On extended and partial real-valued functions in Pointfree Topology

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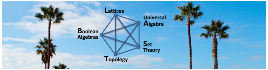

BLAST 2013

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Universidad del País Vasco

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[1] Joint work with Jorge Picado

The ring of continuous real functions on a frame: $\mathrm{C}(L)$

The frame of reals is the frame $\mathfrak{L}(\mathbb{R})$ generated by all ordered pairs (p, q), where $p, q \in \mathbb{Q}$, subject to the following relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,

(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$,

(R3) $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$,

(R4) $\bigvee_{p,q\in\mathbb{Q}}(p, q) = 1$.

The spectrum of $\mathfrak{L}(\mathbb{R})$ is homeomorphic to the space $\mathbb{R}$ of reals endowed with the euclidean topology.

Combining the natural isomorphism $\mathbf{Top}(X, \Sigma L) \simeq \mathbf{Frm}(L, \mathcal{O}X)$ for $L = \mathfrak{L}(\mathbb{R})$ with the homeomorphism $\Sigma\mathfrak{L}(\mathbb{R}) \simeq \mathbb{R}$ one obtains

$$\boxed{\mathrm{C}(X) = \mathbf{Top}(X, \mathbb{R}) \xrightarrow{\sim} \mathbf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{O}X)}$$

Regarding the frame homomorphisms $\mathfrak{L}(\mathbb{R}) \to L$, for a general frame L, as the continuous real functions on L provides a natural extension of the classical notion. They form a lattice-ordered ring that we denote

$$\boxed{\mathrm{C}(L) = \mathbf{Frm}(\mathfrak{L}(\mathbb{R}), L)}$$

Lattice and algebraic operations in $C(L)$

Recall that the operations on the algebra $C(L)$ are determined by the lattice-ordered ring operations of $\mathbb{Q}$ as follows:

(1) For $\diamond = +, \cdot, \wedge, \vee$:

$$(f \diamond g)(p,q) = \bigvee\{f(r,s) \wedge g(t,u) \mid \langle r,s\rangle \diamond \langle t,u\rangle \subseteq \langle p,q\rangle\}$$

where $\langle \cdot,\cdot\rangle$ stands for open interval in $\mathbb{Q}$ and the inclusion on the right means that $x \diamond y \in \langle p,q\rangle$ whenever $x \in \langle r,s\rangle$ and $y \in \langle t,u\rangle$.

(2) $(-f)(p,q) = f(-q,-p)$.

(3) For each $r \in \mathbb{Q}$, a nullary operation $\mathbf{r}$ defined by

$$\mathbf{r}(p,q) = \begin{cases} 1 & \text{if } p < r < q \\ 0 & \text{otherwise.} \end{cases}$$

(4) For each $0 < \lambda \in \mathbb{Q}$, $(\lambda \cdot f)(p,q) = f(\frac{p}{\lambda}, \frac{q}{\lambda})$.

B. Banaschewski,
The real numbers in pointfree topology,
Textos de Matemática, Série B, 12, Univ. de Coimbra, 1997.

Part I: Extended real-valued functions

(based on joint work with Bernhard Banaschewski,)

The frame of extended reals: a first attempt

How to describe the frame $\mathfrak{L}(\overline{\mathbb{R}})$ of extended reals in terms of generators and relations?

The frame of extended reals is the frame $\mathfrak{L}(\mathbb{R})\mathfrak{L}(\overline{\mathbb{R}})$ generated by all ordered pairs (p, q), where $p, q \in \mathbb{Q}$, subject to the following relations:

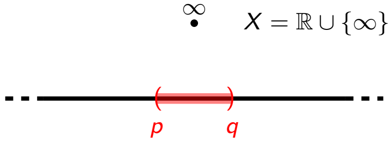
(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,

(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$,

(R3) $(p, q) = \bigvee \{(r, s) \mid p < r < s < q\}$,

~~(R4)~~ $\cancel{\bigvee_{p,q \in \mathbb{Q}} (p, q) = 1}$.

But this frame is precisely the one-point extension of $\mathfrak{L}(\mathbb{R})$!
The spectrum of $\mathfrak{L}(\mathbb{R})$ is <u>not</u> homeomorphic to the space $\overline{\mathbb{R}}$ of extended reals endowed with the euclidean topology. Indeed,



The one-point extension of the real line: $\mathcal{O}X = \mathcal{O}\mathbb{R} \cup \{X\}$

The frame of extended reals

It is useful here to adopt an equivalent description of $\mathfrak{L}(\mathbb{R})$ with the elements

$$(r, —) = \bigvee_{s\in\mathbb{Q}} (r, s) \text{ and } (—, s) = \bigvee_{r\in\mathbb{Q}} (r, s)$$

as primitive notions.

