# Sequential and countability properties in frames

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Consider the interplay of sequences and certain countability, closure and compactness properties in topology.

- Convergence (sequences, filters) characterising notions of closure and closedness.
- X is countably compact  $\Leftrightarrow$  every sequence in X clusters.
- X sequentially compact  $\Rightarrow$  X is countably compact.
- A sequentially closed subspace of a sequentially compact space is sequentially compact.
- The product of a sequentially compact and a countably compact space is countably compact.

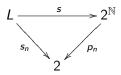
Naive, emboldened by recent results on pseudocompactness.

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A sequence in a space X is a continuous map  $f:\mathbb{N} o X.$  So, as first attempt...

## Definition

A sequence in a frame L is a homomorphism  $s: L \to \mathcal{P}(\mathbb{N})$ .



Thus s is a sequence of points. This will surely be inadequate, none the less the natural definitions and initial results build some intuition.

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A sequence s converges in L if for any cover C of L there exists

 $a \in C$  and  $n \in \mathbb{N}$  such that  $m \ge n \Rightarrow s_m(a) \neq 0$ .

(The filter base of tails of the sequence is convergent; s is eventually non-zero on a.)

## Proposition

If a sequence s converges in L then it has a "limit"  $t: L \to 2$  given by  $t(a) = 1 \Leftrightarrow \exists n \in \mathbb{N} \ \forall m \ge n, \ s_m(a) \ne 0.$ 

One can proceed with natural definitions of subsequence, clustering, sequential closure, sequential compactness and establish initial results relating these concepts. Inevitably, however, the notion is inadequate.

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A (generalised) sequence on a frame L is a collection of frame homomorphisms  $s_n : L \to T_n$  indexed by  $\mathbb{N}$ .

## Definition

- 1. A sequence  $(s_n)$  on L is *convergent* if for any cover C of L there exists  $a \in C$  and  $n \in \mathbb{N}$  such that  $m \ge n \Rightarrow s_m(a) \ne 0$ .
- 2. A sequence  $(s_n)$  on *L* clusters if for any cover *C* of *L* there exists  $a \in C$  such that for all  $n \in \mathbb{N}$  there exists  $m \ge n$  with  $s_m(a) \ne 0$ .

### Proposition

If a sequence  $(s_n)$  has a convergent subsequence then  $(s_n)$  clusters.

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- 1. A sublocale  $L \xrightarrow{h} M$  is sequentially closed if for any sequence  $(s_n)$  on M, if  $(s_nh)$  is convergent then so is  $(s_n)$ .
- 2. A frame *L* is *sequentially compact* if any sequence on *L* has a convergent subsequence.

### Proposition

- 1. If L is sequentially compact and  $h: K \to L$  injective, then K is sequentially compact.
- 2. If L is sequentially compact and  $L \xrightarrow{h} M$  a sequentially closed sublocale, then M is sequentially compact.

#### Lemma

If a sequence  $(s_n)$  on a frame L does not cluster, then there is a countable cover  $\{b_n\}$  of L with  $b_n \leq b_{n+1}$  for each  $n \in \mathbb{N}$  and  $s_m(b_n) = 0$  for any  $m \geq n$  in  $\mathbb{N}$ .

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### Theorem

A frame L is countably compact iff every sequence on L clusters.

### Corollary

L is sequentially compact  $\Rightarrow$  L countably compact.

- 1. A generalised filter on a frame L is a  $(0, \land, 1)$ -homomorphism  $\varphi: L \to T$ .
- 2. A generalised filter  $\varphi : L \to T$  is strongly convergent if there is a frame homomorphism  $h : L \to T$  with  $h \leq \varphi$ .
- 3. A sublocale  $L \xrightarrow{h} M$  is strongly convergence closed if for any generalised filter  $\varphi$  on M,  $\varphi h$  strongly convergent  $\Rightarrow \varphi$  strongly convergent.

### Proposition

A sublocale  $L \xrightarrow{h} M$  is closed iff it is strongly convergence closed.

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- 1. A sublocale  $L \xrightarrow{h} M$  is *extension closed* if for every cover C of M there is a cover D of L such that h[D] = C.
- 2. A sublocale  $L \xrightarrow{h} M$  is *nearly closed* if for every cover C of M there is a cover D of L such that for each  $d \in D$  there is a finite  $A \subseteq C$  with  $h(d) \leq \bigvee A$ .

### Remark

- 1.  $L \xrightarrow{h} M$  is extension closed iff for every cover C of M,  $h_*[C]$  covers L.
- 2.  $L \xrightarrow{h} M$  is nearly closed iff for every directed cover C of M,  $h_*[C]$  covers L.

- 1.  $L \xrightarrow{h} M$  is (countably) extension closed if for every (countable) cover C of M,  $h_*[C]$  covers L.
- 2.  $L \xrightarrow{h} M$  is (countably) nearly closed if for every (countable) directed cover C of M,  $h_*[C]$  covers L.
- 3. An up-set *F* in *L* is *A*-convergent if any A-cover of *L* meets *F*, where  $A \in \{\text{countable, directed}, \text{ countable directed}\}$ .

## Proposition

 $L \xrightarrow{h} M$  is (appropriate notion) closed iff for every up-set F on L,  $h^{-1}(F)$  (obvious)-convergent  $\Rightarrow F$  (obvious)-convergent.

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### Proposition

- 1. If L is countably compact and  $L \xrightarrow{h} M$  countably nearly closed, then M is countably compact.
- 2. If M is countably compact then any  $L \xrightarrow{h} M$  is countably nearly closed.
- 3. For a sublocale, the following closure properties relate:

