SMALL CONGRUENCE LATTICES

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joint work with

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**The Problem**
Characterize congruence lattices of finite algebras.

For an arbitrary algebra, there is essentially no restriction on the shape of its congruence lattice.

**Theorem (Grätzer-Schmidt, 1963)**

Every algebraic lattice is isomorphic to the congruence lattice of an algebra.
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If an algebra is finite, then its congruence lattice is...
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Characterize congruence lattices of finite algebras.

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If an algebra is finite, then its congruence lattice is... finite.

**Problem:** Given an arbitrary finite lattice $L$, does there exist finite algebra $A$ such that $\text{Con } A \cong L$?

**Definition**

We call a finite lattice **representable** if it is (isomorphic to) the congruence lattice of a finite algebra.
A FEW IMPORTANT THEOREMS

THEOREM (Pudlák and Tůma, 1980)

Every finite lattice can be embedded in $Eq(X)$, with $X$ finite.
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*The following statements are equivalent:*

(1) *Every finite lattice is representable.*

(II) *Every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group.*
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(II) Every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group.

THEOREM (Berman, Quackenbush & Wolk, 1970)

Every finite distributive lattice is representable.
Part 1: methods

Given a finite lattice $L$, find a finite algebra $A$ with $\text{Con} A \cong L$. 
HOW TO FIND A FINITE ALGEBRA WITH A GIVEN CONGRUENCE LATTICE?

1. **Use Closure Properties**
   - Relate the given lattice to other lattices known to be representable. If \( L \) is representable, so is \( A \), the dual of \( L \) (Kurzweil 1985, Netter).
   - Any interval sublattice of \( L \) (follows from A.).
   - Any sublattice that is the union of a principal filter and principal idea of \( L \) (Snow 2000).

2. **Combination of Lattices**
   - If \( L_1 \) and \( L_2 \) are representable, so is
     - 1. the direct product of \( L_1 \) and \( L_2 \) (T˚ uma 1989)
     - 2. the ordinal sum of \( L_1 \) and \( L_2 \) (McKenzie 1984, Snow 2000)
     - 3. the parallel sum of \( L_1 \) and \( L_2 \) (Snow 2000)
HOW TO FIND A FINITE ALGEBRA WITH A GIVEN CONGRUENCE LATTICE?

1. USE CLOSURE PROPERTIES

Relate the given lattice to other lattices known to be representable.

- If $L$ is representable, so is
  A. the dual of $L$ (Kurzweil 1985, Netter)
  B. any interval sublattice of $L$ (follows from A.)
  C. any sublattice that is the union of a principal filter and principal idea of $L$ (Snow, 2000)

- If $L_1$ and $L_2$ are representable, so is
  1. the direct product of $L_1$ and $L_2$ (Tůma 1989)
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How to find a finite algebra with a given congruence lattice?

2. The Closure Method

Find a “closed” representation of $L$ in $\text{Eq}(X)$.

YAGC

For $L \leq \text{Eq}(X)$ define

$$\lambda(L) = \{f \in X^X : (\forall \theta \in L) f(\theta) \subseteq \theta\}$$

For $F \subseteq X^X$ define

$$\rho(F) = \{\theta \in \text{Eq}(X) : (\forall f \in F) f(\theta) \subseteq \theta\}$$

The map $\rho \lambda$ is a closure operator on $\text{Sub}[	ext{Eq}(X)]$.
(idempotent, extensive, order preserving)
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**Theorem**

A lattice $L \leq \text{Eq}(X)$ is a congruence lattice if and only if it is closed,

i.e. $\rho \lambda(L) = L$, in which case $L = \text{Con} \langle X, \lambda(L) \rangle$. 
How to find a finite algebra with a given congruence lattice?

2. The Closure Method

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Example: \( M_3 \cong L \leq \text{Eq}(5) \)
3. The G-set method

Find $L$ as an interval in a subgroup lattice of a finite group.

If $H \leq G$ are finite groups, then the filter above $H$ in $\text{Sub}(G)$,

$$[H, G] := \{K : H \leq K \leq G\},$$

is isomorphic to $\text{Con} \langle G/H, \bar{G} \rangle$. 
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4. **THE RABBIT EARS METHOD** (AKA OVERALGEBRAS, AKA EXPANSION-EXTENSION)

Build the required algebra by gluing together isomorphic copies of an algebra and adding new operations.
The G-set method: details

For groups $H \leq G$, let $A = \langle H \backslash G, \bar{G} \rangle$ denote the algebra with

- universe: the right cosets $H \backslash G = \{Hx : x \in G\}$
- operations: $\bar{G} = \{g^A : g \in G\}$, where $g^A(Hx) = Hxg$. 