Specifically, the frame of reals $\mathfrak{L}(\mathbb{R})$ is equivalently given by generators $(r, —)$ and $(—, s)$ for $r, s \in \mathbb{Q}$ subject to the defining relations

(r1) $(r, —) \wedge (—, s) = 0$ whenever $r \geq s$,

(r2) $(r, —) \vee (—, s) = 1$ whenever $r < s$,

(r3) $(r, —) = \bigvee_{s>r}(s, —)$, and $(—, r) = \bigvee_{s<r}(—, s)$, for every $r \in \mathbb{Q}$,

(r4) $\bigvee_{r\in\mathbb{Q}}(r, —) = 1 = \bigvee_{r\in\mathbb{Q}}(—, r)$.

With $(p, q) = (p, —) \wedge (—, q)$ one goes back to (R1)–(R4).

The frame of extended reals and extended continuous real functions

The frame of extended reals is the frame $\mathfrak{L}(\mathbb{R})\mathfrak{L}(\overline{\mathbb{R}})$ generated by generators $(r, —)$ and $(—, s)$ for $r, s \in \mathbb{Q}$ subject to the defining relations

(r1) $(r, —) \wedge (—, s) = 0$ whenever $r \geq s$,

(r2) $(r, —) \vee (—, s) = 1$ whenever $r < s$,

(r3) $(r, —) = \bigvee_{s>r}(s, —)$ and $(—, r) = \bigvee_{s<r}(—, s)$, for every $r \in \mathbb{Q}$,

~~(r4)~~ $\cancel{\bigvee_{r\in\mathbb{Q}}(r, —) = 1 = \bigvee_{r\in\mathbb{Q}}(—, r)}$.

The spectrum of $\mathfrak{L}(\overline{\mathbb{R}})$ is homeomorphic to the space $\overline{\mathbb{R}}$ of extended reals endowed with the euclidean topology.

Combining the natural isomorphism $\mathbf{Top}(X, \Sigma L) \simeq \mathbf{Frm}(L, \mathcal{O}X)$ for $L = \mathfrak{L}(\overline{\mathbb{R}})$ with the homeomorphism $\Sigma\mathfrak{L}(\overline{\mathbb{R}}) \simeq \overline{\mathbb{R}}$ one obtains

$$\boxed{\overline{\mathsf{C}}(X) = \mathbf{Top}(X, \overline{\mathbb{R}}) \xrightarrow{\sim} \mathbf{Frm}(\mathfrak{L}(\overline{\mathbb{R}}), \mathcal{O}X)}$$

Regarding the frame homomorphisms $\mathfrak{L}(\overline{\mathbb{R}}) \to L$, for a general frame L, as the extended continuous real functions on L provides a natural extension of the classical notion. Hence we denote

$$\boxed{\overline{\mathsf{C}}(L) = \mathbf{Frm}(\mathfrak{L}(\overline{\mathbb{R}}), L)}$$

Lattice and algebraic operations in $\mathrm{C}(L)$ (equivalent characterization)

Recall that the operations on the algebra $\mathrm{C}(L)$ are determined by the lattice-ordered ring operations of $\mathbb{Q}$ as follows:

(1) For $\diamond = +, \cdot, \wedge, \vee$:

$$(f \diamond g)(p, —) = \bigvee_{p < r \diamond s} f(r, —) \wedge g(s, —) \quad \text{and} \quad (f \diamond g)(—, q) = \bigvee_{r \diamond s < q} f(—, r) \wedge g(—, s)$$

(2) $(-f)(p, —) = f(—, -p)$ and $(-f)(—, q) = f(-q, —)$.

(3) For each $r \in \mathbb{Q}$, a nullary operation $\mathbf{r}$ defined by

$$\mathbf{r}(p, —) = \begin{cases} 1 & \text{if } p < r \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \mathbf{r}(—, q) = \begin{cases} 1 & \text{if } r < q \\ 0 & \text{otherwise.} \end{cases}$$

(4) For each $0 < \lambda \in \mathbb{Q}$, $(\lambda \cdot f)(p, —) = f(\frac{p}{\lambda}, —)$ and $(\lambda \cdot f)(—, q) = f(—, \frac{q}{\lambda})$.

B. Banaschewski,
The real numbers in pointfree topology,
Textos de Matemática, Série B, 12, Univ. de Coimbra, 1997.

Lattice operations in $\overline{\mathrm{C}}(L)$

An analysis of the proof that $\mathrm{C}(L)$ is an f-ring shows that, by the same arguments, the operations $\vee$, $\wedge$ and $-(\cdot)$ satisfy all identities which hold for the corresponding operations of $\mathbb{Q}$ in $\overline{\mathrm{C}}(L)$.

Hence, $\overline{\mathrm{C}}(L)$ is a distributive lattice with join $\vee$, meet $\wedge$ and an inversion given by $-(\cdot)$. Moreover, it is, of course, bounded, with top $+\infty$ and bottom $-\infty$, where

$$+\infty(p,—) = 1 = -\infty(—,q) \qquad \text{and} \qquad +\infty(—,q) = 0 = -\infty(p,—).$$

Further, the partial order determined by this lattice structure is exactly the one mentioned earlier:

$$\begin{aligned} f \le g \quad &\text{iff} \quad f \vee g = g \quad \text{iff} \quad f \wedge g = f \\ &\text{iff} \quad f(r,—) \le g(r,—) \text{ for all } r \in \mathbb{Q} \\ &\text{iff} \quad f(—,s) \ge g(r,—,s) \text{ for all } s \in \mathbb{Q}. \end{aligned}$$

Algebraic operations in $\overline{\mathsf{C}}(L)$

Things become more complicated in the case of addition and multiplication.

This is not a surprise if we think of the typical indeterminacies

$$-\infty + \infty \quad \text{and} \quad 0 \cdot \infty$$

when dealing with the algebraic operations in $\overline{\mathsf{C}}(X)$

In the classical case, given $f, g \colon X \to \overline{\mathbb{R}}$, the condition

$$f^{-1}(\{+\infty\}) \cap g^{-1}(\{-\infty\}) = \varnothing = f^{-1}(\{-\infty\}) \cap g^{-1}(\{+\infty\})$$

ensures that the addition $f + g$ can be defined for all $x \in X$ just by the natural convention

$$\lambda + (+\infty) = +\infty = (+\infty) + \lambda \quad \text{and} \quad \lambda + (-\infty) = -\infty = (-\infty) + \lambda$$

for all $\lambda \in \mathbb{R}$ together with the usual $(+\infty) + (+\infty) = +\infty$ and the same for $-\infty$.

Clearly enough, this condition is equivalent to

$$(f \vee g)^{-1}(\{+\infty\}) \cap (f \wedge g)^{-1}(\{-\infty\}) = \varnothing.$$

Algebraic operations in $\overline{\mathrm{C}}(L)$

What about the algebraic operations in $\overline{\mathrm{C}}(L)$?: Addition

Let $f, g \in \overline{\mathrm{C}}(L)$, the natural definition of $h = f + g \colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$ on generators would be:

$$h(p, -) = \bigvee_{p<r+s} f(r, -) \wedge g(s, -) \quad \text{and} \quad h(-, q) = \bigvee_{r+s<q} f(-, r) \wedge g(-, s)$$

But $h \notin \overline{\mathrm{C}}(L)$ in general! Indeed, $h \in \overline{\mathrm{C}}(L)$ if and only if

$$\Big(\bigvee_{r\in\mathbb{Q}} f(-, r) \vee \bigvee_{r\in\mathbb{Q}} g(r, -)\Big) \wedge \Big(\bigvee_{r\in\mathbb{Q}} g(-, r) \vee \bigvee_{r\in\mathbb{Q}} f(r, -)\Big) = 1.$$

Notation. For each $f \in \overline{\mathrm{C}}(L)$ let

$$a_f^+ = \bigvee_{r\in\mathbb{Q}} f(-, r), \quad a_f^- = \bigvee_{r\in\mathbb{Q}} f(r, -) \quad \text{and} \quad a_f = a_f^+ \wedge a_f^- = \bigvee_{r<s} f(r, s) = f(\omega).$$

a_f is the pointfree counterpart of the domain of reality $f^{-1}(\mathbb{R})$ of an $f \colon X \to \overline{\mathbb{R}}$.

Note also that $a_f = a_f^+ = a_f^- = 1$ if and only if $f \in \mathrm{C}(L)$.

Algebraic operations in $\overline{\mathsf{C}}(L)$

Definition. Let $f, g \in \overline{\mathsf{C}}(L)$. We say that f and g are sum compatible if

$$a^{+}_{f\vee g} \vee a^{-}_{f\wedge g} = 1 \quad \text{iff} \quad (a^{+}_{f} \vee a^{-}_{g}) \wedge (a^{+}_{g} \vee a^{-}_{f}) = 1.$$

Theorem. Let $f, g \in \overline{\mathsf{C}}(L)$ and $fh = +g\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$ given by

$$(f+g)(p,\text{—}) = \bigvee_{p<r+s} f(r,\text{—}) \wedge g(s,\text{—}) \quad \text{and} \quad (f+g)(\text{—},q) = \bigvee_{r+s<q} f(\text{—},r) \wedge g(\text{—},s).$$

Then $f+g \in \overline{\mathsf{C}}(L)$ if and only if f and g are sum compatible.

Algebraic operations in $\overline{\mathrm{C}}(L)$

What about the algebraic operations in $\overline{\mathrm{C}}(L)$?: Multiplication

In the classical case, given $f, g\colon X \to \overline{\mathbb{R}}$ the condition

$$f^{-1}(\{-\infty,+\infty\}) \cap g^{-1}(\{0\}) = \varnothing = f^{-1}(\{0\}) \cap g^{-1}(\{-\infty,+\infty\})$$

ensures that the multiplication $f \cdot g$ can be defined for all $x \in X$ just by the natural conventions

$$\lambda \cdot (\pm\infty) = \pm\infty = (\pm\infty) \cdot \lambda$$

for all $\lambda > 0$ and

$$\lambda \cdot (\pm\infty) = \mp\infty = (\pm\infty) \cdot \lambda$$

for all $\lambda < 0$ together with the usual

$$(\pm\infty) \cdot (\pm\infty) = +\infty \quad \text{and} \quad (\pm\infty) \cdot (\mp\infty) = -\infty.$$

Notation. Recall that in a frame L, a cozero element is an element of the form

$$\operatorname{coz} f = f((\text{—}, 0) \vee (0, \text{—})) = \bigvee \{ f(p,0) \vee f(0,q) \mid p < 0 < q \text{ in } \mathbb{Q} \}$$

for some $f \in \mathrm{C}(L)$. This is the pointfree counterpart to the notion of a cozero set for ordinary continuous real functions.

Algebraic operations in $\overline{\mathsf{C}}(L)$

Definition. Let $f, g \in \overline{\mathsf{C}}(L)$. We say that f and g are product compatible if

$$(a_f \wedge a_g) \vee (\operatorname{coz} f \wedge \operatorname{coz} g) = 1 \quad \text{iff} \quad (a_f \vee \operatorname{coz} g) \wedge (a_g \vee \operatorname{coz} f) = 1.$$

Theorem. Let $f, g \in \overline{\mathsf{C}}(L)$ and $f \cdot g \colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$ given by

$$(f \cdot g)(p, —) = \bigvee_{p < r \cdot s} f(r, —) \wedge g(s, —) \quad \text{and} \quad (f \cdot g)(—, q) = \bigvee_{r \cdot s < q} f(—, r) \wedge g(—, s).$$

Then $f \cdot g \in \overline{\mathsf{C}}(L)$ if and only if f and g are product compatible.

Extended real functions: an application

Representation Theorem (Johnson, 1962)

Let A be an archimedean f-ring with $N(A) = \{0\}$. Then there is a locally compact Hausdorff space X and an f-ring $\hat{A}$ of almost finite extended real functionsalmost finite extended real functions on X which separates points and closed setswhich separates points and closed sets in X, and an isomorphism $A \to \hat{A}$.

D.J. Johnson,
On a Representation Theory for a Class of Archimedean Lattice-Ordered Rings,
Proc. London Math. Soc, 12 (1962), 207-225.

Question: Is it possible to deal with families of "almost finite extended real functions which separates points and closed sets" in a pointfree setting?

Answer: **Yes, we can!** **!Podemos!**

Extended real functions: an application

Almost finite extended functions.

Recall that we have $\mathsf{C}(L) = \{f \in \overline{\mathsf{C}}(L) \mid a_f = 1\}$. Now, for any frame L, let

$$\boxed{\mathsf{D}(L) = \{f \in \overline{\mathsf{C}}(L) \mid a_f \text{ is dense}\}}$$

This definition extends the familiar classical notion to the pointfree setting:

Given an extended real continuous function $u\colon X \to \overline{\mathbb{R}}$ we have that the corresponding frame homomorphisms $\mathcal{O}u = u^{-1} \in \overline{\mathsf{C}}(\mathcal{O}X)$ and

$$\mathcal{O}u \in \mathsf{D}(\mathcal{O}X) \quad \text{iff} \quad u^{-1}[\mathbb{R}] \text{ is dense in } X \quad \text{iff} \quad u \in \mathsf{D}(X).$$

The correspondence $L \mapsto \mathsf{D}(L)$ is functorial for skeletal homomorphisms, that is, the $h\colon L \to M$ which take dense elements to dense elements

Extended real functions: an application

Theorem. For any L, there exists an inversion lattice embedding $\delta_L \colon \mathrm{D}(L) \to \mathrm{C}(\mathfrak{B}L)$ such that

$$\delta_L(f)(r,—) = f(r,—)^{**} \quad \text{and} \quad \delta_L(f)(—,r) = f(—,r)^{**}$$

which preserves the partial addition and multiplication of $\mathrm{D}(L)$.
Moreover, δ_L is onto if and only if L is extremally disconnected and then the partial operations are total so that δ_L is a lattice-ordered ring isomorphism.

B. Banaschewski, JGG and JP
Extended real functions in Pointfree Topology,
Journal of Pure and Applied Algebra 216 (2012), no. 4, 905-922.

Extended real functions: an application

Subfamilies in $\overline{\mathsf{C}}(X)$ which separates points from closed sets in X.

In Top – the category of all topological spaces – let:

$$f\colon X \to Y_f \quad \text{for all} \quad f \in \mathcal{F}.$$

The family $\mathcal{F}$ separates points from closed sets if for each closed $K \subseteq X$ and $x \in X \setminus K$, there exists an $f \in \mathcal{F}$ with $f(x) \notin \overline{f(K)}$.

Avoiding points. The family $\mathcal{F}$ separates points from closed sets iff for each closed $K \subseteq X$

$$K = \bigcap_{f \in \mathcal{F}} f^{-1}(\overline{f(K)}).$$

Avoiding closed sets. The family $\mathcal{F}$ separates points from closed sets iff for each closed $U \in \mathcal{O}X$

$$U = \bigcup_{f \in \mathcal{F}} f^{-1}(Y_f \setminus \overline{f(X \setminus U)}) = \bigcup_{f \in \mathcal{F}} f^{-1}(f_*(U))$$

(where $f_*\colon \mathcal{O}X \to \mathcal{O}Y_f$ is the right adjoint of the inverse image map $f^{-1}\colon \mathcal{O}Y_f \to \mathcal{O}X$).

Extended real functions: an application

Separating subfamilies in $\overline{\mathsf{C}}(L)$.

In Frm let:

$$h\colon M_h \to L \quad \text{for all} \quad h \in \mathcal{H}.$$

Definition. The family $\mathcal{H}$ is said to be separating if

$$a = \bigvee_{h \in \mathcal{H}} h(h_*(a)) \quad \text{for all } a \in L.$$

(Note that if $\mathcal{H} = \{h\}$ then $\mathcal{H}$ is separating iff h is an embedding.)

This definition extends a familiar classical notion to the pointfree setting:

Let $u\colon X \to Y_u$ be in Top for all $u \in \mathcal{F}$, and let $\mathcal{O}\mathcal{F}$ be the corresponding family of frame homomorphisms $\mathcal{O}u = u^{-1}\colon \mathcal{O}Y_u \to \mathcal{O}X$.

Then

$\mathcal{F}$ separates points from closed sets in Top iff $\mathcal{O}\mathcal{F}$ is separating in Frm.

Part II: Partial real-valued functions

(based on joint work with Imanol Mozo Carollo)

Order completeness of $\mathrm{C}(L)$ and $\overline{\mathrm{C}}(L)$

Certainly both $\mathrm{C}(L)$ and $\overline{\mathrm{C}}(L)$ fail to be Dedekind complete. But. . . why?

Let $\{f_i\}_{i\in I} \subset \mathrm{C}(L)$ and $f \in \mathrm{C}(L)$ be such that $f_i \leq f$ for all $i \in I$.
The natural candidate $h\colon \mathfrak{L}(\mathbb{R}) \to L$ would be defined for each $r \in \mathbb{Q}$ by

$$h(r,—) = \bigvee_{i\in I} f_i(r,—) \quad \text{and} \quad h(—,r) = \bigvee_{s<r} \Big(\bigwedge_{i\in I} f_i(—,s) \Big).$$

Recall that

$$h \in \mathrm{C}(L) \iff \begin{cases} \text{(r1) if } r \leq s, \text{ then } h(—,r) \wedge h(s,—) = 0, & V \\ \text{(r2) if } s < r, \text{ then } h(—,r) \vee h(s,—) = 1, & X \\ \text{(r3) } h(r,—) = \bigvee_{s>r} h(s,—) \text{ and } h(—,r) = \bigvee_{s<r} h(—,s), & V \\ \text{(r4) } \bigvee_{r\in\mathbb{Q}} h(r,—) = 1 = \bigvee_{r\in\mathbb{Q}} h(—,r). & V \end{cases}$$

(r2) if $s < r$, then $h(—,r) \vee h(s,—) \neq 1$ in general. We cannot ensure that $h \in \mathrm{C}(L)$ because of (r2).

$\boxed{\mathrm{C}(L) \text{ fails to be Dedekind complete because of (r2)!}}$

The frame of partial reals $\mathfrak{L}(\mathbb{IR})$

Generators: (p,q), $p,q \in \mathbb{Q}$
Relations:
(R1) $(p,q) \wedge (r,s) = (p \vee r, q \wedge s)$,
(R2) ~~$(p,q) \vee (r,s) = (p,s)$ whenever $p \le r < q \le s$,~~
(R3) $(p,q) = \bigvee \{(r,s) \mid p < r < s < q\}$,
(R4) $\bigvee_{p,q \in \mathbb{Q}} (p,q) = 1$.

