Small Congruence Lattices

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joint work with

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The Problem

Characterize congruence lattices of finite algebras.

For an arbitrary algebra, there is essentially no restriction on the shape of its congruence lattice.

Theorem (Grätzer-Schmidt, 1963)

Every algebraic lattice is isomorphic to the congruence lattice of an algebra.

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If an algebra is finite, then its congruence lattice is... finite.

Problem: Given an arbitrary finite lattice $\mathbf{L}$, does there exist finite algebra $\mathbf{A}$ such that $\mathbf{Con}\,\mathbf{A} \cong \mathbf{L}$?

Definition

We call a finite lattice representable if it is (isomorphic to) the congruence lattice of a finite algebra.

A few important theorems

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(i) *Every finite lattice is representable.*

(ii) *Every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group.*

A FEW IMPORTANT THEOREMS

THEOREM (PUDLÁK AND TŮMA, 1980)

Every finite lattice can be embedded in Eq(X)*, with* X *finite.*

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The following statements are equivalent:

(I) *Every finite lattice is representable.*

(II) *Every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group.*

THEOREM (BERMAN, QUACKENBUSH & WOLK, 1970)

Every finite distributive lattice is representable.

Given a finite lattice $\mathbf{L}$, find a finite algebra $\mathbf{A}$ with $\operatorname{Con} \mathbf{A} \cong \mathbf{L}$.

How to find a finite algebra with a given congruence lattice?

How to find a finite algebra with a given congruence lattice?

1. Use closure properties

Relate the given lattice to other lattices known to be representable.

- If L is representable, so is
  - A. the dual of L (Kurzweil 1985, Netter)
  - B. any interval sublattice of L (follows from A.)
  - C. any sublattice that is the union of a principal filter and principal idea of L (Snow, 2000)
- If L_1 and L_2 are representable, so is
  1. the direct product of L_1 and L_2 (Tůma1989)
  2. the ordinal sum of L_1 and L_2 (McKenzie 1984, Snow 2000)
  3. the parallel sum of L_1 and L_2 (Snow 2000)

2. THE CLOSURE METHOD

Find a “closed” representation of L in Eq(X).

YAGC

For $L \leqslant$ Eq(X) define

$$\lambda(L) = \{f \in X^X : (\forall \theta \in L)\, f(\theta) \subseteq \theta\}$$

For $F \subseteq X^X$ define

$$\rho(F) = \{\theta \in \mathrm{Eq}(X) : (\forall f \in F)\, f(\theta) \subseteq \theta\}$$

The map $\rho\lambda$ is a *closure operator* on Sub$[$Eq$(X)]$.
(idempotent, extensive, order preserving)

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THEOREM

A lattice $L \leqslant$ Eq(X) is a congruence lattice if and only if it is closed, i.e. $\rho\lambda(L) = L$, in which case $L = \text{Con}\,\langle X, \lambda(L)\rangle$.

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Example: $M_3 \cong L \leqslant \mathrm{Eq}(5)$

3. THE G-SET METHOD

Find L as an interval in a subgroup lattice of a finite group.

If $H \leqslant G$ are finite groups, then the filter above H in Sub(G),

$$[\![H, G]\!] := \{K : H \leqslant K \leqslant G\},$$

is isomorphic to $\mathrm{Con}\,\langle G/H, \bar{G}\rangle$.

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4. THE RABBIT EARS METHOD (AKA OVERALGEBRAS, AKA EXPANSION-EXTENSION)

Build the required algebra by gluing together isomorphic copies of an algebra and adding new operations.

The G-set method: details

For groups $H \leqslant G$, let $\mathbf{A} = \langle H\backslash G, \bar{G}\rangle$ denote the algebra with

- universe: the right cosets $H\backslash G = \{Hx : x \in G\}$
- operations: $\bar{G} = \{g^{\mathbf{A}} : g \in G\}$, where $g^{\mathbf{A}}(Hx) = Hxg$.

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The isomorphism $[\![H, G]\!] \ni K \mapsto \theta_K \in \operatorname{Con} \mathbf{A}$ *is given by*

$$\theta_K = \{(Hx, Hy) : xy^{-1} \in K\}.$$

The inverse isomorphism $\operatorname{Con} \mathbf{A} \ni \theta \mapsto K_\theta \in [\![H, G]\!]$ *is*

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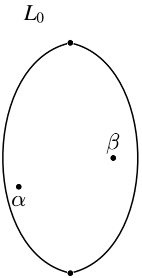
Aside: properties of such congruence lattices correspond to properties of subgroup lattices. For example,

Lemma

In $\operatorname{Con} \langle H\backslash G, \bar{G}\rangle$*, two congruences,* θ_{K_1} *and* θ_{K_2}*,* n*-permute if and only if the corresponding subgroups,* K_1 *and* K_2*,* n*-permute.*

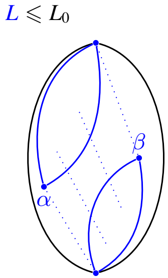
LEMMA

Suppose $L_0 \cong \text{Con}\,\langle A, F\rangle$, *and* $\alpha, \beta \in L_0 \setminus \{0, 1\}$.



Lemma

Suppose $L_0 \cong \operatorname{Con}\langle A, F\rangle$, *and* $\alpha, \beta \in L_0 \setminus \{0, 1\}$. *Consider* $L = \alpha^{\uparrow} \cup \beta^{\downarrow}$.



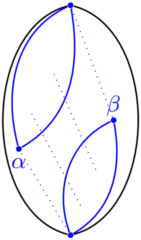
FILTER+IDEAL METHOD: DETAILS

LEMMA

Suppose $L_0 \cong \operatorname{Con}\langle A, F\rangle$, *and* $\alpha, \beta \in L_0 \setminus \{0,1\}$. *Consider* $L = \alpha^{\uparrow} \cup \beta^{\downarrow}$.

There exists a set $F' \subset A^A$ *such that* $L \cong \operatorname{Con}\langle A, F \cup F'\rangle$.

$L \leqslant L_0$



LEMMA

Suppose $L_0 \cong \mathrm{Con}\,\langle A, F\rangle$, *and* $\alpha, \beta \in L_0 \setminus \{0,1\}$. *Consider* $L = \alpha^{\uparrow} \cup \beta^{\downarrow}$.
There exists a set $F' \subset A^A$ *such that* $L \cong \mathrm{Con}\,\langle A, F \cup F'\rangle$.