Aside: properties of such congruence lattices correspond to properties of subgroup lattices. For example,

**Lemma**

In $\text{Con} \langle H \backslash G, \bar{G} \rangle$, two congruences, $\theta_{K_1}$ and $\theta_{K_2}$, $n$-permute if and only if the corresponding subgroups, $K_1$ and $K_2$, $n$-permute.
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Theorem

$$\text{Con } A \cong \llbracket H, G \rrbracket := \{K : H \leq K \leq G\}.$$  

The isomorphism $\llbracket H, G \rrbracket \ni K \mapsto \theta_K \in \text{Con } A$ is given by

$$\theta_K = \{(Hx, Hy) : xy^{-1} \in K\}.$$  

The inverse isomorphism $\text{Con } A \ni \theta \mapsto K_\theta \in \llbracket H, G \rrbracket$ is

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THE G-SET METHOD: DETAILS

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FILTER+IDEAL METHOD: DETAILS

**Lemma**

Suppose $L_0 \cong \text{Con} \langle A, F \rangle$, and $\alpha, \beta \in L_0 \setminus \{0, 1\}$. 

![Diagram with points labeled $\alpha$, $\beta$, and $L_0$]
FILTER+IDEAL METHOD: DETAILS

**Lemma**

 Suppose \( L_0 \cong \text{Con} \langle A, F \rangle \), and \( \alpha, \beta \in L_0 \setminus \{0, 1\} \). Consider \( L = \alpha^\uparrow \cup \beta^\downarrow \).
FILTER+IDEAL METHOD: DETAILS

Lemma

Suppose $L_0 \cong \text{Con} \langle A, F \rangle$, and $\alpha, \beta \in L_0 \setminus \{0, 1\}$. Consider $L = \alpha^\uparrow \cup \beta^\downarrow$.

There exists a set $F' \subset A^A$ such that $L \cong \text{Con} \langle A, F \cup F' \rangle$. 

\[
L \leq L_0
\]
**FILTER+IDEAL METHOD: DETAILS**

**Lemma**

Suppose $L_0 \cong \text{Con} \langle A, F \rangle$, and $\alpha, \beta \in L_0 \setminus \{0, 1\}$. Consider $L = \alpha^\uparrow \cup \beta^\downarrow$.

There exists a set $F' \subset A^A$ such that $L \cong \text{Con} \langle A, F \cup F' \rangle$.

**Proof:**

Fix $\theta \in L_0 \setminus L$. Then $\alpha \not\leq \theta \not\leq \beta$, so

- $\exists (a, b) \in \alpha \setminus \theta$,
- $\exists (u, v) \in \theta \setminus \beta$.

Define $f_\theta : A \to A$ by

$$f_\theta(x) = \begin{cases} 
    a & x \in u/\beta, \\
    b & x \not\in u/\beta.
\end{cases}$$

Then

- $(f_\theta(u), f_\theta(v)) = (a, b) \not\in \theta$, so $f_\theta(\theta) \not\in \theta$,
- $\ker f_\theta \supseteq \beta$, so $f_\theta(\gamma) \subseteq \gamma$ for all $\gamma \leq \beta$,
- $f_\theta(A) \subseteq \{a, b\}$, so $f_\theta(\gamma) \subseteq \gamma$ for all $\gamma \geq \alpha$.

Let $F' = \{f_\theta : \theta \in L_0 \setminus L\}$. 

\[ L \subseteq L_0 \]

\[ \alpha \quad \theta \quad \beta \]

\[ \alpha \quad \theta \quad \beta \]
Which finite lattices are known to be representable?
Lattices with at most 6 elements are representable.


Aschbacher (2008) *JAMS*
Lattices with at most 6 elements are representable.

Theorem: Lattices with at most 6 elements are intervals in subgroup lattices of finite groups.
Are all lattices with at most 7 elements representable?

As of Spring 2011...

Lattices of size $\leq 7$ not yet known to be congruence lattices of finite algebras

Figure courtesy of Peter Jipsen.
Are all lattices with at most 7 elements representable?
Are all lattices with at most 7 elements representable?

$L_{19}$ ✓

$L_{20}$ ✓

$L_{17}$

$L_{13}$

$L_{11}$

$L_{9}$

$L_{10}$
The group $G = (A_4 \times A_4) \rtimes C_2$ has a subgroup $H \sim S_3$ such that $J_H, G_K \sim L_{17}$.

