The finite embeddability property for some noncommutative knotted extensions of RL.

Riquelmi Cardona

University of Denver

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Finite embeddability property

A class of algebras \( \mathcal{K} \) has the finite embeddability property (FEP) if for every \( A \in \mathcal{K} \), every finite partial subalgebra \( B \) of \( A \) can be embedded in a finite \( D \in \mathcal{K} \).
Finite embeddability property

A class of algebras $\mathcal{K}$ has the finite embeddability property (FEP) if for every $A \in \mathcal{K}$, every finite partial subalgebra $B$ of $A$ can be embedded in a finite $D \in \mathcal{K}$.

A *residuated lattice*, is an algebra $L = (L, \wedge, \vee, \cdot, \backslash, /, 1)$ such that

1. $(L, \wedge, \vee)$ is a lattice,
2. $(L, \cdot, 1)$ is a monoid and
3. for all $a, b, c \in L$, $ab \leq c \iff b \leq a \backslash c \iff a \leq c / b$.

RL denotes the variety of residuated lattices.
A (non-trivial) knotted axiom is an inequality of the form $x^m \leq x^n$ for $m \neq n$, $m \geq 1$, $n \geq 0$. 
A (non-trivial) *knotted axiom* is an inequality of the form $x^m \leq x^n$ for $m \neq n$, $m \geq 1$, $n \geq 0$.

Some known examples of these include

- contraction $x \leq x^2$,
- mingle $x^2 \leq x$, and
- integrality $x \leq 1$. 
Some Results

Theorem (Van Alten)

The variety of commutative residuated lattices axiomatized by a knotted axiom has the FEP.
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Theorem

The variety of residuated lattices axiomatized by \( xyx = x^2 y \) and a knotted axiom \( x^m \leq x^n \) has the FEP.
Let’s start with

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The previous equalities can be represented by

\[ xyx = x^{a_0} y x^{a_1}, \]

where \( a_0 + a_1 = 2 \) and \( a_0 a_1 = 0. \)
We consider the generalization

\[ xy_1xy_2x \cdots xy_rx = x^{a_0}y_1x^{a_1}y_2x^{a_2} \cdots x^{a_{r-1}}y_rx^{a_r}, \]

(1)
Generalization

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\[ xy_1 x y_2 x \cdots x y_r x = x^{a_0} y_1 x^{a_1} y_2 x^{a_2} \cdots x^{a_{r-1}} y_r x^{a_r}, \]  

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where at least one of the \( a_i \)'s is equal to 0 and the sum of the \( a_i \)'s is \( r + 1 \).
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where at least one of the $a_i$’s is equal to 0 and the sum of the $a_i$’s is $r + 1$.

Theorem

For $n > m \geq 1, r \geq 1$, the variety $\mathcal{V}_r$ of residuated lattices axiomatized by (1) and a knotted axiom $x^m \leq x^n$ has the FEP.
Let $B$ be a finite partial subalgebra of $A \in \mathcal{V}_r$. Consider $(W, \circ, 1)$, the submonoid of $A$ generated by $B$. 

The relation $N$ is a nuclear relation, because it satisfies the condition $(x \circ y)Nz \iff yN(xz) \iff xNy(z)$. Then $W_{A,B} = (W, W', N, \circ, 1, \{1\})$ is a unital residuated frame.
Let $B$ be a finite partial subalgebra of $A \in \mathcal{V}_r$. Consider $(W, \circ, 1)$, the submonoid of $A$ generated by $B$.

We define $S_W$ to be the set of *unary linear polynomial* (sections) of $(W, \circ, 1)$. Elements of $S_W$ are of the form $u(\_ ) = y \circ \_ \circ w$ for $y, w \in W$. Let $W' = S_W \times B$, and define

$$xN(u, b) \text{ iff } u(x) \leq^A b$$
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$$x N (u, b) \text{ iff } u(x) \leq^A b$$

We define $y \parallel (u, b) = \{(u(y \circ \_), b)\}$ and $(u, b) \parallel y = \{(u(\_ \circ y), b)\}$. The relation $N$ is a nuclear relation, because it satisfies the condition

$$(x \circ y)N z \Leftrightarrow yN(x \parallel z) \Leftrightarrow xN(z \parallel y)$$

Then $W_{A,B} = (W, W', N, \circ, \parallel, \parallel, \{1\})$ is a unital residuated frame.
For $X \subseteq W$ and $Y \subseteq W'$ we define

\[ X^\triangleright = \{ b \in W' : xNb, \text{ for all } x \in X \} \]
\[ Y^\triangleleft = \{ a \in W : aNy, \text{ for all } y \in Y \} \]
Galois algebra

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$$\gamma_N : \mathcal{P}(W) \to \mathcal{P}(W), \quad \gamma_N(X) = X \triangleright \triangleleft,$$

is a closure operator.
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X^{\triangleright} = \{ b \in W' : xNb, \text{ for all } x \in X \}
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$\gamma_N : \mathcal{P}(W) \to \mathcal{P}(W)$, $\gamma_N(X) = X^{\triangleright\triangleleft}$, is a closure operator.

