Duality Theories for Boolean Algebras with Operators

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People who have contributed to the subject

- Marshall Stone.
- Bjarni Jónsson and Alfred Tarski.
- Paul Halmos.
- Georges Hansoul.
- Giovanni Sambin and Vicenzo Vaccaro.
- Robert Goldblatt.
- Sergio Celani.
- Steven Givant.

Existing dualities for Boolean algebras

Algebra Dual

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▶ Boolean algebra of subsets of U.	* Stone-Čech compactification of discrete space U.

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- *Complete* operator: distributes (in each coordinate) over all infinite sums that exist.

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Algebraic duality summary

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- Complete and atomic Boolean algebras with complete operators.

Dual

* Relational structures.

Algebraic duality summary

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Direct products.	Disjoint unions.

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- Example: complex algebra of group $(G\,,\,\circ)$,

$$X \circ Y = \{h \in G : h = f \circ g \text{ for some } f \in X \text{ and } g \in Y\}.$$

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- Atom structure (Jónsson-Tarski):

$$\mathfrak{U} = (U\,,\ R).$$

Duality for algebras and structures

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Identify a complete and atomic algebra with complete operators with its second dual, the complex algebra of its atom structure.

Morphisms

- Algebra: complete homomorphisms

$$\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$$

preserve fundamental operations and all existing infinite sums,

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- Structures: bounded homomorphisms (Goldblatt, modal logic)

$$\vartheta : (U\,,\; R) \longrightarrow (V\,,\; S)$$

preserve fundamental relations, and

$$S(r, s, \vartheta(w)) \qquad \text{implies} \qquad R(u, v, w)$$

for some u, v in U such that $\vartheta(u) = r$ and $\vartheta(v) = s$.

Complete homomorphisms from bounded homomorphisms

- Given: bounded homomorphism

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- Definition (Goldblatt): for each X in $\mathfrak{Cm}(V)$ (so $X \subseteq V$),

$$\varphi(X) = \vartheta^{-1}(X) = \{u \in U : \vartheta(u) \in X\}.$$

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- Definition (Jónsson): for each u in $\mathfrak{U}$,

$$\vartheta(u) = r \qquad \text{if and only if} \qquad r \in \bigcap\{X \subseteq V : u \in \varphi(X)\}$$

Duality for morphisms

Theorem [Strengthens Jónsson]

There is a bijective correspondence between:
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- The duality reverses compositions.

Categories

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Dually equivalent categories [Jónsson]:

- Category of structures with bounded homomorphisms,
- Category of complete and atomic Boolean algebras with complete operators, and with complete homomorphisms.

Complete ideals

- Definition: An *ideal* in $\mathfrak{A}$ is a Boolean ideal M such that

$$r \in M \quad \text{and} \quad s \in A \quad \text{implies} \quad r \circ s,\ s \circ r \in M.$$

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- The complete ideals form a complete lattice under inclusion.

Inner subuniverses

- Definition (Goldblatt, modal logic): An inner subuniverse of $\mathfrak{U}$ is a subuniverse V such that for r, s, t in $\mathfrak{U}$,

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Inner subuniverses from complete ideals

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- Definition: If W is the generator of M, then $V = \sim W$.

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- Given: inner subuniverse V of $\mathfrak{U}$.
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- Definition: M consists of the subsets of $\sim V$.

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- The function mapping each complete ideal to its dual inner subuniverse is a dual lattice isomorphism.

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$\mathfrak{V}$ is an inner substructure of $\mathfrak{U}$,
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- The dual structure of $\mathfrak{Cm}(U)/M$ is isomorphic to $\mathfrak{V}$.

Complete subalgebras and bounded quotients

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Complete subalgebras and bounded quotients

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- Definition: $U = \bigcup_i U_i$ and $R = \bigcup_i R_i$.
- Construct: system $(\mathfrak{Cm}(U_i) : i \in I)$ of dual complex algebras.

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Theorem [strengthens Goldblatt]

The dual of the union $\mathfrak{U}$ is the internal product $\prod_i \mathfrak{Cm}(U_i)$.

- $\mathfrak{Cm}(U) = \prod_i \mathfrak{Cm}(U_i)$.

Topological duality summary

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Topological duality summary

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- Homomorphisms.

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* Continuous bounded homomorphisms.

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Direct products.	Stone-Čech compactifications of unions.