The group $G = (C_2 \times C_2 \times C_2 \times C_2) \rtimes A_5$ has a subgroup $H \sim A_4$ such that $J_H, G_K \sim L_{13}$.
Finding representations...  
...as intervals in subgroup lattices

The group $G = (A_4 \times A_4) \rtimes C_2$ has a subgroup $H \cong S_3$ such that $[H, G] \cong L_{17}$.

...so the dual $L_{16}$ is also representable.
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The group $G = (C_2 \times C_2 \times C_2 \times C_2) \rtimes A_5$ has a subgroup $H \cong A_4$ such that $[H, G] \cong L_{13}$. 

SmallGroup(288,1025)  
$|G : H| = 48$

SmallGroup(960,11358)  
$|G : H| = 80$
Are all lattices with at most 7 elements representable?
Finding representations...

...using subgroup lattice intervals and the filter+ideal lemma.

$L_{11}$
Finding representations...
...using subgroup lattice intervals and the filter+ideal lemma.

SmallGroup(288,1025)

- Let $G = (A_4 \times A_4) \rtimes C_2$.
- $G$ has a subgroup $H \cong C_6$ with $[H, G] \cong N_5$.
- Let $[H, G] = \{H, \alpha, \beta, \gamma, G\} \cong N_5$. 

$|G : H| = 48$
Let $G = (A_4 \times A_4) \rtimes C_2$.

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Let $[H, G] = \{H, \alpha, \beta, \gamma, G\} \cong N_5$.

Sub($G$) is a congruence lattice, so if there exists a subgroup $K \triangleright 1$, below $\beta$ and not below $\gamma$, then

$$L_{11} \cong K\downarrow \cup H\uparrow.$$

SmallGroup(288,1025)
Finding representations...

...using subgroup lattice intervals and the filter+ideal lemma.

- Let $G = (A_4 \times A_4) \rtimes C_2$.
- $G$ has a subgroup $H \cong C_6$ with $[H, G] \cong N_5$.
- Let $[H, G] = \{H, \alpha, \beta, \gamma, G\} \cong N_5$.
- Sub($G$) is a congruence lattice, so if there exists a subgroup $K \triangleright 1$, below $\beta$ and not below $\gamma$, then

$$L_{11} \cong K^\downarrow \cup H^\uparrow.$$ 

- Sub($A_4$) is a congruence lattice (of $A_4$ acting regularly on itself).
- Therefore,

$$L_{17} \cong V_4^\downarrow \cup P^\uparrow$$

is a congruence lattice.
Are all lattices with at most 7 elements representable?
Are all lattices with at most 7 elements representable?
Construction of an algebra $A$ with $\text{Con} \ A \cong L_9$.

Example: Let $B = \{0, 1, \ldots, 5\}$ index the elements of $S_3$ and consider the right regular action of $S_3$ on itself.

\[
g_0 = (0, 4)(1, 3)(2, 5)
g_1 = (0, 1, 2)(3, 4, 5)\]

generate this action group, the image of $S_3 \hookrightarrow S_6$.

$\text{Con} \langle B, \{g_0, g_1\} \rangle \cong M_4$ with congruences $\alpha = \langle 0, 1, 2 \rangle \langle 3, 4, 5 \rangle$, $\beta = \langle 0, 3 \rangle \langle 1, 4 \rangle \langle 2, 5 \rangle$, $\gamma = \langle 0, 4 \rangle \langle 1, 5 \rangle \langle 2, 3 \rangle$, $\delta = \langle 0, 5 \rangle \langle 1, 3 \rangle \langle 2, 4 \rangle$.

Goal: expand $B$ to an algebra $A$ that has $\alpha$ “doubled” in $\text{Con} \ A$.

Since $\alpha = Cg_B(0, 2)$, we let $A = B_0 \cup B_1 \cup B_2$ where $B_0 = \{0, 1, 2, 3, 4, 5\} = B$, $B_1 = \{0, 6, 7, 8, 9, 10\}$, $B_2 = \{11, 12, 13, 14, 15\}$.

Define unary operations $e_0, e_1, e_2, s, g_0 e_0$, and $g_1 e_0$. 

$M_4$ $L_9$
Construction of an algebra $A$ with $\text{Con} A \cong L_9$.

**Step 1** Take a permutational algebra $B = \langle B, F \rangle$ with congruence lattice $\text{Con} B \cong M_4$.

**STEP 2** Define unary operations $e_0, e_1, e_2, s, g_0 e_0,$ and $g_1 e_0$. 

**STEP 3** $L_9$
**CONSTRUCTION OF AN ALGEBRA $\mathbf{A}$ WITH $\text{Con} \mathbf{A} \cong L_9$.**

**STEP 1** Take a permutational algebra $\mathbf{B} = \langle B, F \rangle$ with congruence lattice $\text{Con} \mathbf{B} \cong M_4$.