Proof:

Fix $\theta \in L_0 \setminus L$. Then $\alpha \not\leqslant \theta \not\leqslant \beta$, so

- $\exists (a,b) \in \alpha \setminus \theta$,
- $\exists (u,v) \in \theta \setminus \beta$.

Define $f_\theta : A \to A$ by

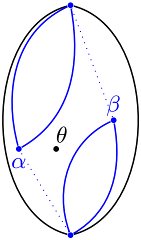
$$f_\theta(x) = \begin{cases} a & x \in u/\beta, \\ b & x \notin u/\beta. \end{cases}$$

Then

- $(f_\theta(u), f_\theta(v)) = (a,b) \notin \theta$, so $f_\theta(\theta) \nsubseteq \theta$,
- $\ker f_\theta \geqslant \beta$, so $f_\theta(\gamma) \subseteq \gamma$ for all $\gamma \leqslant \beta$,
- $f_\theta(A) \subseteq \{a,b\}$, so $f_\theta(\gamma) \subseteq \gamma$ for all $\gamma \geqslant \alpha$.

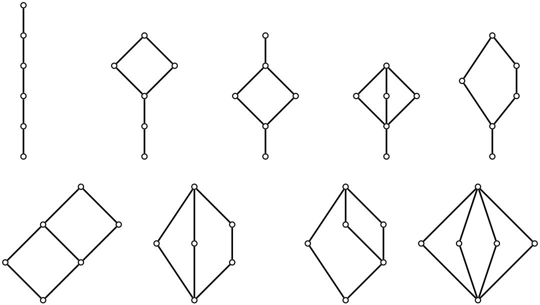
Let $F' = \{f_\theta : \theta \in L_0 \setminus L\}$.

$L \leqslant L_0$

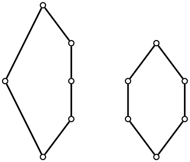


Which finite lattices are known to be representable?

Lattices with at most 6 elements are representable.



Watatani (1996) *J. Funct. Anal.*



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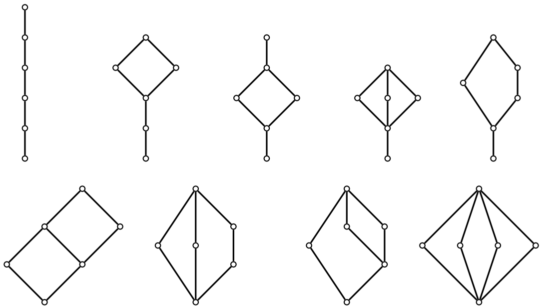

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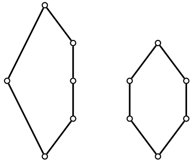

Aschbacher (2008) *JAMS*

Theorem: *Lattices with at most 6 elements are intervals in subgroup lattices of finite groups.*

Are all lattices with at most 7 elements representable?

As of Spring 2011...

LATTICES OF SIZE ≤ 7 NOT YET KNOWN TO BE CONGRUENCE LATTICES OF FINITE ALGEBRAS

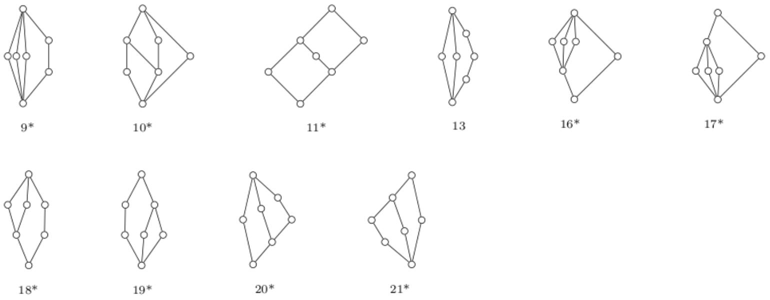

Figure courtesy of Peter Jipsen.

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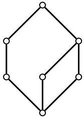
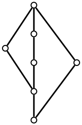
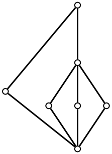


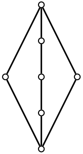
L_{19}



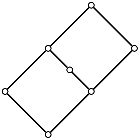
L_{20}



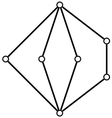
L_{17}



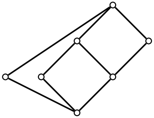
L_{13}



L_{11}



L_9



L_{10}

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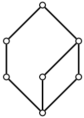

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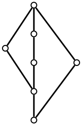
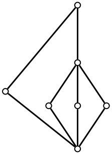
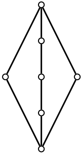


L_{20} ✓



L_{17}



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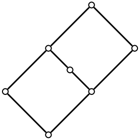

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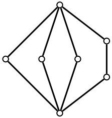

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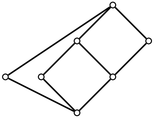
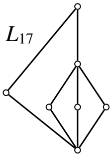


L_{10}

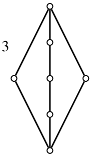
Finding representations…

…as intervals in subgroup lattices

L_{17}

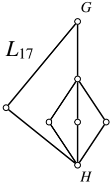


L_{13}



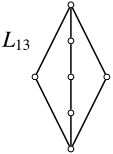
FINDING REPRESENTATIONS...

...AS INTERVALS IN SUBGROUP LATTICES



SmallGroup(288,1025)

$|G : H| = 48$



- The group $G = (A_4 \times A_4) \rtimes C_2$ has a subgroup $H \cong S_3$ such that $[\![H, G]\!] \cong L_{17}$.
- ...so the dual L_{16} is also representable.

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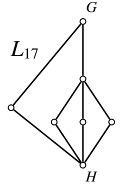

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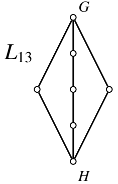
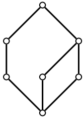


`SmallGroup(960,11358)`

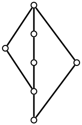
$|G : H| = 80$

- The group $G = (C_2 \times C_2 \times C_2 \times C_2) \rtimes A_5$ has a subgroup $H \cong A_4$ such that $[\![H, G]\!] \cong L_{13}$.