The Galois algebra of $W_{A,B}$ is

\[
W_{A,B}^{+} = (\gamma_N[\emptyset(W)], \cap, \cup \gamma_N, \circ \gamma_N, \setminus, /, \gamma_N(\{1\}))
\]

which is a complete residuated lattice.
The embedding

The map $b \mapsto \{(id, b)\}^\triangleleft$ is an embedding of the partial subalgebra $B$ of $A$ into $W^+_A B$ [Galatos, Jipsen].
The map $b \mapsto \{(id, b)\}^\triangleleft$ is an embedding of the partial subalgebra $B$ of $A$ into $W_{A,B}^+$ [Galatos, Jipsen].

Furthermore, $W_{A,B}^+$ and $A$ belong to $\mathcal{V}_k$ and the closed sets $\{(u, b)\}^\triangleleft$ for $u \in S_W, b \in B$ form a basis for $W_{A,B}^+$. 
The setting

$F$ is a pomonoid and $h$ is an order preserving homomorphism. Furthermore, $h$ is surjective and $F$ is a well partially ordered set.
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\( \mathbf{F} \) is a pomonoid and \( h \) is an order preserving homomorphism. Furthermore, \( h \) is surjective and \( \mathbf{F} \) is a well partially ordered set.
A poset is said to be well partially ordered if it has no infinite antichains and no infinite descending chains. For instance, \( \langle \mathbb{N}, \leq \rangle \) is well partially ordered.
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If $\langle P, \leq \rangle$ is well partially ordered, then it is known that for each $k \in \mathbb{N}$, $P^k$ is well partially ordered under the direct product ordering. Furthermore, homomorphic images, finite disjoint unions, and subposets of well partially ordered sets are well partially ordered.
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Consider the poset \( \langle P, \leq \rangle \). An infinite sequence \( p_1, p_2, \ldots \) of elements of \( P \) is called *bad* when \( i < j \) implies that \( p_i \nleq p_j \). Note that an infinitely descending chain or antichain would be a bad sequence. A poset is well partially ordered if and only if it has no bad sequences.
The proof

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**Lemma**

For each \( b \in B \), \( \langle C_b, \supseteq \rangle \) is well partially ordered.
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For each $b \in B$, define $C_b = \{(u, b)\} : u \in S_W\}$. 

**Lemma**

For each $b \in B$, $\langle C_b, \sqsupseteq \rangle$ is well partially ordered.

**Proof.**

$\langle C_b, \sqsupseteq \rangle$ is a homomorphic image of $\langle F^2, \leq^F \rangle$. 
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Proof.

$\langle C_b, \supseteq \rangle$ is a homomorphic image of $\langle F^2, \leq^F \rangle.$ Define $\varphi : F^2 \to C_b$ by $\varphi(y, w) = \{(h(y) \circ_\circ h(w), b)\} \triangleleft. \varphi$ is surjective.
The proof

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Let $(y_1, w_1), (y_2, w_2) \in F^2$ such that $(y_1, w_1) \leq^F (y_2, w_2)$
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\varphi(y, w) = \{(h(y) \circ _{-} \circ h(w), b)\}^\langle. \quad \varphi \text{ is surjective.}
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Let \((y_1, w_1), (y_2, w_2) \in F^2\) such that \((y_1, w_1) \leq^F (y_2, w_2)\). For all \( x \in F \),
\[
y_1 \cdot^F x \cdot^F w_1 \leq^F y_2 \cdot^F x \cdot^F w_2 \quad \text{and} \quad h(y_1) \circ h(x) \circ h(w_1) \leq h(y_2) \circ h(x) \circ h(w_2).
\]
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Assume that we have $F$ and $h$ satisfy the given conditions.

For each $b \in B$, define $C_b = \{(u, b)\}^{\triangledown} : u \in S_W\}$. 

