have type:
A variety $\mathcal{V}$ is said to have type:

- 1 if every finitely presented $A$ in $\mathcal{V}$ has unification type 1; 

- $\omega$ if every finitely presented $A$ in $\mathcal{V}$ has finite unification type and at least one finitely presented $A_0$ in $\mathcal{V}$ has unification type 1 and at least one finitely presented $A_0$ in $\mathcal{V}$ has unification type $\infty$;

- $\infty$ if every finitely presented $A$ in $\mathcal{V}$ has unification type $1$, $n$, or $\infty$ and at least one finitely presented $A_0$ in $\mathcal{V}$ has unification type $\infty$;

- 0 if at least one finitely presented $A_0$ in $\mathcal{V}$ has unification type 0.
Introduction
Algebraic Unification

A variety $\mathcal{V}$ is said to have type:

- $1$ if every finitely presented $A$ in $\mathcal{V}$ has unification type $1$;
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Introduction
Algebraic Unification

A variety $\mathcal{V}$ is said to have type:

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- $0$ if at least one finitely presented $A_0$ in $\mathcal{V}$ has unification type $0$. 
## Introduction
Fragments of Heyting algebras

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<td>Heyting algebras</td>
<td>$\omega$</td>
<td>(Ghilardi)</td>
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<td>Hilbert algebras</td>
<td>1</td>
<td>(Prucnal)</td>
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<td>(Ghilardi)</td>
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<td>($\to, \neg$)-Fragment</td>
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<td>Bounded Distributive Lattices</td>
<td>0</td>
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<td>Pseudocomplemented Lattices</td>
<td>0</td>
<td>(Ghilardi)</td>
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Introduction

Fragments of Heyting algebras: Bounded Distributive lattices

Introduction
Fragments of Heyting algebras: Bounded Distributive lattices


Theorem
Let $L$ be a finitely presented (equivalently finite) bounded distributive lattice and $H(L)$ be its Priestley dual. Then the unification type of $L$ is:

1. $1$ iff $H(L)$ is a lattice;
2. finite iff for every $x, y \in H(L)$ the interval $[x, y]$ is a lattice;
3. $0$ otherwise.
Pseudocomplemented Lattices

Definition

An algebra \((A, \lor, \land, \neg, 0, 1)\) is a pseudocomplemented distributive lattice if \((A, \lor, \land, 0, 1)\) is a bounded distributive lattice and it satisfies

\[
a \land b = 0 \iff a \leq \neg b
\]
Pseudocomplemented Lattices

Duality

[2] H.A. Priestley,
The construction of spaces dual to pseudocomplemented distributive lattices,

[3] A. Urquhart,
Projective distributive p-algebras,
Pseudocomplemented Lattices

Duality

<table>
<thead>
<tr>
<th>Algebra</th>
<th>Spaces</th>
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<tr>
<td>((L, \lor, \land, \neg, 0, 1))</td>
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Pseudocomplemented Lattices

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### Pseudocomplemented Lattices

#### Duality

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<td><strong>Homomorphisms</strong></td>
<td><strong>Order preserving maps commute with</strong> (\min)</td>
</tr>
<tr>
<td><strong>Projective</strong></td>
<td>((\ast)) <strong>Join-Semilattice</strong> ((J, \leq))</td>
</tr>
<tr>
<td></td>
<td>(\min: J \to \mathcal{P}(J))</td>
</tr>
<tr>
<td></td>
<td>is join preserving</td>
</tr>
</tbody>
</table>
Pseudocomplemented Lattices

Duality

Algebraic Unifiers

\[ A \xrightarrow{h_1} P_1 \]
\[ h_2 \downarrow \]
\[ P_2 \]
\[ h_\downarrow \]

Dual Unifiers

\[ R_1 \xrightarrow{\eta_1} Q \]
\[ \mu \uparrow \]
\[ R_2 \]
\[ \eta_2 \uparrow \]
Pseudocomplemented Lattices

Main Result

Definition
Let \((X, \leq)\) be a finite poset and

\[ X' = \bigcup \{ \eta(Y) \mid \eta : Y \to X \text{ and } Y \text{ satisfies } (*) \} \).

Then the subposet \((X', \leq_{X'})\) with the order inherited from \((X, \leq)\) is called the unification core of \((X, \leq)\).
Definition
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\[
X' = \bigcup\{\eta(Y) \mid \eta: Y \to X \text{ and } Y \text{ satisfies } (*)\}.
\]

Then the subposet \((X', \leq_{X'})\) with the order inherited from \((X, \leq)\) is called the \textit{unification core of } \((X, \leq)\).

Definition
Let \((X, \leq)\) be a finite poset and \(Y \subseteq X\). We say that \(Y\) is \textit{connected} if it satisfies

\[(i) \quad \min(Y) \subseteq Y;\]

\[(ii) \quad \text{for each } x, y \in Y \text{ there exists } z \in Y \text{ such that } x, y \leq z \text{ and } \min(x) \cup \min(y) = \min(z).\]
Pseudocomplemented Lattices

Main Result

Theorem

Let $A$ be a finitely presented pseudocomplemented lattice and $(X, \leq)$ be it dual space. If $X'$ is its unification core

\[ \text{Type}(U_P(A)) = \begin{cases} \text{finite} & \text{if each } Y \in \max(\text{Con}(X')), \\ 0 & \text{otherwise} \end{cases} \]

where $\text{Con}(X') \subseteq P(X')$ denotes the family of connected subsets of $(X', \leq)$. 
Theorem

Let $A$ be a finitely presented pseudocomplemented lattice and $(X, \leq)$ be its dual space. If $X'$ is its unification core, then

$$\text{Type}(\mathcal{U}_P(A)) = \begin{cases} 
\text{finite} & \text{if each } Y \in \max(\text{Con}(X')), \\
\text{satisfies } (\ast) & \\
0 & \text{otherwise};
\end{cases}$$

where $\text{Con}(X') \subseteq \mathcal{P}(X')$ denotes the family of connected subsets of $(X', \leq_{X'})$.
Pseudocomplemented Lattices

Sketch of the proof
Pseudocomplemented Lattices

Other Results

- Classification of unification problems in each subvariety of pseudocomplemented algebras.

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<th>Variety</th>
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<tbody>
<tr>
<td>Boolean algebras</td>
<td>1</td>
</tr>
<tr>
<td>Stone Algebras</td>
<td>0</td>
</tr>
<tr>
<td>$B_n$ ($n \geq 2$)</td>
<td>0</td>
</tr>
</tbody>
</table>
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Thank you for your attention!

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