Sheaf representations of MV-algebras via Stone duality

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History of MV-algebras

- MV-algebras were introduced by C.C. Chang as algebraic semantics of Łukasiewicz infinite-valued logic (a propositional calculus with truth values in \([0, 1]\))

- MV-algebras have been studied extensively, mainly by Mundici and co-workers. They have developed the structure theory as well as links with other areas and applications

- The category of MV-algebras is equivalent to the category of unital lattice-ordered abelian groups
**MV-algebras**

An **MV-algebra** is an algebra \((A, \oplus, \neg, 0)\) such that

- \((A, \oplus, 0)\) is a commutative monoid,
- \(\neg \neg x = x\), that is, \(\neg\) is an involution,
- \(x \oplus 1 = 1\) where \(1 := \neg 0\),
- \(\neg (\neg x \oplus y) \oplus y = \neg (\neg y \oplus x) \oplus x\).

MV-algebras are **bounded distributive lattices** in the term-definable operations:

\[
\begin{align*}
a \lor b & := \neg (\neg a \oplus b) \oplus b, \\
a \land b & := \neg (\neg a \lor \neg b).
\end{align*}
\]

A variety of residuated lattices with the property that \(x \oplus x = x\) implies A is a BA
Examples of MV-algebras

- Boolean algebras
- The unit interval \([0, 1]\) in its natural order is an MV-algebra with
  \[
a \oplus b = \min\{a + b, 1\}, \quad \neg a = \min\{1 - a\}
\]
- The MV-algebra \(C(X, [0, 1])\) of continuous functions from a space \(X\) to \([0, 1]\) with point-wise operations
- Free MV-algebras, consisting of the McNaughton functions on the unit cube
- Nonstandard extensions of \([0, 1]\) (ultrapowers) are examples of MV-algebras with infinitesimal elements
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MV-algebras

Duality between finitely presented MV-algebras and rational polyhedra [Marra and Spada 2012]


Sheaf representations over the maximum MV-spectrum [Keimel 1968; Filipoiu-Georgescu 1995] and over the MV-spectrum [Kennison 1976; Cornish 1980; Yang 2006; Dubuc-Poveda 2010]
The lattice spectrum

Let $D$ be a distributive lattice, the points, $x \in X_D$, of the lattice spectrum of $D$ are in one-to-one correspondence with each of the following:

- $h_x : D \to 2$ a lattice homomorphism
- $I_x$ a prime ideal of $D \ (= h^{-1}(0))$
- $F_x$ a prime filter of $D \ (= h^{-1}(1))$

If, in addition, $D$ is finite, then the above are equivalent to

- $F_x = \uparrow p$ where $p \in J(D)$ and $I_x = \downarrow m$ where $m \in M(D)$
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The different dual spaces of an MV-algebra

Stone’s representation theorem and duality

Let $D$ be a distributive lattice, $X_D$ the spectrum of $D$, then

$$\eta : D \rightarrow \mathcal{P}(X_D)$$

$$a \mapsto \hat{a} = \{ x \in X_D \mid h_x(a) = 1 \}$$

is an injective lattice homomorphism

Topologies on $X_D$:

- $\tau_D = \langle \hat{a} \mid a \in D \rangle$ (spectral or Stone topology)
- $\tau_D^{\partial} = \langle (\hat{a})^c \mid a \in D \rangle$ (dual spectral topology)
- $\pi_D = \langle \hat{a}, (\hat{a})^c \mid a \in D \rangle$ (Priestley topology)

In all three $\text{Im}(\hat{\cdot})$ can be characterized (order-)topologically
The MV-spectrum

A simple but important fact in the theory of MV-algebras is that

\[ \theta : A \rightarrow Con(A) \]

\[ a \rightarrow \theta(a) = \langle (0, a) \rangle_{Con(A)} \]

is a bounded lattice homomorphism.

The image of this map is the lattice \( Con_{fin}(A) \) of finitely generated MV-algebra congruences of \( A \) (and thus these congruences are pairwise permuting).

The MV-spectrum of \( A \), is the dual space, \( Y \), of \( Con_{fin}(A) \)
The MV-spectrum

Since $A \twoheadrightarrow Con_{fin}(A)$ is a bounded distributive lattice quotient, by duality, $Y \hookrightarrow X$ may be seen as a closed subspace of $X$

The MV-spectrum may also be seen as the set of those MV-ideals (non-empty downsets closed under $\oplus$) that are prime in the sense that one of $a \ominus b (: = \neg(\neg a \oplus b))$ and $b \oplus a$ is a member for all $a, b \in A$. This is the same set $Y \subseteq X$.