Generators: $(r,\text{—}), (\text{—},s)$, $r,s \in \mathbb{Q}$
Relations:
(r1) $(r,\text{—}) \wedge (\text{—},s) = 0$ whenever $r \ge s$,
(r2) ~~$(r,\text{—}) \vee (\text{—},s) = 1$ whenever $r < s$,~~
(r3) $(r,\text{—}) = \bigvee_{s>r} (s,\text{—})$ and $(\text{—},s) = \bigvee_{r<s} (\text{—},r)$,
(r4) $\bigvee_{r \in \mathbb{Q}} (r,\text{—}) = 1 = \bigvee_{s \in \mathbb{Q}} (\text{—},s)$.

They both generate the same frame, the frame of partial reals $\mathfrak{L}(\mathbb{R})$. **Question.** Do they generate the same frame?

Answer. Yes, they do.

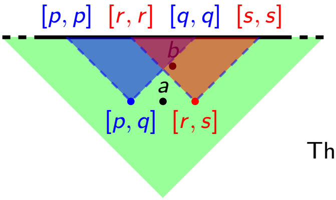
We will call it the frame of partial reals and denote by $\mathfrak{L}(\mathbb{IR})$.

The frame of partial reals $\mathfrak{L}(\mathbb{IR})$

Generators:	$(p, q), \quad p, q \in \mathbb{Q}$
Relations:	
(R1)	$(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,
~~(R2)~~	$\cancel{(p, q) \vee (r, s) = (p, s) \text{ whenever } p \le r < q \le s}$,
(R3)	$(p, q) = \bigvee \{(r, s) \mid p < r < s < q\}$,
(R4)	$\bigvee_{p,q\in\mathbb{Q}} (p, q) = 1$.

Generators:	$(r, —), (—, s), \quad r, s \in \mathbb{Q}$
Relations:	
(r1)	$(r, —) \wedge (—, s) = 0$ whenever $r \ge s$,
~~(r2)~~	$\cancel{(r, —) \vee (—, s) = 1 \text{ whenever } r < s}$,
(r3)	$(r, —) = \bigvee_{s>r} (s, —)$ and $(—, s) = \bigvee_{r<s} (—, r)$,
(r4)	$\bigvee_{r\in\mathbb{Q}} (r, —) = 1 = \bigvee_{s\in\mathbb{Q}} (—, s)$.

The spectrum $\Sigma\mathfrak{L}(\mathbb{IR})$ is the partial real line!



$\mathbb{IR} = \{a := [\underline{a}, \overline{a}] \subset \mathbb{R} \mid \underline{a}, \overline{a} \in \mathbb{R} \text{ and } \underline{a} \le \overline{a}\}$

$a \sqsubseteq b \quad \text{iff} \quad [\underline{a}, \overline{a}] \supseteq [\underline{b}, \overline{b}]$

$(\mathbb{IR}, \sqsubseteq)$ is the partial real line (or interval-domain)

The Scott topology on $(\mathbb{IR}, \sqsubseteq)$ is isomorphic to $\mathfrak{L}(\mathbb{IR})$

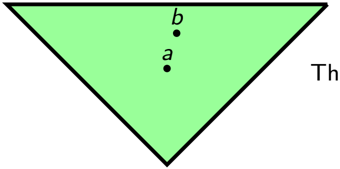
$$(p, q) \equiv \{a \in \mathbb{IR} \mid [p, q] \ll a\}$$

The frame of extended partial reals $\mathfrak{L}(\overline{\mathbb{IR}})$

Generators: $\quad (p,q), \quad p,q \in \mathbb{Q}$
Relations:
(R1) $(p,q) \wedge (r,s) = (p \vee r, q \wedge s)$,
~~(R2)~~ $\cancel{(p,q) \vee (r,s) = (p,s)}$ ~~whenever~~ $\cancel{p \le r < q \le s}$,
(R3) $(p,q) = \bigvee \{(r,s) \mid p < r < s < q\}$,
~~(R4)~~ $\cancel{\bigvee_{p,q \in \mathbb{Q}} (p,q) = 1}$.

Generators: $\quad (r, —), (—, s), \quad r,s \in \mathbb{Q}$
Relations:
(r1) $(r, —) \wedge (—, s) = 0$ whenever $r \ge s$,
~~(r2)~~ $\cancel{(r, —) \vee (—, s) = 1}$ ~~whenever~~ $\cancel{r < s}$,
(r3) $(r, —) = \bigvee_{s>r} (s, —)$ and $(—, s) = \bigvee_{r<s} (—, r)$,
~~(r4)~~ $\cancel{\bigvee_{r \in \mathbb{Q}} (r, —) = 1 = \bigvee_{s \in \mathbb{Q}} (—, s)}$.