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$\langle C_b, \triangledown \rangle$ is a homomorphic image of $\langle F^2, \leq_F \rangle$. Define $\varphi : F^2 \to C_b$ by $\varphi(y, w) = \{(h(y) \circ \ _ \circ h(w), b)\}^{\triangledown}$. $\varphi$ is surjective.

Let $(y_1, w_1), (y_2, w_2) \in F^2$ such that $(y_1, w_1) \leq_F (y_2, w_2)$. For all $x \in F$, $y_1 \cdot F x \cdot F w_1 \leq_F y_2 \cdot F x \cdot F w_2$ and $h(y_1) \circ h(x) \circ h(w_1) \leq h(y_2) \circ h(x) \circ h(w_2)$.

Now if $z \in \{(h(y_2) \circ \ _ \circ h(w_2), b)\}^{\triangledown}$, then $h(y_2) \circ h(x) \circ h(w_2) \leq b$. Hence $h(y_1) \circ h(x) \circ h(w_1) \leq b$ and $z = h(x) \in \{(h(y_1) \circ \ _ \circ h(w_1), b)\}^{\triangledown}$.
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Assume that we have $F$ and $h$ satisfy the given conditions.

For each $b \in B$, define $C_b = \{(u, b)\}^\prec : u \in S_W\}.$

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$\langle C_b, \supseteq \rangle$ is a homomorphic image of $\langle F^2, \leq^F \rangle$. Define $\varphi : F^2 \to C_b$ by

$\varphi(y, w) = \{(h(y) \circ _\circ h(w), b)\}^\prec$. $\varphi$ is surjective.

Let $(y_1, w_1), (y_2, w_2) \in F^2$ such that $(y_1, w_1) \leq^F (y_2, w_2)$. For all $x \in F$, $y_1 \cdot F x \cdot F w_1 \leq^F y_2 \cdot F x \cdot F w_2$ and $h(y_1) \circ h(x) \circ h(w_1) \leq h(y_2) \circ h(x) \circ h(w_2)$.

Now if $z \in \{(h(y_2) \circ _\circ h(w_2), b)\}^\prec$, then $h(y_2) \circ h(x) \circ h(w_2) \leq b$. Hence $h(y_1) \circ h(x) \circ h(w_1) \leq b$ and $z = h(x) \in \{(h(y_1) \circ _\circ h(w_1), b)\}^\prec$. So $\{(h(y_1) \circ _\circ h(w_1), b)\}^\prec \supseteq \{(h(y_2) \circ _\circ h(w_2), b)\}^\prec$. $\square$
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Assume there exists an infinite chain \((u_1, b) \supset \supset (u_2, b) \supset \ldots \) in \(C_b\).
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For each \( i \in \mathbb{Z}^+ \), choose \( w_i \in \{(u_i, b)\} \subset \supseteq \{(u_{i+1}, b)\} \subset \supseteq \).
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\( C_b \) is finite for every \( b \in B \). Thus, there are finitely many sets \( \{(u, b)\} \).
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\( C_b \) is finite for every \( b \in B \). Thus, there are finitely many sets \( \{(u, b)\} \triangleleft \).
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For instance, in the monoid on generators \(\{z_1, z_2, z_3, z_4, z_5\}\), \(z_5^3z_1^4z_3^2\) will be represented by \(((4, 0, 2, 0, 3), 513)\)
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For instance, in the monoid on generators \(\{z_1, z_2, z_3, z_4, z_5\}\), \(z_5^3 z_1^4 z_3^2\) will be represented by \(((4, 0, 2, 0, 3), 513)\)

For that purpose we need to look at the defining equation and obtain information out of it.
For instance, consider the equation

\[ xy_1 xy_2 xy_3 xy_4 xy_5 x = x^2 y_1 y_2 y_3 x^3 y_4 y_5 x. \]
Example

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We can use it rewrite expressions like

\[ xy_1 xy_2 xy_3 xy_4 xy_5 xy_6 xy_7 xy_8 xy_9 x = (x^2 y_1 y_2 y_3 x^3 y_4 y_5 x) y_6 xy_7 xy_8 xy_9 x \]
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In general, we can gather generators together when we have enough of them.
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\[ = x^6y_1y_2y_3x^3y_4y_5y_6y_7y_8y_9x \]
\[ = xx^8y_1y_2y_3y_4y_5y_6y_7y_8y_9x \]

In general, we can gather generators together when we have enough of them.
Further work
Thank you for your attention.