Developments on higher levels of the substructural hierarchy

Nick Galatos
University of Denver
ngalatos@du.edu

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Residuated lattices

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A *residuated lattice*, or *residuated lattice-ordered monoid*, is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$ such that

- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot, 1)$ is a monoid and
- for all $a, b, c \in A$,

$$ab \leq c \;\Leftrightarrow\; b \leq a\backslash c \;\Leftrightarrow\; a \leq c/b.$$

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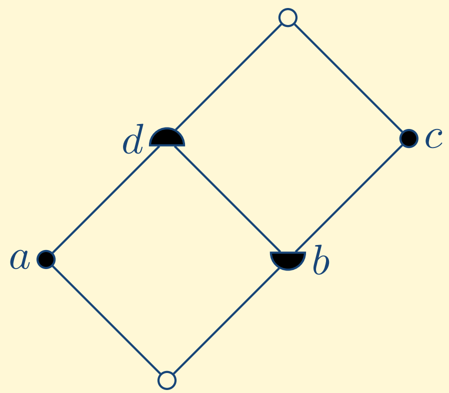
Residuated lattice appear in both

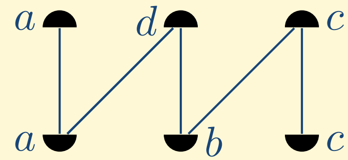
- **Algebra**: Lattice-ordered groups, relation algebras, ideals of a ring, quantales.

- **Logic**: As models of various logics: Classical, intuitionistic, many-valued, linear, relevance logic.

N. Galatos, P. Jipsen, T. Kowalski and H. Ono. Residuated Lattices: an algebraic glimpse at substructural logics, Studies in Logics and the Foundations of Mathematics, Elsevier, 2007.

Lattice frames





$\leq$	a	d	c
a	$\times$	$\times$	
b		$\times$	$\times$
c			$\times$

We obtain an oriented bipartite graph/*lattice frame* $\mathbf{F} = (L, R, N)$, where $N \subseteq L \times R$. This is an algebraic rendering of sequents!

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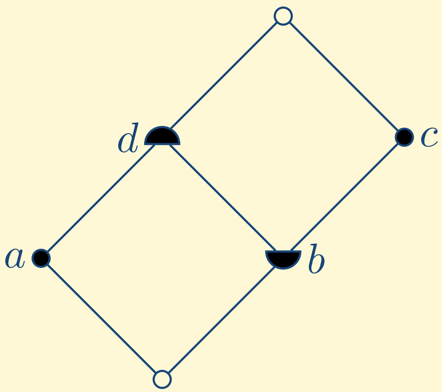

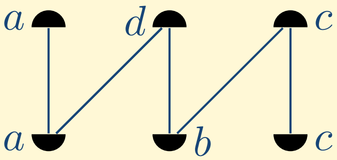

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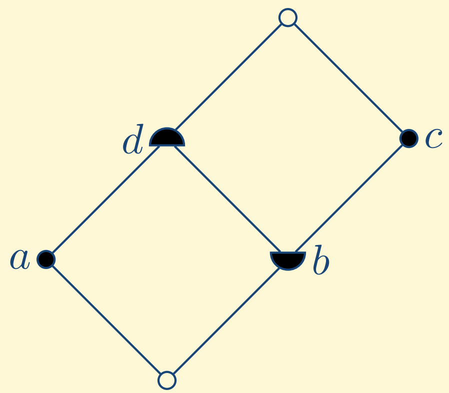
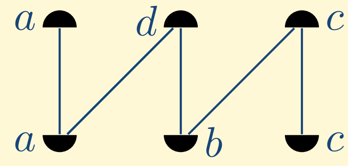
Given $X \subseteq L$ and $Y \subseteq R$ we define

$$X^{\rhd} = \{b \in R : x \; N \; b, \text{ for all } x \in X\}$$
$$Y^{\lhd} = \{a \in L : a \; N \; y, \text{ for all } y \in Y\}$$

Then $\gamma_N(X) = X^{\rhd\lhd}$ defines a closure operator on $\mathcal{P}(L)$ and the *Galois algebra* $\mathbf{F}^+ = (\gamma_N[\mathcal{P}(L)], \cap, \cup_{\gamma_N})$ is a complete lattice.