**Example:**

- Let $B = \{0, 1, \ldots, 5\}$ index the elements of $S_3$ and consider the right regular action of $S_3$ on itself.
- $g_0 = (0, 4)(1, 3)(2, 5)$ and $g_1 = (0, 1, 2)(3, 4, 5)$ generate this action group, the image of $S_3 \hookrightarrow S_6$.
- $\text{Con} \langle B, \{g_0, g_1\} \rangle \cong M_4$ with congruences $
\alpha = |012|345|, \quad \beta = |03|14|25|, \quad \gamma = |04|15|23|, \quad \delta = |05|13|24|.$
**Construction of an algebra A with Con A \cong L_9.**

**Step 1** Take a permutational algebra \( B = \langle B, F \rangle \) with congruence lattice \( \text{Con} B \cong M_4 \).

**Example:**

- Let \( B = \{ 0, 1, \ldots, 5 \} \) index the elements of \( S_3 \) and consider the right regular action of \( S_3 \) on itself.
- \( g_0 = (0, 4)(1, 3)(2, 5) \) and \( g_1 = (0, 1, 2)(3, 4, 5) \) generate this action group, the image of \( S_3 \hookrightarrow S_6 \).
- \( \text{Con} \langle B, \{ g_0, g_1 \} \rangle \cong M_4 \) with congruences
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  \alpha = |012|345|, \quad \beta = |03|14|25|, \quad \gamma = |04|15|23|, \quad \delta = |05|13|24|.
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**Goal:** expand \( B \) to an algebra \( A \) that has \( \alpha \) “doubled” in \( \text{Con} A \).
**Construction of an Algebra $A$ with $\text{Con } A \cong L_9$.**

**Step 1** Take a permutational algebra $B = \langle B, F \rangle$ with congruence lattice $\text{Con } B \cong M_4$.

**Example:**
- Let $B = \{0, 1, \ldots, 5\}$ index the elements of $S_3$ and consider the right regular action of $S_3$ on itself.
- $g_0 = (0, 4)(1, 3)(2, 5)$ and $g_1 = (0, 1, 2)(3, 4, 5)$ generate this action group, the image of $S_3 \hookrightarrow S_6$.
- $\text{Con } \langle B, \{g_0, g_1\} \rangle \cong M_4$ with congruences $\alpha = |012345|, \beta = |031425|, \gamma = |041523|, \delta = |051324|$.

**Goal:** expand $B$ to an algebra $A$ that has $\alpha$ “doubled” in $\text{Con } A$.

**Step 2** Since $\alpha = \text{Cg}^B(0, 2)$, we let $A = B_0 \cup B_1 \cup B_2$ where

- $B_0 = \{0, 1, 2, 3, 4, 5\} = B$
- $B_1 = \{0, 6, 7, 8, 9, 10\}$
- $B_2 = \{11, 12, 2, 13, 14, 15\}$. 
CONSTRUCTION OF AN ALGEBRA A WITH Con A \cong L_9.

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**Example:**
- Let B = \{0, 1, \ldots, 5\} index the elements of S_3 and consider the right regular action of S_3 on itself.
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\begin{align*}
B_0 &= \{0, 1, 2, 3, 4, 5\} = B \\
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**Step 3** Define unary operations \(e_0, e_1, e_2, s, g_0e_0, \text{ and } g_1e_0\).
Construction of an algebra $A$ with $\text{Con} A \cong L_9$.

Con $\langle B, \{g_0, g_1\} \rangle$

- $\alpha = |0, 1, 2|3, 4, 5|$
- $\beta = |0, 3|1, 4|2, 5|$
- $\gamma = |0, 4|1, 5|2, 3|$
- $\delta = |0, 5|1, 3|2, 4|$

Con $\langle A, F_A \rangle$

- $\hat{\alpha} = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10, 13, 14, 15|$
- $\alpha^* = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10|13, 14, 15|$
- $\beta^* = |0, 3, 8|1, 4|2, 5, 15|6, 9|7, 10|11, 13|12, 14|$
- $\gamma^* = |0, 4, 9|1, 5|2, 3, 13|6, 10|7, 8|11, 14|12, 15|$
- $\delta^* = |0, 5, 10|1, 3|2, 4, 14|6, 8|7, 9, 11, 15|12, 13|
Construction of an algebra $A$ with $\text{Con } A \cong L_9$.