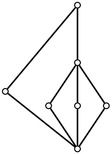
Are all lattices with at most 7 elements representable?



L_{19} ✓



L_{20} ✓



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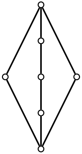

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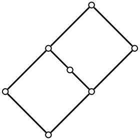
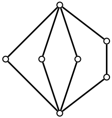
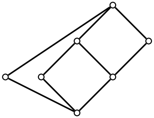


L_{11}



L_{9}

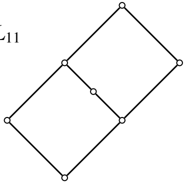


L_{10}

Finding representations…

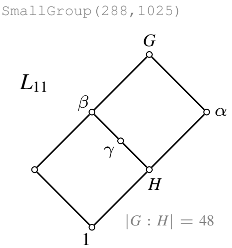
…using subgroup lattice intervals and the filter+ideal lemma.

L_{11}



FINDING REPRESENTATIONS…

…USING SUBGROUP LATTICE INTERVALS AND THE FILTER+IDEAL LEMMA.

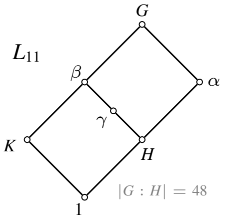


- Let $G = (A_4 \times A_4) \rtimes C_2$.
- G has a subgroup $H \cong C_6$ with $[\![H, G]\!] \cong N_5$.
- Let $[\![H, G]\!] = \{H, \alpha, \beta, \gamma, G\} \cong N_5$.

FINDING REPRESENTATIONS...

...USING SUBGROUP LATTICE INTERVALS AND THE FILTER+IDEAL LEMMA.

SmallGroup(288,1025)



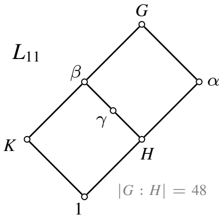
- Let $G = (A_4 \times A_4) \rtimes C_2$.
- G has a subgroup $H \cong C_6$ with $[\![H, G]\!] \cong N_5$.
- Let $[\![H, G]\!] = \{H, \alpha, \beta, \gamma, G\} \cong N_5$.
- Sub(G) is a congruence lattice, so if there exists a subgroup $K \succ 1$, below β and not below γ, then

$$L_{11} \cong K^{\downarrow} \cup H^{\uparrow}.$$

FINDING REPRESENTATIONS...

...USING SUBGROUP LATTICE INTERVALS AND THE FILTER+IDEAL LEMMA.

SmallGroup(288,1025)



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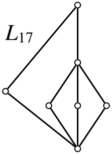
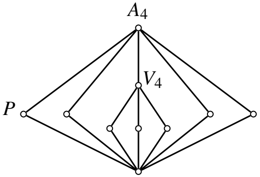


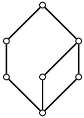


- Sub(A_4) is a congruence lattice (of A_4 acting regularly on itself).
- Therefore,

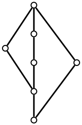
$$L_{17} \cong V_4^{\downarrow} \cup P^{\uparrow}$$

is a congruence lattice.

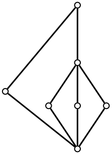
Are all lattices with at most 7 elements representable?



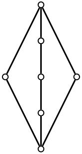
L_{19} ✓



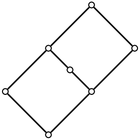
L_{20} ✓



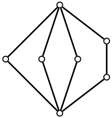
L_{17} ✓



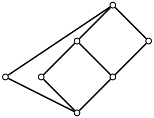
L_{13} ✓



L_{11} ✓

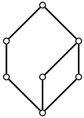


L_9

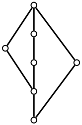


L_{10}

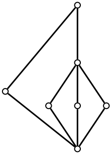
Are all lattices with at most 7 elements representable?



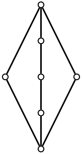
L_{19}



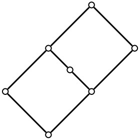
L_{20}



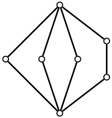
L_{17}



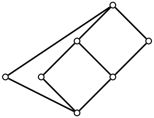
L_{13}



L_{11}

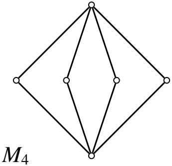


L_{9}

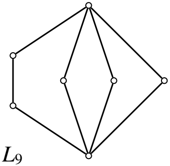


L_{10}

Construction of an algebra $\mathbf{A}$ with $\operatorname{Con}\mathbf{A} \cong L_9$.



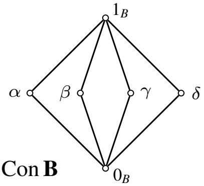
M_4



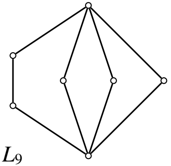
L_9

Construction of an algebra $\mathbf{A}$ with $\operatorname{Con}\mathbf{A} \cong L_9$.

Step 1 Take a permutational algebra $\mathbf{B} = \langle B, F \rangle$ with congruence lattice $\operatorname{Con}\mathbf{B} \cong M_4$.



Con **B**



L_9

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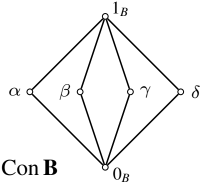


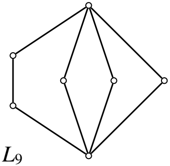
Con **B**

Step 1 Take a permutational algebra $\mathbf{B} = \langle B, F\rangle$ with congruence lattice $\operatorname{Con}\mathbf{B} \cong M_4$.

<u>Example:</u>

- Let $B = \{0, 1, \dots, 5\}$ index the elements of S_3 and consider the right regular action of S_3 on itself.
- $g_0 = (0,4)(1,3)(2,5)$ and $g_1 = (0,1,2)(3,4,5)$ generate this action group, the image of $S_3 \hookrightarrow S_6$.
- $\operatorname{Con}\langle B, \{g_0, g_1\}\rangle \cong M_4$ with congruences

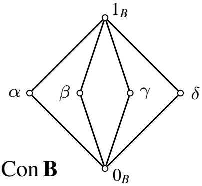
$$\alpha = |012|345|,\ \beta = |03|14|25|, \gamma = |04|15|23|,\ \delta = |05|13|24|.$$



L_9

CONSTRUCTION OF AN ALGEBRA $\mathbf{A}$ WITH $\text{Con}\,\mathbf{A} \cong L_9$.