Lattice frames





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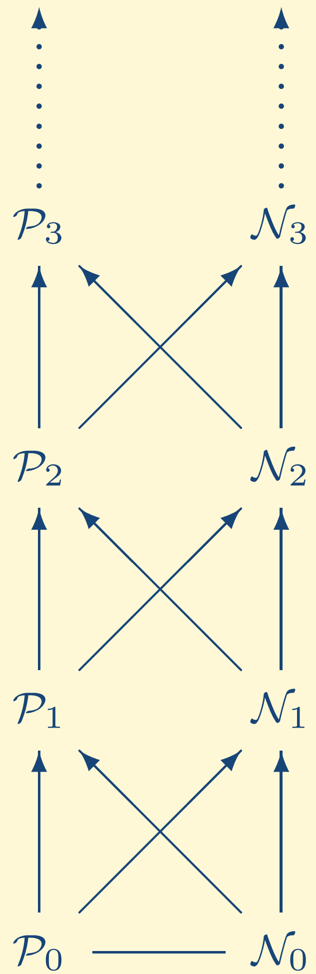
If $\mathbf{A}$ is a lattice, $\mathbf{F_A} = (A, A, \leq)$ is a lattice frame. Also, $\mathbf{F}^+_{\mathbf{A}}$ is the Dedekind-MacNeille completion of $\mathbf{A}$ and $x \mapsto \{x\}^{\lhd}$ is an embedding.

Formula hierarchy



- We separate our signature into positive $\{\vee, \cdot, 1\}$ and negative $\{\wedge, \backslash, /\}$.
- The sets $\mathcal{P}_n, \mathcal{N}_n$ of formulas are defined by:

(0) $\mathcal{P}_0 = \mathcal{N}_0 =$ the set of variables

(P) $\mathcal{P}_{n+1} = \langle \mathcal{N}_n \rangle_{\vee, \prod}$

(N) $\mathcal{N}_{n+1} = \langle \mathcal{P}_n \rangle_{\wedge, \mathcal{P}_{n+1} \backslash, / \mathcal{P}_{n+1}}$

- $\mathcal{P}_1$-reduced: $\bigvee \prod p_i$
- $\mathcal{N}_1$-reduced: $\bigwedge (p_1 p_2 \cdots p_n \backslash r / q_1 q_2 \cdots q_m)$

$p_1 p_2 \cdots p_n q_1 q_2 \cdots q_m \leq r$

- Sequent: $a_1, a_2, \ldots, a_n \Rightarrow a_0 \ (a_i \in Fm)$

A. Ciabattoni, NG, K. Terui. From axioms to analytic rules in nonclassical logics, Proceedings of LICS'08, 229-240, 2008.

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Theorem. If $\mathbf{F}$ is a residuated frame then the *Galois algebra* $\mathbf{F}^+$ expands to a residuated lattice. $X \cdot Y = \gamma_N(\{x \circ y : x \in X, y \in Y\})$.

If $\mathbf{A}$ is a RL, $\mathbf{F_A} = (A, A, \leq, \cdot, \{1\})$ is a residuated frame. $\mathbf{F_A}^+$ is the *Dedekind-MacNeille completion* of $\mathbf{A}$. Moreover, for $\mathbf{F_A}$, $x \mapsto \{x\}^{\triangleleft}$ is an embedding.

N. Galatos and P. Jipsen. Residuated frames and applications to decidability, Transactions of the AMS (2013).

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If the following conditions hold for a common subset/subalgebra $\mathbf{B}$ of L and R (Gentzen frame), then we can get a (quasi) embedding from $\mathbf{B}$ to $\mathbf{F}^+$.