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  - clopen sets form a base for the topology.
- Fundamental relations are clopen (Halmos, Hansoul, Goldblatt): if F, G are clopen subsets of U, then the image of $F \times G$ under R is clopen in U.

$$R^*(F \times G) = \{t \in U : R(r, s, t) \text{ for some } (r, s) \in F \times G\}.$$

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- Fundamental relations are continuous: inverse images under R of open subsets G of U are open in product space $U \times U$.

$$R^{-1}(G) = \{(r,s) \in U \times U : R(r,s,t) \text{ implies } t \in G\}.$$

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- Definition (Halmos, Hansoul, Goldblatt): $\mathfrak{A}$ is corresponding subalgebra of clopen sets.

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- $X \circ Y = \{r \circ s : r \in X \text{ and } s \in Y\}$.
- Topology: Stone topology, where clopen sets have form

$$F_t = \{X \in U : t \in X\}$$

for elements t in $\mathfrak{A}$.

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Theorem [other versions due to Goldblatt, Hansoul, Halmos]

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Identify every Boolean algebra with operators with its second dual, the algebra of clopen sets of ultrafilters.

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- Relational spaces: continuous, bounded homomorphisms

$$\vartheta : \mathfrak{V} \longrightarrow \mathfrak{U},$$

inverse images under ϑ of open sets are open.

Homomorphisms from continuous bounded homomorphisms

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- Definition (Goldblatt): for each F in $\mathfrak{A}$ (so F is a clopen subset of $\mathfrak{U}$),

$$\varphi(F) = \vartheta^{-1}(F) = \{u \in U : \vartheta(u) \in F\}.$$

Continuous bounded homomorphisms from homomorphisms

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- $X_r = \{F \in A : r \in F\} \qquad \text{and} \qquad Y_s = \{G \in B : s \in G\}$.

Duality for morphisms

Theorem [strengthens Goldblatt; see also Halmos, Hansoul]

There is a bijective correspondence between:
continuous bounded homomorphisms $\vartheta : \mathfrak{V} \longrightarrow \mathfrak{U}$, and
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- The duality reverses compositions.

Categories

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Dually equivalent categories [Goldblatt]:

- Category of relational spaces with continuous bounded homomorphisms,
- Category of Boolean algebras with normal operators, and with homomorphisms.

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- Definition: An open set H in a relational space $\mathfrak{U}$ is *special* if for every pair of clopen sets F, G,

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  where
- $R(F \times G) = \{t \in U : R^*(r, s, t) \text{ for some } r \in F \text{ and } s \in G\}$.
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- Given: ideal M in $\mathfrak{A}$.
- Construct: dual special open subset H of $\mathfrak{U}$.
- Definition: H is the union of the clopen sets that belong to M.

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- The function mapping each ideal to its dual special open set is a lattice isomorphism.

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- Result: a dual lattice isomorphism from ideals to special closed sets.

Duality for quotients and inner subspaces

Theorem

$\mathfrak{U}$ a relational space and $\mathfrak{A}$ its dual algebra of clopen sets.
$\mathfrak{V}$ is an inner subspace of $\mathfrak{U}$,
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- The dual relational space of $\mathfrak{A}/M$ is homeo-isomorphic to $\mathfrak{V}$.

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- Subalgebras and quotient relational spaces (modulo bounded Boolean congruences)

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- Intermediate subdirect products form lattice under subalgebra relation.

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- Definition: relativization of $\mathfrak{B}$ to U:
- $B_0 = \{F \cap U : F \in B\}$ is subuniverse between $\mathfrak{D}$ and $\mathfrak{A}$.
- $\mathfrak{B}$ is isomorphic to $\mathfrak{B}_0$ via relativization isomorphism

$$F \longrightarrow F \cap U.$$

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- Let $\mathfrak{V}$ be dual relational space of $\mathfrak{C}$.
- Construct: homeo-isomorphic copy of $\mathfrak{U}$ as a dense, locally compact inner subspace of $\mathfrak{V}$.
- Use the Exchange Theorem to get a compactification of $\mathfrak{U}$ with dual algebra isomorphic to $\mathfrak{C}$ via relativization.

Duality for subdirect products and compactifications

Theorem

$\mathfrak{U}$ is the union of a disjoint system of relational spaces.
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- The dual algebra of the Stone-Čech compactification of $\mathfrak{U}$ is isomorphic via relativization to the direct product $\mathfrak{A}$.

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- The Stone-Čech compactification of U endowed with the discrete topology.
- Problem: generalize this result to complex algebras of relational structures.

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- Goal: characterize the dual relational space of $\mathfrak{Cm}(U)$ and some of its subalgebras.

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- Part of solution: develop duality theory for weakly bounded homomorphisms and homomorphisms.

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- (Dwinger) $\mathfrak{A}$ is the dual algebra of the Stone-Čech compactification of U.

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