STEP 1 Take a permutational algebra $\mathbf{B} = \langle B, F\rangle$ with congruence lattice $\text{Con}\,\mathbf{B} \cong M_4$.



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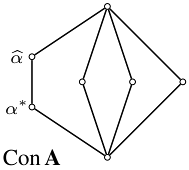
Goal: *expand* $\mathbf{B}$ *to an algebra* $\mathbf{A}$ *that has* α *"doubled" in* $\text{Con}\,\mathbf{A}$.



CONSTRUCTION OF AN ALGEBRA $\mathbf{A}$ WITH $\operatorname{Con}\mathbf{A} \cong L_9$.

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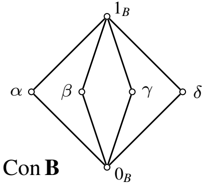

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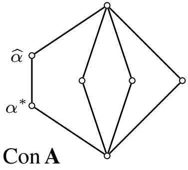
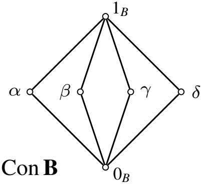


STEP 2 Since $\alpha = \operatorname{Cg}^{\mathbf{B}}(0, 2)$, we let $A = B_0 \cup B_1 \cup B_2$ where

$$B_0 = \{0, 1, 2, 3, 4, 5\} = B$$
$$B_1 = \{0, 6, 7, 8, 9, 10\}$$
$$B_2 = \{11, 12, 2, 13, 14, 15\}.$$

CONSTRUCTION OF AN ALGEBRA $\mathbf{A}$ WITH $\operatorname{Con}\mathbf{A} \cong L_9$.



$\operatorname{Con}\mathbf{B}$

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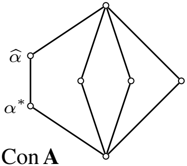

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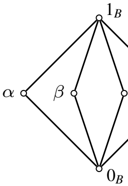
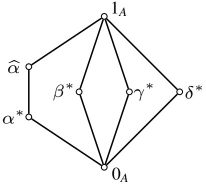
STEP 3 Define unary operations $e_0, e_1, e_2,\ s,\ g_0e_0$, and g_1e_0.

CONSTRUCTION OF AN ALGEBRA $\mathbf{A}$ WITH $\operatorname{Con}\mathbf{A} \cong L_9$.



Con $\langle B, \{g_0, g_1\}\rangle$



Con $\langle A, F_A\rangle$

$$\alpha = |0, 1, 2|3, 4, 5|$$

$$\beta = |0, 3|1, 4|2, 5|$$

$$\gamma = |0, 4|1, 5|2, 3|$$

$$\delta = |0, 5|1, 3|2, 4|$$

$$\widehat{\alpha} = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10, 13, 14, 15|$$

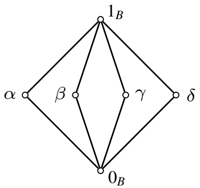
$$\alpha^* = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10|13, 14, 15|$$

$$\beta^* = |0, 3, 8|1, 4|2, 5, 15|6, 9|7, 10|11, 13|12, 14|$$

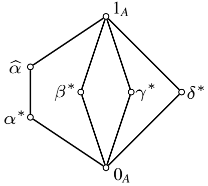
$$\gamma^* = |0, 4, 9|1, 5|2, 3, 13|6, 10|7, 8|11, 14|12, 15|$$

$$\delta^* = |0, 5, 10|1, 3|2, 4, 14|6, 8|7, 9, 11, 15|12, 13|$$

Construction of an algebra $\mathbf{A}$ with $\mathrm{Con}\,\mathbf{A} \cong L_9$.



Con $\langle B, \{g_0, g_1\}\rangle$



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$$\beta^* = |0, 3, 8|1, 4|2, 5, 15|6, 9|7, 10|11, 13|12, 14|$$
$$\gamma^* = |0, 4, 9|1, 5|2, 3, 13|6, 10|7, 8|11, 14|12, 15|$$
$$\delta^* = |0, 5, 10|1, 3|2, 4, 14|6, 8|7, 9, 11, 15|12, 13|$$

$$\alpha = \alpha^* \cap B^2 = \widehat{\alpha} \cap B^2, \quad \beta = \beta^* \cap B^2, \quad \ldots$$

Why does it work?

$$\text{Con}\,\langle B, \{g_0, g_1\}\rangle$$

$$\alpha = |0, 1, 2|3, 4, 5|$$

$$\beta = |0, 3|1, 4|2, 5|$$

$$\gamma = |0, 4|1, 5|2, 3|$$

$$\delta = |0, 5|1, 3|2, 4|$$

0	1	2
3	4	5

B_0

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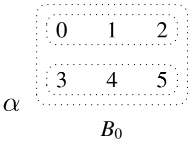

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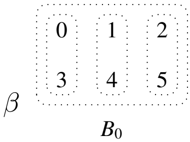

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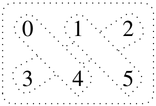
$$\delta = |0,5|1,3|2,4|$$

γ



B_0

WHY DOES IT WORK?

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$\beta = |0, 3|1, 4|2, 5|$

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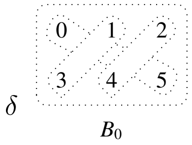

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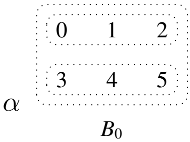

$B_0 = \{\ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5\ \}$

Why does it work?