\[
\begin{align*}
\text{Con } \langle B, \{g_0, g_1\} \rangle \quad &\text{Con } \langle A, F_A \rangle \\
\alpha = |0, 1, 2|3, 4, 5| \quad &\hat{\alpha} = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10, 13, 14, 15| \\
\beta = |0, 3|1, 4|2, 5| \quad &\beta^* = |0, 3, 8|1, 4|2, 5, 15|6, 9|7, 10|11, 13|12, 14| \\
\gamma = |0, 4|1, 5|2, 3| \quad &\gamma^* = |0, 4, 9|1, 5|2, 3, 13|6, 10|7, 8|11, 14|12, 15| \\
\delta = |0, 5|1, 3|2, 4| \quad &\delta^* = |0, 5, 10|1, 3|2, 4, 14|6, 8|7, 9, 11, 15|12, 13|
\end{align*}
\]

$\alpha = \alpha^* \cap B^2 = \hat{\alpha} \cap B^2, \quad \beta = \beta^* \cap B^2, \quad \ldots$
Why does it work?

\[ \text{Con} \langle B, \{g_0, g_1\} \rangle \]

\[ \alpha = |0, 1, 2|3, 4, 5| \]
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\delta

B_0
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\[
B_0 = \{ \begin{array}{cccccc}
0 & 1 & 2 \\
3 & 4 & 5 \\
\end{array} \}
\]
**Why does it work?**

\[ \text{Con} \langle B, \{g_0, g_1\} \rangle \]

\[ \alpha = |0, 1, 2|3, 4, 5| \]
\[ \beta = |0, 3|1, 4|2, 5| \]
\[ \gamma = |0, 4|1, 5|2, 3| \]
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- \( A = B_0 \cup B_1 \cup B_2 \)

\[ B_1 = \{ 0, 6, 7, 8, 9, 10 \} \]
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- \( A = B_0 \cup B_1 \cup B_2 \)
- **Unary operations**

\[
\begin{align*}
  e_0: A &\rightarrow B_0 \\
  e_1: A &\rightarrow B_1 \\
  e_2: A &\rightarrow B_2 \\
  s: A &\rightarrow B_0 \\
  ge_0: A &\xrightarrow{e_0} B_0 \xrightarrow{g} B_0 \\
\end{align*}
\]

\[
B_1 = \{ 0, 6, 7, 8, 9, 10 \} \\
B_1 = \{ 7, 6, 0, 1 \} \\
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- $A = B_0 \cup B_1 \cup B_2$
- Unary operations

$e_0: A \to B_0$

$e_1: A \to B_1$

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\[
\begin{array}{cccc}
B_0 &=& \{ & 0 & 1 & 2 & 3 & 4 & 5 & \} \\
& & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
B_1 &=& \{ & 0 & 6 & 7 & 8 & 9 & 10 & \} \\
& & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\
B_2 &=& \{ & 11 & 12 & 2 & 13 & 14 & 15 & \} \\
\end{array}
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B_0 = \{ 0, 1, 2, 3, 4, 5 \}
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\[
B_2 = \{ 11, 12, 2, 13, 14, 15 \}
\]

\[
B_0 = \begin{array}{ccc}
0 & 6 & 7 \\
9 & 10 & 15 \\
12 & 11 & 14
\end{array}
\]

\[
B_1 = \begin{array}{ccc}
0 & 3 & 2 \\
4 & 5 & 1
\end{array}
\]
**WHY DOES IT WORK?**

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\text{Con } \langle B, \{g_0, g_1\} \rangle
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\]

\[
\begin{array}{ccccccc}
    & 7 & 10 & \\
6 & 9 & & \\
0 & 1 & 8 & 2 & \\
3 & 4 & 5 & \\
\end{array}
\]

\[
\begin{array}{ccccccc}
7 & 10 & \\
6 & 9 & \\
0 & 1 & 8 & 2 & \\
3 & 4 & 5 & \\
\end{array}
\]

\[
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7 & 10 & \\
6 & 9 & \\
0 & 1 & 8 & 2 & \\
3 & 4 & 5 & \\
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  \]

