Unification on Subvarieties of Pseudocomplemented lattices

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Introduction

Algebraic Unification

[1] S. Ghilardi,
Unification through projectivity,
Journal of Logic and Comp. **7**(6) 733-752, 1997.

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Solution (Unifier):	$h\colon A \to P$
	P is projective

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Pre-order:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[dr, "h'"'] & P \arrow[d, "f"] \\ & P' \end{tikzcd}$$

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Algebraic Unification

Let $A \in \mathcal{V}$ a finitely presented algebra of a variety V and $\mathfrak{U}_{\mathcal{V}}(A)$ the pre-order of its unifiers. Then A is said to have unification type:

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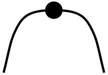
1 

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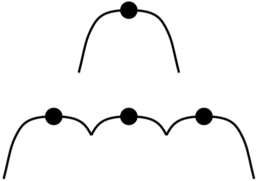

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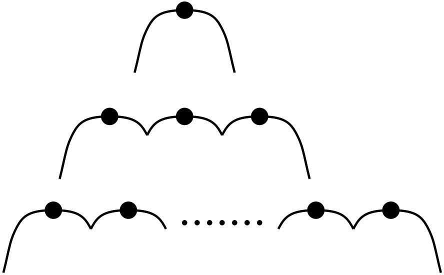

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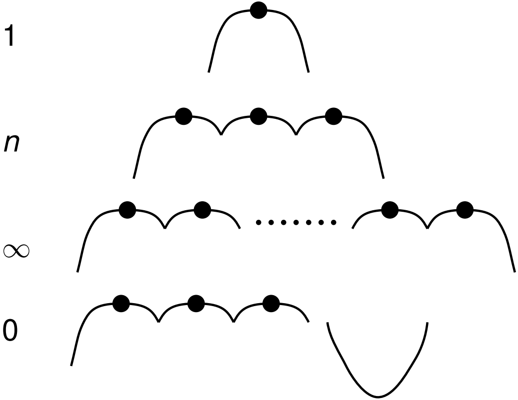

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Algebraic Unification

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- ω if every finitely presented A in $\mathcal{V}$ has finite unification type

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- ∞ if every every finitely presented A of $\mathcal{V}$ has unification 1, n or ∞ and at least one finitely presented A_0 in $\mathcal{V}$ has unification has type ∞;
- 0 if at least one finitely presented A_0 in $\mathcal{V}$ has unification type 0.

Introduction

Fragments of Heyting algebras

Fragment	Type	
Heyting algebras	ω	(Ghilardi)
Hilbert algebras	1	(Prucnal)
Browerian semilattices	1	(Ghilardi)
$(\rightarrow, \neg)$-Fragment	ω	(Cintula-Metcalfe)
Bounded Distributive Lattices	0	(Ghilardi)
Pseudocomplemented Lattices	0	(Ghilardi)

Introduction

Fragments of Heyting algebras: Bounded Distributive lattices

[4] S. Bova and LMC,
Unification and Projectivity in
De Morgan and Kleene Algebras
Order (published online June 2013).

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Theorem

Let **L** *be a finitely presented (equivalently finite) bounded distributive lattice and* $\mathrm{H}(\mathbf{L})$ *be its Priestley dual. Then the unification type of* **L** *is:*

- 1 *iff* $\mathrm{H}(\mathbf{L})$ *is a lattice;*
- finite *iff for every* $x, y \in \mathrm{H}(\mathbf{L})$ *the interval* $[x, y]$ *is a lattice;*
- 0 *otherwise.*

Pseudocomplemented Lattices
Definition

An algebra $(A, \vee, \wedge, \neg, 0, 1)$ is a pseudocomplemented distributive lattice if $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and it satisfies

$$a \wedge b = 0 \quad \Leftrightarrow \quad a \leq \neg b$$

Pseudocomplemented Lattices

Duality

[2] H.A. Priestley,
The construction of spaces dual to pseudocomplemented distributive lattices,
Quarterly Journal of Mathematics
Oxford Series. 26(2) (1975), 215–228.

[3] A. Urquhart,
Projective distributive p-algebras,
Bulletin of the Australian Mathematical Society
24 (1981), 269–275.

Pseudocomplemented Lattices

Duality

Algebra	Spaces
$(L, \vee, \wedge, \neg, 0, 1)$	$(X, \leq)$

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Pseudocomplemented Lattices
Duality

Algebra	Spaces
$(L, \vee, \wedge, \neg, 0, 1)$	$(X, \leq)$
Homomorphisms	Order preserving maps commute with $\min$
Projective	$(*)$ Join-Semilattice $(J, \leq)$ $\min\colon J \to \mathcal{P}(J)$ is join preserving

Pseudocomplemented Lattices
Duality

Algebraic Unifiers	Dual Unifiers
$\begin{tikzcd} A \arrow[r, "h_1"] \arrow[dr, "h_2"'] & P_1 \arrow[d, "f"] \\ & P_2 \end{tikzcd}$	$\begin{tikzcd} R_1 \arrow[r, "\eta_1"] & Q \\ R_2 \arrow[u, "\mu"] \arrow[ur, "\eta_2"'] & \end{tikzcd}$

Pseudocomplemented Lattices

Main Result

Definition

Let $(X, \leq)$ be a finite poset and

$$X' = \bigcup \{\eta(Y) \mid \eta\colon Y \to X \text{ and } Y \text{ satisfies } (*)\}.$$

Then the subposet $(X', \leq_{X'})$ with the order inherited from $(X, \leq)$ is called the *unification core of* $(X, \leq)$.

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Definition

Let $(X, \leq)$ be a finite poset and $Y \subseteq X$. We say that Y is *connected* if it satisfies

(i) $\min(Y) \subseteq Y$;

(ii) for each $x, y \in Y$ there exists $z \in Y$ such that $x, y \leq z$ and $\min(x) \cup \min(y) = \min(z)$.

Pseudocomplemented Lattices

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Theorem

Let A be a finitely presented pseudocomplemented lattice and $(X, \leq)$ be it dual space. If X' is its unification core

Pseudocomplemented Lattices

Main Result

Theorem

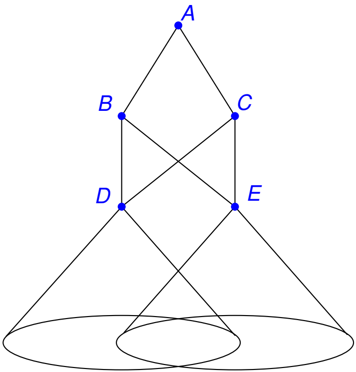
Let A be a finitely presented pseudocomplemented lattice and $(X, \leq)$ be it dual space. If X' is its unification core, then

$$\mathrm{Type}(\mathfrak{U}_{\mathcal{P}}(A)) = \begin{cases} \textit{finite} & \textit{if each } Y \in \max(\mathrm{Con}(X')), \\ & \textit{satisfies } (*) \\ 0 & \textit{otherwise}; \end{cases}$$

where $\mathrm{Con}(X') \subseteq \mathcal{P}(X')$ denotes the family of conected subsets of $(X', \leq_{X'})$

Pseudocomplemented Lattices

Sketch of the proof



Pseudocomplemented Lattices

Other Results

- Classification of unification problems in each subvariety of pseudocomplemented algebras.

Variety	Type
Boolean algebras	1
Stone Algebras	0
$\mathcal{B}_n$ $(n \geq 2)$	0

Unification on Subvarieties of Pseudocomplemented lattices

Thank you for your attention!

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