Con $\langle B, \{g_0, g_1\}\rangle$

$\alpha = |0, 1, 2|3, 4, 5|$

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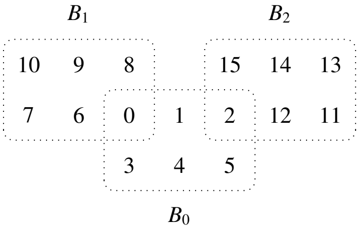

- $A = B_0 \cup B_1 \cup B_2$

$$B_1 = \{\ 0 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10\ \}$$

$$B_0 = \{\ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5\ \}$$

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Why does it work?

$$\mathrm{Con}\,\langle B, \{g_0, g_1\}\rangle$$

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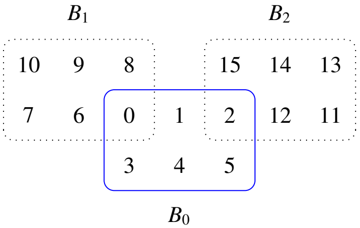


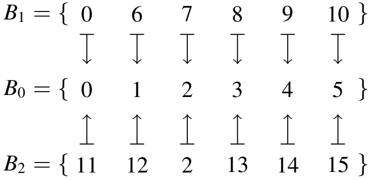
- $A = B_0 \cup B_1 \cup B_2$
- Unary operations

$$e_0 \colon A \twoheadrightarrow B_0$$

$$e_1 \colon A \twoheadrightarrow B_1$$

$$e_2 \colon A \twoheadrightarrow B_2$$

$$s \colon A \twoheadrightarrow B_0$$



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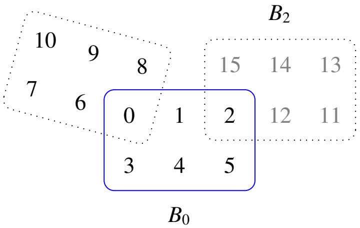

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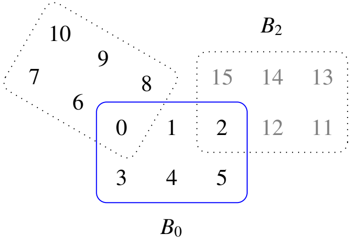

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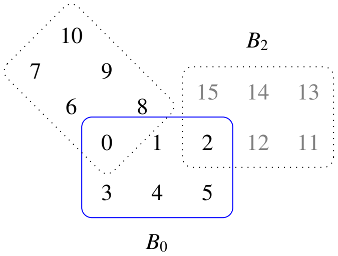

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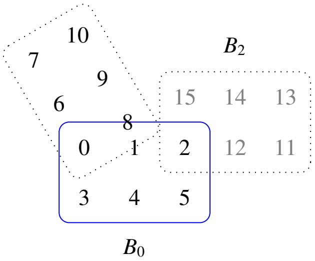

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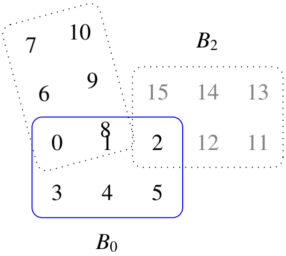

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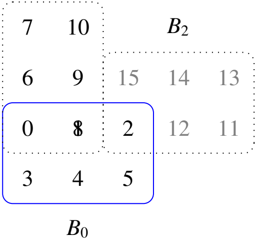

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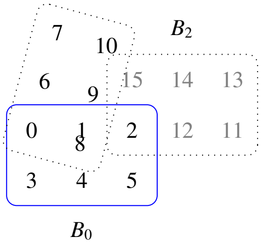

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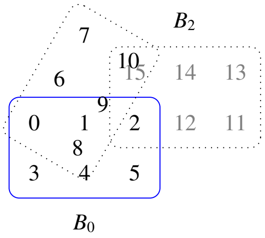

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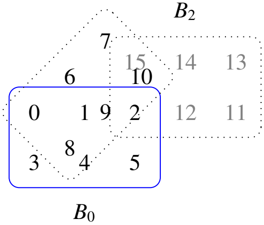

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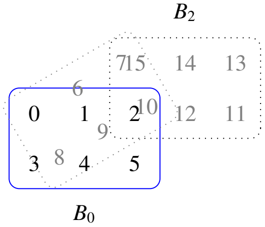

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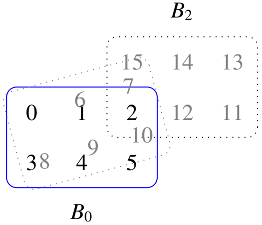

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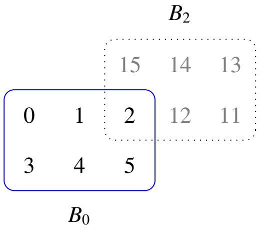

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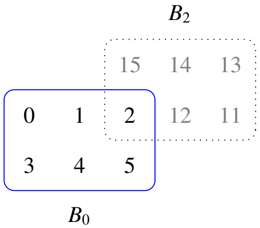

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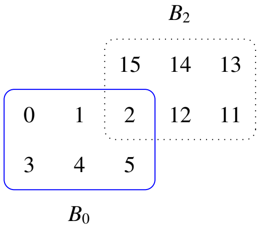

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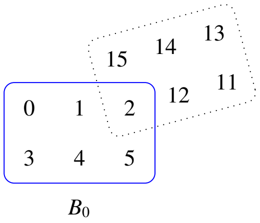

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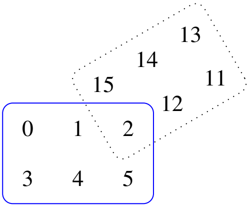

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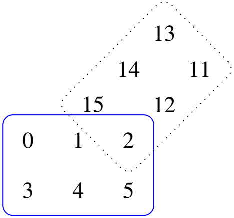

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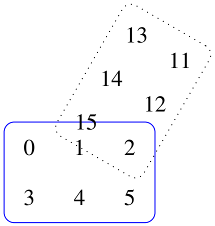

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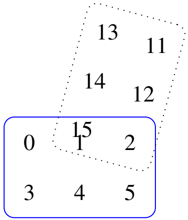

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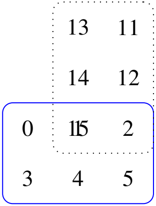

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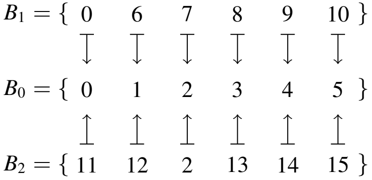

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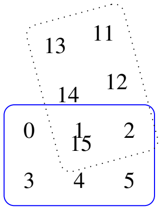

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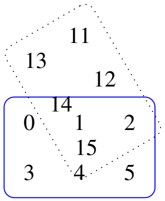

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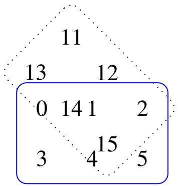

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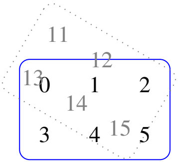

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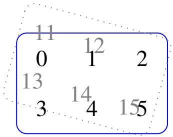