The spectrum $\Sigma\mathfrak{L}(\overline{\mathbb{IR}})$ is the extended partial real line.



$$\overline{\mathbb{IR}} = \{a := [\underline{a}, \overline{a}] \subset \overline{\mathbb{R}} \mid \underline{a}, \overline{a} \in \overline{\mathbb{R}} \text{ and } \underline{a} \le \overline{a}\}$$

$$a \sqsubseteq b \quad \text{iff} \quad [\underline{a}, \overline{a}] \supseteq [\underline{b}, \overline{b}]$$

The Scott topology on $(\overline{\mathbb{IR}}, \sqsubseteq)$ is isomorphic to $\mathfrak{L}(\overline{\mathbb{IR}})$

The frame of partial reals and partial continuous real functions

The frame of partial reals is the frame $\mathfrak{L}(\mathbb{R})\mathfrak{L}(\mathbb{IR})$ generated by generators $(r, —)$ and $(—, s)$ for $r, s \in \mathbb{Q}$ subject to the defining relations

- (r1) $(r, —) \wedge (—, s) = 0$ whenever $r \geq s$,
- ~~(r2) $(r, —) \vee (—, s) = 1$ whenever $r < s$,~~
- (r3) $(r, —) = \bigvee_{s>r}(s, —)$ and $(—, r) = \bigvee_{s<r}(—, s)$, for every $r \in \mathbb{Q}$,
- (r4) $\bigvee_{r\in\mathbb{Q}}(r, —) = 1 = \bigvee_{r\in\mathbb{Q}}(—, r)$.

The spectrum of $\mathfrak{L}(\mathbb{IR})$ is homeomorphic to the space $\mathbb{IR}$ of partial reals endowed with the Scott topology.

Combining the natural isomorphism $\mathbf{Top}(X, \Sigma L) \simeq \mathbf{Frm}(L, \mathcal{O}X)$ for $L = \mathfrak{L}(\mathbb{IR})$ with the homeomorphism $\Sigma\mathfrak{L}(\mathbb{IR}) \simeq \mathbb{IR}$ one obtains

$$\boxed{\mathrm{IC}(X) = \mathbf{Top}(X, \mathbb{IR}) \xrightarrow{\sim} \mathbf{Frm}(\mathfrak{L}(\mathbb{IR}), \mathcal{O}X)}$$

Regarding the frame homomorphisms $\mathfrak{L}(\mathbb{IR}) \to L$, for a general frame L, as the partial continuous real functions on L provides a natural extension of the classical notion. Hence we denote

$$\boxed{\mathrm{IC}(L) = \mathbf{Frm}(\mathfrak{L}(\mathbb{IR}), L)}$$

Dedekind completeness of $\mathrm{IC}(L)$

Let $\{f_i\}_{i\in I} \subset \mathrm{IC}(L)$ and $f \in \mathrm{IC}(L)$ be such that $f_i \leq f$ for all $i \in I$.
Does there exist $\bigvee_{i\in I} f_i$ in $\mathrm{IC}(L)$?

Here again, the natural candidate would be defined for each $r \in \mathbb{Q}$ by

$$h(r,—) = \bigvee_{i\in I} f_i(r,—) \quad \text{and} \quad h(—,r) = \bigvee_{s<r} \Big(\bigwedge_{i\in I} f_i(—,s) \Big).$$

Recall that

$$h \in \mathrm{IC}(L) \iff \begin{cases} \text{(r1) if } r \leq s, \text{ then } h(—,r) \wedge h(s,—) = 0, & V \\ \text{(r3) } f(r,—) = \bigvee_{s>r} f(s,—) \text{ and } f(—,r) = \bigvee_{s<r} f(—,s), & V \\ \text{(r4) } \bigvee_{r\in\mathbb{Q}} f(r,—) = 1 = \bigvee_{r\in\mathbb{Q}} f(—,r). & V \end{cases}$$

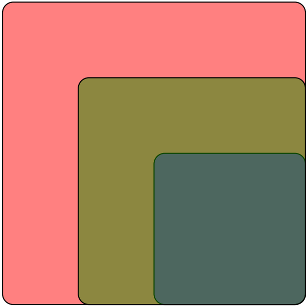
Hence $h \in \mathrm{IC}(L)$. Moreover, $h = \bigvee_{i\in I}^{\mathrm{IC}(L)} h_i$.

Theorem. $\mathrm{IC}(L)$ is Dedekind complete.

Dedekind completion of $\mathrm{C}(L)$

Recall that we can consider $\mathrm{C}(L)$ as a subset of $\mathrm{IC}(L)$.

$\mathrm{IC}(L)$



$$\mathrm{C}(L)^{\#} = \{h \in \mathrm{IC}(L) \mid \ ??? \ \}$$

$$\mathrm{C}(L) = \{h \in \mathrm{IC}(L) \mid h(-, r) \vee h(s, -) = 1 \text{ for each } s < r\}$$

Now, since $\mathrm{IC}(L)$ is Dedekind complete it follows that it contains the Dedekind completion of all its subsets, in particular $\mathrm{C}(L)$.