GN

$$\frac{xNa \quad aNz}{xNz}\ (\mathsf{CUT}) \qquad \frac{}{aNa}\ (\mathsf{Id})$$

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$$\frac{aNz \quad bNz}{a \vee bNz}\ (\vee\mathsf{L}) \qquad \frac{xNa}{xNa \vee b}\ (\vee\mathsf{R}\ell) \qquad \frac{xNb}{xNa \vee b}\ (\vee\mathsf{R}r)$$

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$$\frac{a \circ bNz}{a \cdot bNz}\ (\cdot\mathsf{L}) \qquad \frac{xNa \quad yNb}{x \circ yNa \cdot b}\ (\cdot\mathsf{R}) \qquad \frac{\varepsilon Nz}{1Nz}\ (1\mathsf{L}) \qquad \frac{}{\varepsilon N1}\ (1\mathsf{R})$$

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$$\frac{a \circ b N z}{a \cdot b N z}\ (\cdot\text{L}) \qquad \frac{x N a \quad y N b}{x \circ y N a \cdot b}\ (\cdot\text{R}) \qquad \frac{\varepsilon N z}{1 N z}\ (1\text{L}) \qquad \frac{}{\varepsilon N 1}\ (1\text{R})$$

$$\frac{x N a \quad b N z}{a \backslash b N x \mathbin{\backslash\!\backslash} z}\ (\backslash\text{L}) \qquad \frac{x N a \mathbin{\backslash\!\backslash} b}{x N a \backslash b}\ (\backslash\text{R})$$

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$$\frac{xNa \quad bNz}{x \circ (a\backslash b)Nz} \qquad \frac{xNa \quad bN(v \mathbin{\backslash\!\!\backslash} c \mathbin{/\!\!/} u)}{x \circ (a\backslash b)N(v \mathbin{\backslash\!\!\backslash} c \mathbin{/\!\!/} u)}$$

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We obtain $\mathbf{FL}$ for $a, b, c \in Fm$, $x, y, u, v \in Fm^*$, $z \in Fm^* \times Fm \times Fm^*$.

$$\frac{x \Rightarrow a \quad y \circ a \circ z \Rightarrow c}{y \circ x \circ z \Rightarrow c}\ (\text{cut}) \qquad \frac{}{a \Rightarrow a}\ (\text{Id})$$

$$\frac{y \circ a \circ z \Rightarrow c}{y \circ a \wedge b \circ z \Rightarrow c}\ (\wedge \text{L}\ell) \quad \frac{y \circ b \circ z \Rightarrow c}{y \circ a \wedge b \circ z \Rightarrow c}\ (\wedge \text{L}r) \quad \frac{x \Rightarrow a \quad x \Rightarrow b}{x \Rightarrow a \wedge b}\ (\wedge \text{R})$$

$$\frac{y \circ a \circ z \Rightarrow c \quad y \circ b \circ z \Rightarrow c}{y \circ a \vee b \circ z \Rightarrow c}\ (\vee \text{L}) \quad \frac{x \Rightarrow a}{x \Rightarrow a \vee b}\ (\vee \text{R}\ell) \quad \frac{x \Rightarrow b}{x \Rightarrow a \vee b}\ (\vee \text{R}r)$$

$$\frac{x \Rightarrow a \quad y \circ b \circ z \Rightarrow c}{y \circ x \circ (a \backslash b) \circ z \Rightarrow c}\ (\backslash \text{L}) \qquad \frac{a \circ x \Rightarrow b}{x \Rightarrow a \backslash b}\ (\backslash \text{R})$$

$$\frac{x \Rightarrow a \quad y \circ b \circ z \Rightarrow c}{y \circ (b/a) \circ x \circ z \Rightarrow c}\ (/\text{L}) \qquad \frac{x \circ a \Rightarrow b}{x \Rightarrow b/a}\ (/\text{R})$$

$$\frac{y \circ a \circ b \circ z \Rightarrow c}{y \circ a \cdot b \circ z \Rightarrow c}\ (\cdot \text{L}) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x \circ y \Rightarrow a \cdot b}\ (\cdot \text{R})$$

$$\frac{y \circ z \Rightarrow a}{y \circ 1 \circ z \Rightarrow a}\ (1\text{L}) \qquad \frac{}{\varepsilon \Rightarrow 1}\ (1\text{R})$$

where $a, b, c \in Fm$, $x, y, z \in Fm^*$.