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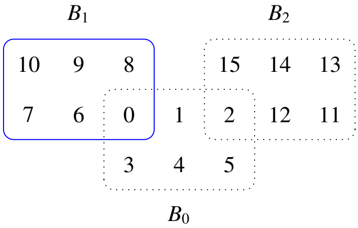

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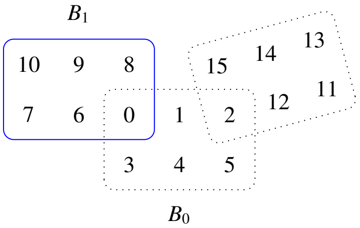

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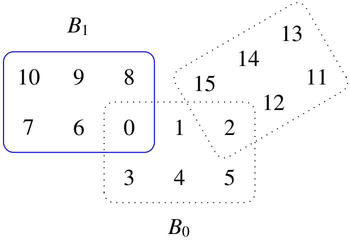

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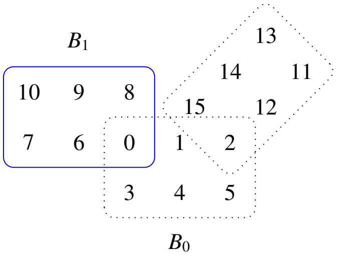

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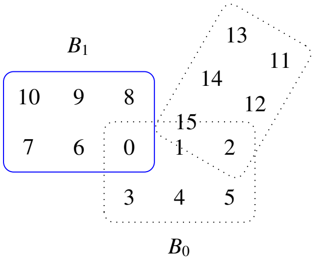

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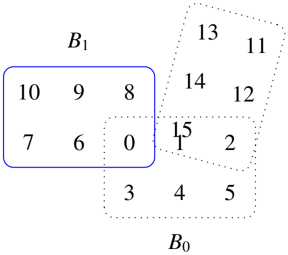

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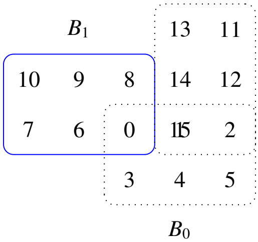

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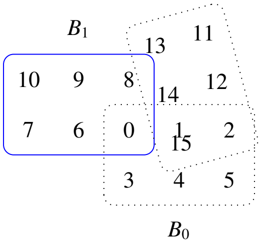

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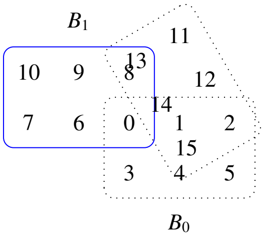

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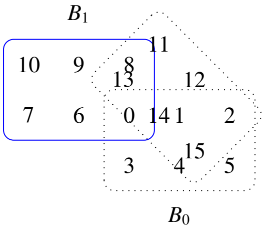

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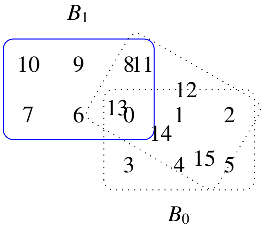

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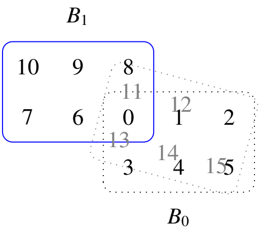

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- Unary operations

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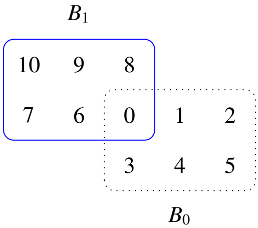

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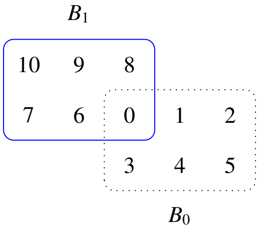

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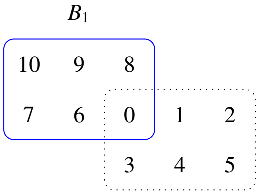

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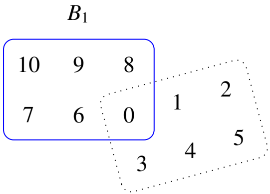

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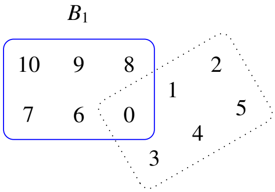

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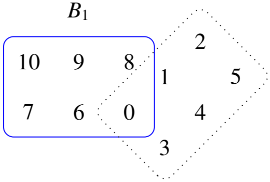

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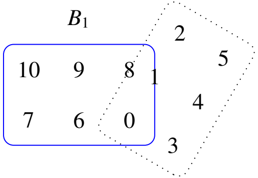

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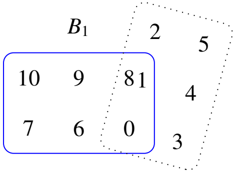

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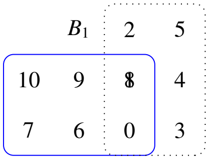

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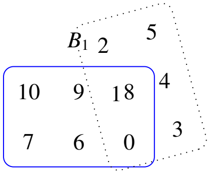

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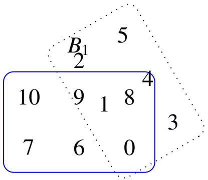

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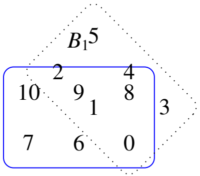

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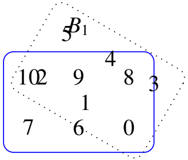

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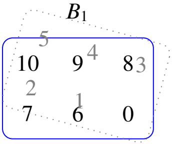

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B_1

10	9	8
7	6	0

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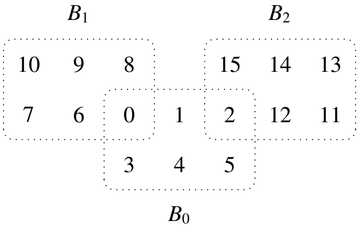

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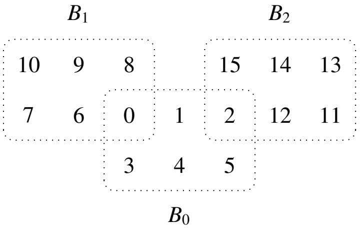