Dedekind completion of $\mathrm{C}(L)$ and $\overline{\mathrm{C}}(L)$

There is no essential loss of generality if we restrict ourselves to *completely regular* frames, so L will denote a completely regular frame in what follows.

Recall that if $f \in \mathrm{C}(L)$ then

$$\text{(r2)}\ f(\text{—}, r) \vee f(s, \text{—}) = 1 \quad \forall s < r \qquad \Longrightarrow \qquad \text{(r2)'} \begin{cases} f(s, \text{—})^* \leq f(\text{—}, r) \\ f(\text{—}, r)^* \leq f(s, \text{—}) \end{cases} \qquad \forall s < r$$

If L extremally disconnected then (r2) $\Longleftrightarrow$ (r2)'.

Theorem. Let L be a frame. Then the Dedekind completion $\mathrm{C}(L)^\#$ of $\mathrm{C}(L)$ is given by

$$\boxed{\begin{aligned} \mathrm{C}(L)^\# = \{h \in \mathrm{IC}(L) \mid\ & (1)\ \exists f, g \in \mathrm{C}(L) : f \leq h \leq g \\ & (2)\ h(s, \text{—})^* \leq h(\text{—}, r) \text{ and } h(\text{—}, r)^* \leq h(s, \text{—}) \text{ if } s < r\} \end{aligned}}$$

Corollary. $\mathrm{C}(L)$ is Dedekind complete if and only if L is extremally disconnected.

Dedekind completion of $\mathrm{C}^*(L)$, $\mathrm{C}(L,\mathbb{Z})$, ...

Let

$$\mathrm{C}^*(L) = \{h \in \mathrm{C}(L) \mid \text{ there exists } r \in \mathbb{Q} \text{ such that } h(-r,r) = 1\}$$

$$\mathrm{IC}^*(L) = \{h \in \mathrm{IC}(L) \mid \text{ there exists } r \in \mathbb{Q} \text{ such that } h(-r,r) = 1\}.$$

Corollary. Let L be a completely regular frame. Let L be a frame. Then the Dedekind completion $\mathrm{C}^*(L)^\#$ of $\mathrm{C}^*(L)$ is given by

$$\mathrm{C}^*(L)^\# = \mathrm{C}(L)^\# \cap \mathrm{IC}^*(L).$$

Corollary. $\mathrm{C}^*(L)$ is Dedekind complete if and only if L is extremally disconnected.

The integer-valued case follows similarly:

An $h \in \mathrm{IC}(L)$ is said to be integer-valued if $f(r,s) = f(\lfloor r \rfloor, \lceil s \rceil)$ for all $r, s \in \mathbb{Q}$, (where $\lfloor r \rfloor$ denotes the biggest integer $\leq r$ and $\lceil s \rceil$ the smallest integer $\geq s$).

Let

$$\mathfrak{Z}L \simeq \mathrm{C}(L,\mathbb{Z}) = \mathrm{C}(L) \cap \{h \in \mathrm{IC}(L) \mid h \text{ is integer-valued}\}.$$

Corollary. For any zero-dimensional frame L, $\mathrm{C}(L,\mathbb{Z})^\# = \mathrm{C}(L)^\# \cap \mathrm{IC}(L,\mathbb{Z})$ is the Dedekind completion of $\mathrm{C}(L,\mathbb{Z})$.

Corollary. For any zero-dimensional frame L, $\mathrm{C}(L,\mathbb{Z})$ is Dedekind complete if and only if L is extremally disconnected.

Summary

Generators: $(r,—),(—,s)$, $\quad r,s\in\mathbb{Q}$
Relations:
(r1) $(r,—)\wedge(—,s)=0$ whenever $r\geq s$,
(r2) $(r,—)\vee(—,s)=1$ whenever $r<s$,
(r3) $(r,—)=\bigvee_{s>r}(s,—)$ and $(—,s)=\bigvee_{r<s}(—,r)$,
~~(r4)~~ $\cancel{\bigvee_{r\in\mathbb{Q}}(r,—)=1=\bigvee_{s\in\mathbb{Q}}(—,s)}$.

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The frame of extended reals $\mathfrak{L}(\overline{\mathbb{R}})$.

Extended continuous real functions:

$$\boxed{\overline{\mathrm{C}}(L)=\mathbf{Frm}(\mathfrak{L}(\overline{\mathbb{R}}),L)}$$

The frame of partial reals $\mathfrak{L}(\mathbb{IR})$.

Partial continuous real functions:

$$\boxed{\mathrm{IC}(L)=\mathbf{Frm}(\mathfrak{L}(\mathbb{IR}),L)}$$