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B_0 & & & & & \\
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- \( A = B_0 \cup B_1 \cup B_2 \)
- **Unary operations**

\[ e_0: A \rightarrow B_0 \]

\[ e_1: A \rightarrow B_1 \]

\[ e_2: A \rightarrow B_2 \]

\[ s: A \rightarrow B_0 \]

\[ ge_0: A \xrightarrow{e_0} B_0 \rightarrow^g B_0 \]

\[ B_1 = \{ 0 \, 6 \, 7 \, 8 \, 9 \, 10 \} \]

\[ B_0 = \{ 0 \, 1 \, 2 \, 3 \, 4 \, 5 \} \]

\[ B_2 = \{ 11 \, 12 \, 2 \, 13 \, 14 \, 15 \} \]
Why does it work?

\[
\text{Con} \langle B, \{g_0, g_1\} \rangle
\]

\[
\alpha = |0, 1, 2, 3, 4, 5|
\]

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\]

- \( A = B_0 \cup B_1 \cup B_2 \)
- Unary operations
  
  \[
e_0: A \rightarrow B_0
  \]
  
  \[
e_1: A \rightarrow B_1
  \]
  
  \[
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  \]
  
  \[
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  \]

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Why does it work?

\[ \text{Con} \langle B, \{g_0, g_1\} \rangle \]

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- \( A = B_0 \cup B_1 \cup B_2 \)
- Unary operations
  \[ e_0: A \to B_0 \]
  \[ e_1: A \to B_1 \]
  \[ e_2: A \to B_2 \]
  \[ s: A \to B_0 \]

\[ g e_0: A \xrightarrow{e_0} B_0 \xrightarrow{g} B_0 \]

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- \( A = B_0 \cup B_1 \cup B_2 \)
- **Unary operations**
  
  \[ e_0: A \rightarrow B_0 \]
  
  \[ e_1: A \rightarrow B_1 \]
  
  \[ e_2: A \rightarrow B_2 \]
  
  \[ s: A \rightarrow B_0 \]

\[ ge_0: A \xrightarrow{e_0} B_0 \xrightarrow{g} B_0 \]

**B_1**

\[
\begin{array}{ccc}
10 & 9 & 8 \\
7 & 6 & 0 \\
3 & 4 & 5 \\
\end{array}
\]

**B_2**

\[
\begin{array}{ccc}
15 & 14 & 13 \\
2 & 12 & 11 \\
\end{array}
\]

**B_0**

\[
\begin{array}{ccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
\end{array}
\]

\[ B_1 = \{ 0, 6, 7, 8, 9, 10 \} \]

\[ B_0 = \{ 0, 1, 2, 3, 4, 5 \} \]

\[ B_2 = \{ 11, 12, 2, 13, 14, 15 \} \]
WHY DOES IT WORK?

Con $\langle B, \{g_0, g_1\} \rangle$

$\alpha = |0, 1, 2|3, 4, 5|$
$\beta = |0, 3|1, 4|2, 5|$
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- $A = B_0 \cup B_1 \cup B_2$
- Unary operations
  
  $e_0: A \rightarrow B_0$
  $e_1: A \rightarrow B_1$
  $e_2: A \rightarrow B_2$
  $s: A \rightarrow B_0$

$B_0 = \{ 0 1 2 3 4 5 \}$

$B_1 = \{ 0 6 7 8 9 10 \}$

$B_2 = \{ 11 12 2 13 14 15 \}$
**Why does it work?**

Con \( \langle B, \{g_0, g_1\} \rangle \)

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\]

\[B_1\]

\[
\begin{array}{ccc}
10 & 9 & 8 \\
7 & 6 & 0 \\
3 & 4 & 5 \\
\end{array}
\]

\[B_2\]

\[
\begin{array}{ccc}
15 & 14 & 13 \\
1 & 2 & 12 \\
6 & 9 & 7 \\
\end{array}
\]

Con \( \langle A, F_A \rangle \)

\[
\hat{\alpha} = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10, 13, 14, 15|
\]

\[
\alpha^* = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10|13, 14, 15|
\]

\[
\beta^* = |0, 3, 8|1, 4|2, 5, 15|6, 9|7, 10|11, 13|12, 14|
\]

\[
\gamma^* = |0, 4, 9|1, 5|2, 3, 13|6, 10|7, 8|11, 14|12, 15|
\]

\[
\delta^* = |0, 5, 10|1, 3|2, 4, 14|6, 8|7, 9, 11, 15|12, 13|
\]
Why does it work?

\[
\text{Con } \langle B, \{g_0, g_1\} \rangle
\]

\[
\alpha = \{0, 1, 2, 3, 4, 5\}
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\text{Con } \langle A, F_A \rangle
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**Why does it work?**