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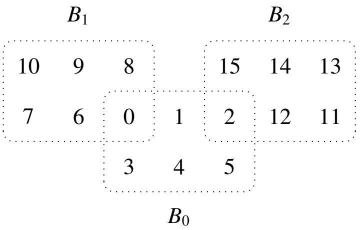

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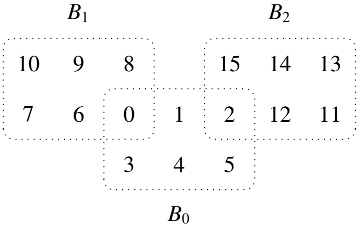

Con $\langle A, F_A\rangle$

$$\widehat{\alpha} = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10, 13, 14, 15|$$
$$\alpha^* = |0, 1, 2, 6, 7, 11, 12|3, 4, 5|8, 9, 10|13, 14, 15|$$
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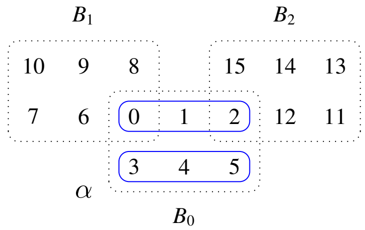

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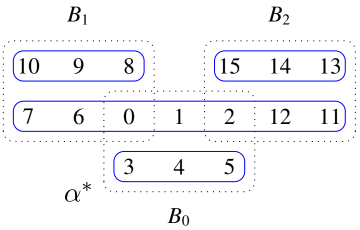

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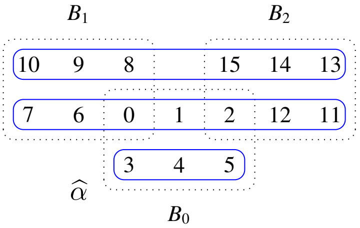

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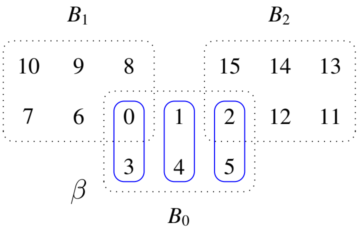

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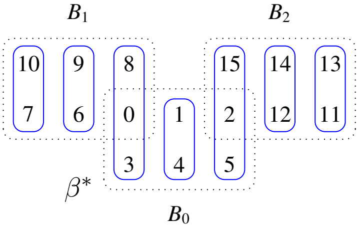

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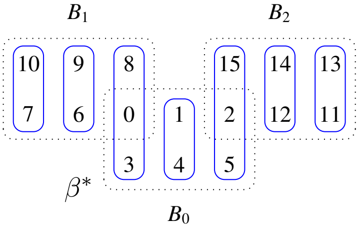

$\text{Con}\,\langle A, F_A\rangle$

Why don't the β classes of B_1 and B_2 mix?

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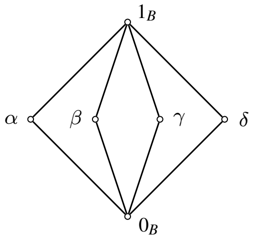

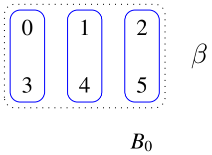

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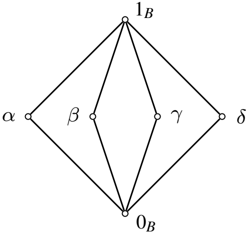

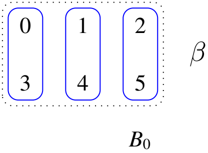

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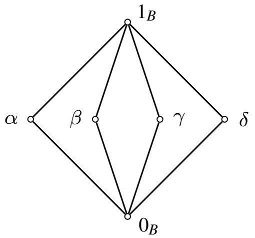

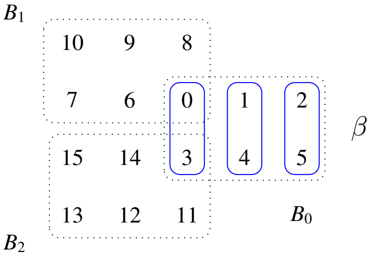

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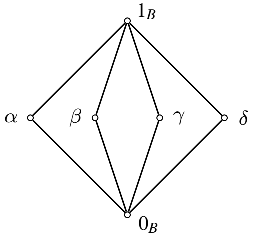

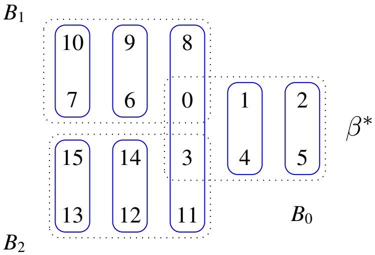

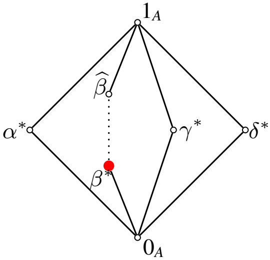

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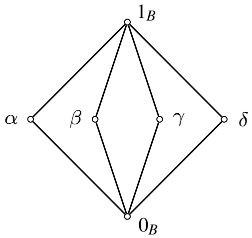

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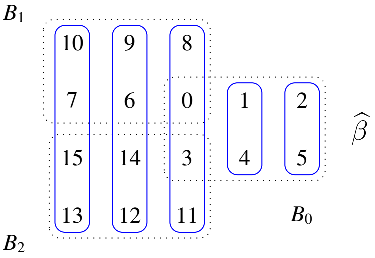

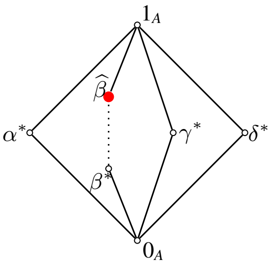

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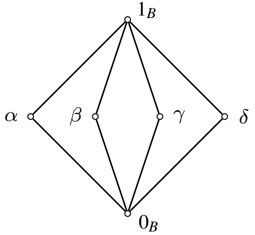