Frame applications

- DM-completion
- Completeness of the calculus
- Cut elimination
- Finite model property
- Finite embeddability property
- (Generalized super-)amalgamation property (Transferable injections, Congruence extension property)
- (Craig) Interpolation property
- Disjunction property
- Strong separation
- Stability under linear structural rules/equations over $\{\vee, \cdot, 1\}$.

NG and H. Ono, APAL.

NG and P. Jipsen, TAMS.

NG and P. Jipsen, manuscript.

A. Ciabattoni, NG and K. Terui, APAL.

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For every rule (on the left) of $\mathbf{FL}$, the system $\mathbf{HFL}$ is defined to contain the rule (on the right)

$$\frac{s_1 \quad s_2}{s} \qquad \rightsquigarrow \qquad \frac{H \mid s_1 \quad H \mid s_2}{H \mid s}$$

where H is a (meta)variable for hypersequents.

Hypersequents

Structural rules correspond to $\mathcal{N}_2$-equations. They are based on sequents, which stem from $\mathcal{N}_1$-normal formulas.To handle $\mathcal{P}_3$-equations, we define hypersequents, based on $\mathcal{P}_2$-normal formulas: $(x_1 \dots x_n \to x_0) \vee (y_1 \dots y_n \to y_0) \vee \dots$.

A hypersequent is a multiset $s_1 \mid \cdots \mid s_m$ of sequents s_i.

For every rule (on the left) of $\mathbf{FL}$, the system $\mathbf{HFL}$ is defined to contain the rule (on the right)

$$\frac{s_1 \quad s_2}{s} \qquad \rightsquigarrow \qquad \frac{H \mid s_1 \quad H \mid s_2}{H \mid s}$$

where H is a (meta)variable for hypersequents. A hyperstructural rule is of the form

$$\frac{H \mid s_1' \quad H \mid s_2' \quad \dots \quad H \mid s_n'}{H \mid s_1 \mid \cdots \mid s_m}$$

Frame constructions

Given a $\mathbf{L}$-biset R under $\backslash\!\backslash, /\!/$, and an *index* set G, the set $R \times G$ becomes an $\mathbf{L}$-biset in a natural way. We act only on the R coordinate: $x \backslash\!\backslash (y, g) = (x \backslash\!\backslash y, g)$.

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An extension of the $\mathbf{L}$-biset $R \times G$ to a residuated frame can be obtained by a collection of *indexed* residuated frames $(L, R \times \{g\}, N_g, \circ, 1, \backslash\!\backslash, /\!/)$, one for each $g \in G$.

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(The basic closed sets of the new frame are then the basic closed sets of each index frame put together. The associated closure operator is the meet of all the indexed closure operators.)

Given any (commutative) monoid $\mathbf{H} = (H, |, \emptyset)$ such that G is an $\mathbf{H}$-biset under the action $\_$, we extend the product $(\mathbf{L} \times \mathbf{H})$-biset $R \times G$ to a residuated frame with

$$(x, h) \; N \; (z, g) \;\Leftrightarrow\; x \; N_{\frac{g}{h}} \; (z, \tfrac{g}{h})$$

A *hyper-residuated frame* based on the $\mathbf{L}$-biset R is given by the above construction, where $G = H = (L \times R)^*$, the set/commutative monoid of (L, R)-hyper-sequents, the multiplication and the action(s) coincide, and for all $h \in H$, $s \in L \times R$,

- N_h is nuclear
- $N_h \subseteq N_{h|s}$
- $N_{h|s|s} \subseteq N_{h|s}$
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Equivalently, a *hyperresiduated frame* is a structure of the form $\mathbf{H} = (L, R, \vdash, \circ, \varepsilon, \backslash\!\!\backslash, /\!\!/)$, where