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\[ \alpha = |0, 1, 2|3, 4, 5| \]
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\[ \delta = |0, 5|1, 3|2, 4| \]

\[ \begin{array}{ccc} 10 & 9 & 8 \\ 7 & 6 & 0 & 1 & 2 & 12 & 11 \\ \alpha^* & 3 & 4 & 5 \\ \end{array} \]

\[ \text{Con} \langle A, F_A \rangle \]

\[ \hat{\alpha} = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10, 13, 14, 15| \]
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\[ \gamma^* = |0, 4, 9|1, 5|2, 3, 13|6, 10|7, 8|11, 14|12, 15| \]
\[ \delta^* = |0, 5, 10|1, 3|2, 4, 14|6, 8|7, 9, 11, 15|12, 13| \]
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\]

\[
\begin{array}{ccc}
B_1 & & B_2 \\
10 & 9 & 8 & 15 & 14 & 13 \\
7 & 6 & 0 & 1 & 2 & 12 & 11 \\
\end{array}
\]

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\delta^* = |0, 5, 10|1, 3|2, 4, 14|6, 8|7, 9, 11, 15|12, 13|
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\]

\[
\delta = |0, 5|1, 3|2, 4|
\]

*Why don’t the \( \beta \) classes of \( B_1 \) and \( B_2 \) mix?*

\[
\text{Con } \langle A, F_A \rangle
\]

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\]
VARIATIONS ON THE SAME EXAMPLE...

- Suppose we want $\beta = C_g^B(0, 3) = |0, 3|2, 5|1, 4|$ to have non-trivial inverse image $\beta^{-1}_B = [\beta^*, \widehat{\beta}]$. 

\[\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\alpha
\end{array}
\end{array}
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\end{array}\]

\[\begin{array}{c}
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\beta
\end{array}
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\[\begin{array}{c}
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\gamma
\end{array}
\end{array}
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\end{array}\]

\[\begin{array}{c}
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\delta
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\[\begin{array}{c}
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\beta
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\[\begin{array}{c}
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\beta
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\[\begin{array}{c}
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\beta
\end{array}
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\[\begin{array}{c}
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\[\begin{array}{c}
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\beta
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\[\begin{array}{c}
\begin{array}{c}
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\end{array}
\end{array}
\end{array}
\end{array}\]
Suppose we want \( \beta = Cg^B(0, 3) = |0, 3|2, 5|1, 4| \) to have non-trivial inverse image \( \beta|_B^{-1} = [\beta^*, \hat{\beta}] \).

Select elements 0 and 3 as intersection points:

\[
A = B_0 \cup B_1 \cup B_2 \quad \text{where}
\]

\[
B_0 = \{0, 1, 2, 3, 4, 5\}
\]

\[
B_1 = \{0, 6, 7, 8, 9, 10\}
\]

\[
B_2 = \{11, 12, 13, 3, 14, 15\}.
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Select elements 0 and 3 as intersection points:

$$A = B_0 \cup B_1 \cup B_2$$

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Suppose we want $\beta = C_g^B(0, 3) = |0, 3|2, 5|1, 4|$ to have non-trivial inverse image $\beta^{-1}_B = [\beta^*, \hat{\beta}]$.

Select elements 0 and 3 as intersection points:

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Variations on the same example...

- Suppose we want $\beta = C_g^B(0, 3) = |0, 3|2, 5|1, 4|$ to have non-trivial inverse image $\beta|^{-1}_B = [\beta^*, \hat{\beta}]$.
- Select elements 0 and 3 as intersection points:

$$A = B_0 \cup B_1 \cup B_2$$

where

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- $B_1 = \{0, 6, 7, 8, 9, 10\}$
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Suppose we want $\beta = C_g^B(0, 3) = |0, 3|2, 5|1, 4|$ to have non-trivial inverse image $\beta|_B^{-1} = [\beta^*, \hat{\beta}]$.

Select elements 0 and 3 as intersection points:

$A = B_0 \cup B_1 \cup B_2$ where

$B_0 = \{0, 1, 2, 3, 4, 5\}$
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Variations on the same example...

- Suppose we want \( \beta = C_{g^B}(0, 3) = |0, 3|2, 5|1, 4| \) to have non-trivial inverse image \( \beta^{-1}_B = [\beta^*, \hat{\beta}] \).

- Select elements 0 and 3 as intersection points:

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Variations on the same example...