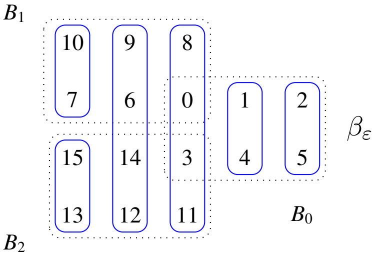

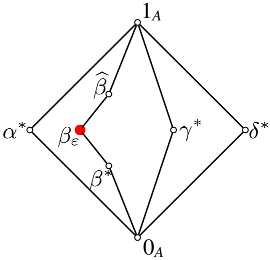

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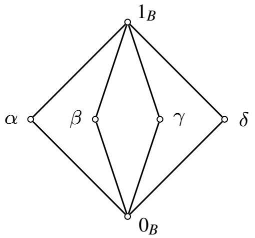

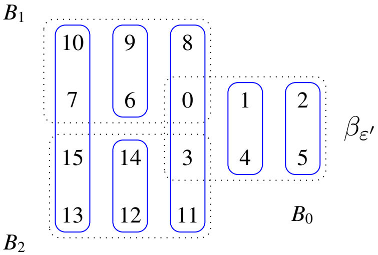

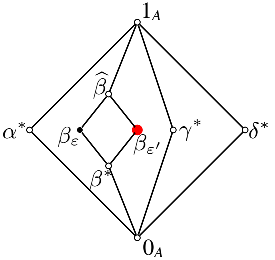

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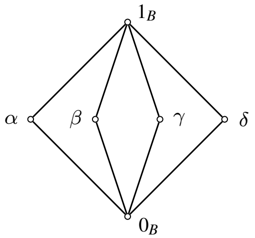
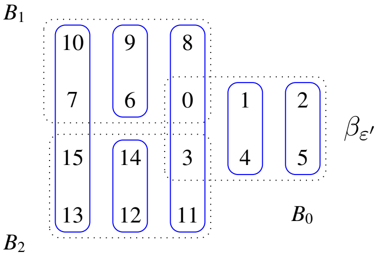
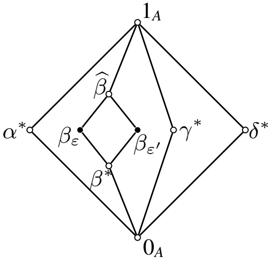
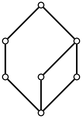


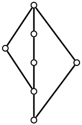




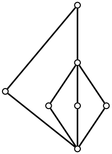
Seven element lattices: summary



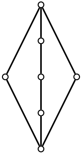
L_{19} ✓



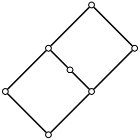
L_{20} ✓



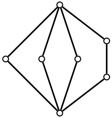
L_{17} ✓



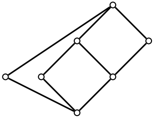
L_{13} ✓



L_{11} ✓

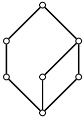


L_9 ✓

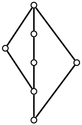


L_7

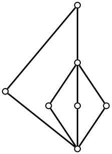
SEVEN ELEMENT LATTICES: SUMMARY



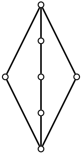
L_{19}



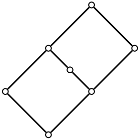
L_{20}



L_{17}



L_{13}



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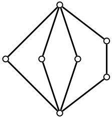
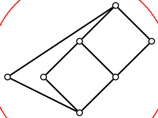


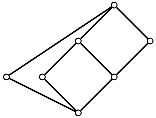
L_9



L_7

Interval enforceable properties

- A minimal representation of L_7 must come from a transitive G-set.



L_7

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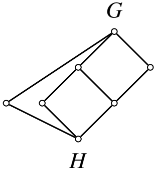
- Suppose $L_7 \cong [\![H, G]\!]$ for some finite groups $H < G$.

  What can we say about the group G?



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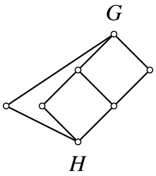
- If we prove G must have certain properties, then FLRP has a positive answer iff every finite lattice is an interval in the subgroup lattice of a group *satisfying all of these properties*.



INTERVAL ENFORCEABLE PROPERTIES

- A minimal representation of L_7 must come from a transitive G-set.
- Suppose $L_7 \cong [\![H, G]\!]$ for some finite groups $H < G$.

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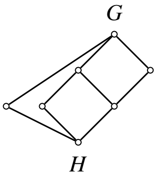

PROPOSITION

Suppose $H < G, \quad \mathrm{core}_G(H) = 1, \quad L_7 \cong [\![H, G]\!]$.

(I) G is a primitive permutation group.
(II) If $N \lhd G$, then $C_G(N) = 1$.
(III) G contains no non-trivial abelian normal subgroup.
(IV) G is not solvable.
(V) G is subdirectly irreducible.
(VI) With the possible exception of at most one maximal subgroup, all proper subgroups in the interval $[\![H, G]\!]$ are core-free.

Open Problems

1. Are homomorphic images of a representable lattices representable?

2. Are subdirect products of representable lattices representable?

3. Does representable imply "group representable?"

   i.e., is the congruence lattice of a finite algebra isomorphic to an interval in the subgroup lattice of a finite group?

4. Is the class of representable lattices recursive?

Remarks on the problems

- Are homomorphic images of representable lattices representable?

  If $L = \operatorname{Con}\langle A, F\rangle$ and $\tilde{L}$ is the homomorphic image $\{\theta \cap B^2 \mid \theta \in L\}$, where $B = e(A)$ for some operation $e^2 = e \in F$, then $\tilde{L} = \operatorname{Con}\langle B, F|_B\rangle$.

REMARKS ON THE PROBLEMS

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  In other words, is the membership problem for this class decidable?

  (Note that the class is recursively enumerable by the closure method.)

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  In other words, is the membership problem for this class decidable?

  (Note that the class is recursively enumerable by the closure method.)

  This question asks whether it's possible to write a program that, when given a finite lattice L, halts with output True if L is representable and False otherwise.

  A negative answer would solve the finite lattice representation problem.

OPEN PROBLEMS

One approach: try to find a computable function f such that, if L is a representable lattice of size n, then L is representable as the congruence lattice of an algebra of cardinality $f(n)$ or smaller.

This would answer the decidability question, as follows:

```
IsRepresentable( L ) {

    n:= Size( L )
    N:= f( n )

    for each L' in Sub[Eq(N)] {

        if ( L' isomorphic to L ) and ( L' is closed ):
            return True

    }

    return False

}
```