The spectral topology on $Y$ is also the hull-kernel or spectral topology corresponding to the MV-ideals of $A$. 
Relative normality of the dual of the MV-spectrum

The following are equivalent:

- A bounded distributive lattice $D$ is normal: For all $a, b \in D$, if $a \vee b = 1$ then there are $c, d \in A$ with $c \wedge d = 0$ and $a \vee d = 1$ and $c \vee b = 1$
- Each point in the dual space of $D$ is below a unique maximal point
- The inclusion of the maximal points of the dual space of $D$ admits a continuous retraction

For any MV-algebra $A$, the lattice $\text{Con}_{\text{fin}}(A)$ is relatively normal (that is, each interval $[a, b]$ is a normal lattice)

As a consequence $Y$ is always a root-system, that is, $\uparrow y$ is a chain for each $y \in Y$
The maximal MV-spectrum

Given an MV-algebra, $A$, the subspace $Z$ of $Y$ of maximal MV-ideals of $A$ is called the maximal MV-spectrum.

Since $Con_{fin}(A)$ is relatively normal for any MV-algebra $Y$ is a root-system and the map

$$m : Y \rightarrow Z$$

$$y \mapsto \text{unique maximal point above } y$$

is a continuous retraction. The maximal MV-spectrum is compact Hausdorff, but not in general spectral.
Extended Priestley duality for $MV$-algebras

(from BLAST’08 on [G-Priestley’07-’08])

The dual spaces of $MV$-algebras are the spaces $(X, \leq, \tau, \cdot)$ satisfying:

- $(X, \leq, \tau)$ is a Priestley space with bounds
- $(X, \tau, \leq, \cdot, 1)$ is an ordered topological monoid
- The $\cdot$ is open, commutative, and has a lower adjoint
- The element $0$ is absorbant
- $\forall x, y \ [ x \neq y \ast x \Rightarrow \forall z \ (x \leq z \text{ or } y \cdot z \leq y \ast x )]$ where $y \ast x$ is the least $z \in X$ such that

$$\forall x' \ [ y \not\leq x' \Rightarrow x' \cdot x \leq z]$$
A bit of explanation

To best make sense of this we need canonical extension, but there is not enough time

- On an MV-algebra we have $\oplus$ and $\neg$, but also $\ominus$, and $\odot$ and $\rightarrow$, which are de Morgan duals of $\oplus$ and $\ominus$
  - $\ominus$ is lower adjoint to $\oplus$
  - $\odot$ is lower adjoint to $\rightarrow$

- In [G-Priestley] we took $\rightarrow$ as basic. It is witnessed dually by $\odot$, which by DQA restricts to $X \cup \{0\}$

- In order to get a First-Order characterization, we also needed $\star$ (which plays no role in this work)
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Duality for $\oplus$ and $\neg$

What we need here

- For lattice ideals $I$ and $J$
  \[ I \oplus J = \downarrow \{ a \oplus b \mid a \in I, b \in J \} \]

- For $x, x' \in X$ we have $I_x \oplus I_{x'}$ is either prime or all of $A$

- Define a partial operation on $X$ by $x + x' = z$ iff $I_x \oplus I_{x'} = I_z$

- The partial operation $+$ has domain $\{(x, y) \mid i(x) \geq y\}$, where $i$ is the involution dual to $\neg$

This partial operation $+$ witnesses fully the MV-algebra structure of $A$ relative to its lattice reduct
Relation to the MV-spectrum

For each $x \in X$ we have that $x + x$ is defined and

$$x \in Y \iff x + x = x$$

For each $x \in X$, there is a largest $x' \in X$ with

$$x + x' \leq x$$

and this $x'$ satisfies $x' + x' = x'$

This defines a retraction $k : X \to Y$, which is continuous with respect to the Priestley topology on $X$ and the spectral topology on $Y$. A function named $k$ (defined differently, but with the same action) was present in Martinez’ work.
Interpolation Lemma

NB! The map $k$ is neither order preserving nor reversing.

However it satisfies the following **Interpolation Lemma**:

If $x \leq x'$ then there is $x''$ with

\[
x \leq x'' \leq x' \quad \text{and} \quad k(x'') \geq k(x) \quad \text{and} \quad k(x'') \geq k(x')
\]
From $X$ to $Z$ with $m \circ k$

Combining the two earlier retractions we get

$$m \circ k : (X, \pi) \rightarrow (Z, \tau)$$

The kernel of this map is given by the relation $x_1 W x_2$ iff there are $x'_1, x'_2, x_0 \in X$ with

Proof: If $mk(x_1) = mk(x_2)$, then take $x'_i = k(x_i)$ and $x_0 = mk(x_i)$.