- $\vdash \subseteq H = (L \times R)^*$. We write $\vdash h$ instead of $h \in \vdash$.
- $(L, \circ, \varepsilon)$ is a monoid.
- $\vdash h$ implies $\vdash (x, y) \mid h$ for any $(x, y) \in L \times R$.
- $\vdash (x, y) \mid (x, y) \mid h$ implies $\vdash (x, y) \mid h$ for any $(x, y) \in L \times R$.
- $\vdash (x \circ y, z) \mid h \Leftrightarrow \vdash (y, x \backslash\!\!\backslash z) \mid h \Leftrightarrow \vdash (x, z /\!\!/ y) \mid h$.

Example. Based on $\mathbf{HFL}$ we define a hyperresiduated frame $\mathbf{H_{HFL}} = (L, R, \vdash, \circ, \varepsilon)$, where

$$\vdash s_1 \mid \ldots \mid s_n \iff \vdash_{\mathbf{HFL}} s_1 \mid \cdots \mid s_n$$

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$$\vdash (x_1, y_1)| \ldots |(x_n, y_n) \iff 1 \leq \gamma_1(x_1\backslash y_1) \vee \cdots \vee \gamma_n(x_n\backslash y_n).$$

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Example. Given a residuated frame $\mathbf{F} = (L, R, N, \circ, \varepsilon, \backslash\!\!\backslash, /\!\!/)$, we obtain a hyperresiduated frame $h(\mathbf{F}) = (L, R, \vdash, \circ, \varepsilon, \backslash\!\!\backslash, /\!\!/)$ by defining

$$\vdash (x_1, y_1) \mid \ldots \mid (x_n, y_n) \iff x_1 \; N \; y_1 \text{ or } \cdots \text{ or } x_n \; N \; y_n.$$

Extra structure

Given $X, Y \subseteq L \times H$ and $H_1, H_2 \subseteq H$, we define:

$$
\begin{array}{rcl}
X \rightharpoonup Y & = & \{(x, z) \mid h_1 \mid h_2 : (x; h_1) \in X, (z; h_2) \in Y^{\triangleright}\} \\
H_1 \mid H_2 & = & \{h_1 \mid h_2 : h_1 \in H_1, h_2 \in H_2\} \\
\vdash H_1 & \iff & \vdash h \text{ for every } h \in H_1.
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Key Lemma. For every hyperresiduated frame $\mathbf{H}$ and every $X, Y \in \gamma[\mathcal{P}(L \times H)]$ and $H_0 \subseteq H$,

$$\vdash X \rightharpoonup Y \mid H_0 \iff (\varepsilon; H_0) \subseteq X \backslash Y.$$

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Then we obtain the quasiembedding lemma. We can prove cut elimination for $\mathbf{HFL}$, closure under (hyper-MacNeille) completions, etc.

A. Ciabattoni, NG, K. Terui. Algebraic proof theory for substructural logics: cut elimination and completions.

Being a subdirect product of chains (PUF: $x \leq y$ or $y \leq x$) is captured by the following (γ's denote iterated conjugates):

- $(a \to b) \vee (b \to a)$, in $\mathsf{FL_{ew}}$.
- $(a \to b)_{\wedge 1} \vee (b \to a)_{\wedge 1}$, in $\mathsf{FL_e}$.
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Theorem. [Ciabattoni-G-Terui] For any set of *normal* rules R, any set of hypersequents H and any sequent s we have

$$H \vdash_{\mathbf{HFL}(R)} s \Leftrightarrow H \models_{RL(R)_{SI}} s.$$

Extensions

The system **InFL** has cut elimination, FMP (and is decidable). Its simple extension all have cut elimination.

N. Galatos and P. Jipsen. Residuated frames and applications to decidability, Transactions of the AMS.

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Diagrams

Lattice-ordered groups are axiomatized relative to $\mathbf{FL}$ by the following structural rule [G.-Metcalfe] . Unfortunately, this rule is *cyclic* and is not preserved from the frame to the Galois algebra.