- Suppose we want $\beta = C_g^B(0, 3) = |0, 3|2, 5|1, 4|$ to have non-trivial inverse image $\beta|_B^{-1} = [\beta^*, \hat{\beta}]$.
- Select elements 0 and 3 as intersection points:

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SEVEN ELEMENT LATTICES: SUMMARY

$L_{19}$ ✓

$L_{17}$ ✓

$L_{13}$ ✓

$L_{20}$ ✓

$L_{11}$ ✓

$L_9$ ✓

$L_7$
SEVEN ELEMENT LATTICES: SUMMARY

$L_{19}$ ✓

$L_{17}$ ✓

$L_{13}$ ✓

$L_{11}$ ✓

$L_{9}$ ✓

$L_{7}$

$L_{20}$ ✓
**Interval Enforceable Properties**

- A minimal representation of $L_7$ must come from a transitive $G$-set.

![Diagram of $L_7$](image)
INTERVAL ENFORCEABLE PROPERTIES

- A minimal representation of $L_7$ must come from a transitive $G$-set.
- Suppose $L_7 \cong [H, G]$ for some finite groups $H < G$. What can we say about the group $G$?
INTERVAL ENFORCEABLE PROPERTIES

- A minimal representation of $L_7$ must come from a transitive $G$-set.
- Suppose $L_7 \cong [H, G]$ for some finite groups $H < G$. What can we say about the group $G$?
- If we prove $G$ must have certain properties, then FLRP has a positive answer iff every finite lattice is an interval in the subgroup lattice of a group satisfying all of these properties.
INTERVAL ENFORCEABLE PROPERTIES

- A minimal representation of $L_7$ must come from a transitive $G$-set.
- Suppose $L_7 \cong [H, G]$ for some finite groups $H < G$. What can we say about the group $G$?
- If we prove $G$ must have certain properties, then FLRP has a positive answer iff every finite lattice is an interval in the subgroup lattice of a group satisfying all of these properties.

PROPOSITION

Suppose $H < G$, $\text{core}_G(H) = 1$, $L_7 \cong [H, G]$.

(I) $G$ is a primitive permutation group.
(II) If $N \triangleleft G$, then $C_G(N) = 1$.
(III) $G$ contains no non-trivial abelian normal subgroup.
(IV) $G$ is not solvable.
(V) $G$ is subdirectly irreducible.
(VI) With the possible exception of at most one maximal subgroup, all proper subgroups in the interval $[H, G]$ are core-free.
Open Problems

1. Are homomorphic images of a representable lattices representable?

2. Are subdirect products of representable lattices representable?

3. Does representable imply “group representable?”
   i.e., is the congruence lattice of a finite algebra isomorphic to an interval in the subgroup lattice of a finite group?

4. Is the class of representable lattices recursive?
Remarks on the problems

- Are homomorphic images of representable lattices representable?
  If \( L = \text{Con} \langle A, F \rangle \) and \( \tilde{L} \) is the homomorphic image \( \{ \theta \cap B^2 \mid \theta \in L \} \), where \( B = e(A) \) for some operation \( e^2 = e \in F \), then \( \tilde{L} = \text{Con} \langle B, F \mid_B \rangle \).
Remarks on the problems

- Are homomorphic images of representable lattices representable?
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- Is the class of representable lattices recursive?
  In other words, is the membership problem for this class decidable?
  (Note that the class is recursively enumerable by the closure method.)
Remarks on the problems

- Are homomorphic images of representable lattices representable?
  If \( L = \text{Con} \langle A, F \rangle \) and \( \tilde{L} \) is the homomorphic image \( \{ \theta \cap B^2 \mid \theta \in L \} \), where \( B = e(A) \) for some operation \( e^2 = e \in F \), then \( \tilde{L} = \text{Con} \langle B, F \mid_B \rangle \).

- Is the class of representable lattices recursive?
  In other words, is the membership problem for this class decidable?
  (Note that the class is recursively enumerable by the closure method.)
  This question asks whether it’s possible to write a program that, when given a finite lattice \( L \), halts with output True if \( L \) is representable and False otherwise.
  A negative answer would solve the finite lattice representation problem.
Open Problems

One approach: try to find a computable function \( f \) such that, if \( L \) is a representable lattice of size \( n \), then \( L \) is representable as the congruence lattice of an algebra of cardinality \( f(n) \) or smaller.

This would answer the decidability question, as follows:

\[
\text{IsRepresentable}(L) \left\{ \\
\quad n := \text{Size}(L) \\
\quad N := f(n) \\
\quad \text{for each } L' \text{ in } \text{Sub}[\text{Eq}(N)] \left\{ \\
\quad \quad \text{if ( } L' \text{ is isomorphic to } L \text{ ) and ( } L' \text{ is closed )}: \\
\quad \quad \quad \text{return True} \\
\quad \quad \text{)} \\
\quad \} \\
\quad \text{return False} \\
\right\}
\]


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