For the converse note that if $x \leq x'$, then by (Int) there is $x''$ between with greater $k$-image than both, but then $mk(x) = mk(x'') = mk(x')$. So all the elements of $X$ in one order component have the same $mk$-image
Kaplansky’s theorem

[Kaplansky 1947]
Let $Z_1, Z_2$ be compact Hausdorff spaces such that the lattices $C(Z_1, [0, 1])$ and $C(Z_2, [0, 1])$ are isomorphic. Then $Z_1$ and $Z_2$ are homeomorphic spaces.
Kaplansky theorem for arbitrary MV-algebras

Theorem

If $A_1$ and $A_2$ are MV-algebras having isomorphic lattice reducts, then the max MV-spectra of $A_1$ and $A_2$ are homeomorphic.

- Note that the max MV-spectrum of an MV-algebra of the form $C(Z, [0, 1])$ is $Z$ so that our result generalizes Kaplansky’s result.

Proof (sketch).

The maximal MV-spectrum can be constructed from the lattice spectrum by taking the topological quotient w.r.t. the relation $W$ and $W$ is definable from just the order structure of $X$. □
Spectral sum

We have seen that there is a continuous retraction

\[ k : (X, \pi) \longrightarrow (Y, \tau) \]

It follows that \( k^{-1}(\uparrow y) \), call it \( X_y \), is a closed subspace of \( X \) in the Priestley topology. In fact, \( X_y \) is a chain and

\[ X_y = k^{-1}(\uparrow y) \text{ is the dual of the MV quotient } A_y = A/I_y \]

Moreover, one can show (Patch):

For any finite cover \( (U_i)_{i=1}^n \) of \( Y \) by \( \tau^\partial \)-open sets, and any collection \( (\widehat{a}_i)_{i=1}^n \) of clopen downsets of \( X \) such that

\[ \widehat{a}_i \cap k^{-1}(U_i \cap U_j) = \widehat{a}_j \cap k^{-1}(U_i \cap U_j) \quad (*) \]

holds for any \( i, j \in \{1, \ldots, n\} \). Then the set \( \bigcup_{i=1}^n (\widehat{a}_i \cap q^{-1}(U_i)) \) is a clopen downset in \( X \)
Patching property
Patching property
There is at most one \( b \in A \),

It must satisfy \( \hat{b} = \bigcup_{i=1}^{n}(\hat{a}_i \cap k^{-1}(U_i)) =: K \), and \( K \) is closed

Prove that \( K \) is a open: \( K^c = \bigcup_{i=1}^{n}(\hat{a}_i^c \cap k^{-1}(U_i)) \)

Prove that \( K \) is a downset: Interpolation Lemma!!!

This also yields a formula for \( b \) by compact approximation:

\[
b = \bigvee_{i=1}^{n}(a_i \odot \neg m u_i),
\]

where \( m, n \in \mathbb{N} \) and \( u_i \in A \) such that \( \hat{u}_i^c = U_i \).
From spectral sums to sheaves

**Theorem**

Let $D$ be a distributive lattice with dual space $X$. Suppose that $q: (X, \pi) \to (I, \rho)$ is a continuous surjection onto a stably compact space which satisfies the property (Patch) as given earlier. Then the associated étale space $p: (E, \sigma) \to (I, \rho^\partial)$ is a sheaf representation of $D$ over $Y$.

- We saw above that $k: (X, \pi) \to (Y, \tau)$ satisfies the hypothesis of the above theorem.
- One can also show that $m \circ k: (X, \pi) \to (Z, \tau)$ satisfies the hypothesis of the above theorem.

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1See Sam van Gool’s talk tomorrow morning.
Sheaf decompositions of MV-algebras

As a consequence of the sum decompositions we obtain:

**Theorem**

Any MV-algebra is representable as the global sections of a sheaf

1. over the maximal MV-spectrum, which is a compact Hausdorff space, with stalks that are *local MV-algebras* (i.e. having a unique maximal ideal)
   
   [Keimel 1968; Filipoiu-Georgescu 1995]

2. over the MV-spectrum with the *dual spectral* topology with stalks that are *MV-chains*
   
   [Kennison 1976; Cornish 1980; Yang 2006; Dubuc-Poveda 2010]
Conclusion

We have obtained the sheaf representations of MV-algebras over their prime and maximal MV-spectra from a corresponding decomposition of their Priestley dual spaces. This allows a direct treatment of all MV-algebras using only simple facts from the theory of MV-algebras.

If sheaf representations are interesting to MV-algebraists, then so is Priestley duality.