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The variety of ℓ-groups is generated by $\mathbf{Aut}(\mathbb{R})$, so we can test validity of equations there. This can be done efficiently by studying *diagrams* [Holland-McCleary].

In analogy to semantical tableux, one can turn this into a form of (hypersequent) calculus. [G.-Metcalfe]

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We also provide a cut-free analytic calculus. The price: modified logical rules.

ALG and MV

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Thus the decidability of the equational theory of abelian ℓ-groups can be proved using geometric/linear-programming tools.

Nevertheless in [G.-Jipsen-Marra] we show that one can also use a diagram refutation system (by implementing Fourier-Motzkin into diagrams).

Pregroups are ordered monoids $(A, \cdot, 1, \leq)$ with two additional unary operations ^{l}, ^{r} that satisfy the inequations

$$x^l x \leq 1 \leq x x^l \qquad \text{and} \qquad x x^r \leq 1 \leq x^r x.$$

(InRL's with $x \cdot y = x + y$.) Introduced in mathematical linguistics, and studied from algebraic and proof-theoretic points of view (W. Buskowski). *ℓ-pregroups* are lattice-based. ℓ-groups are exactly the ℓ-pregroups that satisfy $x^l = x^r$.

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Given a chain $\mathbf{C}$, the collection of all maps on $\mathbf{C}$ that have arbitrary residuals and arbitrary dual residuals form an ℓ-pregroup $\mathbf{F}(\mathbf{C})$.

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Goal: a diagram refutation system for distributive ℓ-pregroups.

The only group elements in $\mathbf{F}(\mathbb{Z})$ are the translations (isomorphic to $\mathbb{Z}$). However, we obtain the following surprising result.

Theorem [G.-Jipsen-Ball] The variety of ℓ-groups is contained in the variety generated by $\mathbf{F}(\mathbb{Z})$.

Embedding theorems

A map f in a poset $\mathbf{C}$ is residuated iff there exists f^* on $\mathbf{C}$ such that for all $x, y \in C$ $f(x) \leq y \Leftrightarrow x \leq f^*(y)$. Recal that if $\mathbf{C}$ is a complete join semilattice, then f is residuated iff it preserves all joins.

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We denote the set of all residuated maps on $\mathbf{C}$ by $Res(\mathbf{C})$. If $\mathbf{C}$ is a complete join semilattice, then $\mathbf{Res}(\mathbf{C})$ is a residuated lattice, under composition and pointwise join and meet.

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A *conucleus* σ on a residuated $\mathbf{L}$ is an interior operator on L, such that its image is a submonoid: for all $x, y \in L$, $\sigma(1) = 1$ and $\sigma(xy) = \sigma(\sigma(x)\sigma(y))$.

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(Given a residuated lattice $\mathbf{L}$ and a conucleus σ on it, the image $\sigma[L]$ supports a residuated lattice $\mathbf{L}_\sigma$, where $\vee, \cdot$ and 1 are the restrictions from $\mathbf{L}$, while$\backslash, /$ and $\wedge$ are the σ-images from $\mathbf{L}$.)

Holland's representation for RL [G.-Horčík] If a residuated lattice satisfies

$$(h \vee ca) \wedge (h \vee db) \leq h \vee cb \vee da$$

then $\mathbf{C}$ can be taken to be a chain.

Maps on a chain

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If $\mathbf{C}$ is a chain, the order-preserving bijections on $\mathbf{C}$ form an ℓ-group $\mathbf{Aut}(\mathbf{C})$. We can obtain the following celebrated theorem as a corollary.

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[G.-Horčík] Cayley's and Holland's Theorems for Idempotent Semirings and Their Applications to Residuated Lattices, Semigroup Forum.

Maps on a chain

Holland's representation for RL [G.-Horčík] If a residuated lattice satisfies

$$(h \vee ca) \wedge (h \vee db) \leq h \vee cb \vee da$$

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The proof can be presented in a way that involves *action hyper-residuated frames*.

We hope to extract a new notion of a frame and of proof theory from these representation theorems.