Sequential and countability properties in frames

David Holgate

University of the Western Cape
South Africa

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Consider the interplay of sequences and certain countability, closure and compactness properties in topology.

- Convergence (sequences, filters) characterising notions of closure and closedness.
- X is countably compact $\Leftrightarrow$ every sequence in X clusters.
- X sequentially compact $\Rightarrow$ X is countably compact.
- A sequentially closed subspace of a sequentially compact space is sequentially compact.
- The product of a sequentially compact and a countably compact space is countably compact.

Naive, emboldened by recent results on pseudocompactness.

A sequence in a space X is a continuous map $f : \mathbb{N} \to X$. So, as first attempt...

Definition

A sequence in a frame L is a homomorphism $s : L \to \mathcal{P}(\mathbb{N})$.

$$\begin{tikzcd} L \arrow[rr, "s"] \arrow[dr, "s_n"'] & & 2^{\mathbb{N}} \arrow[dl, "p_n"] \\ & 2 & \end{tikzcd}$$

Thus s is a sequence of points. This will surely be inadequate, none the less the natural definitions and initial results build some intuition.

Definition

A sequence s converges in L if for any cover C of L there exists $a \in C$ and $n \in \mathbb{N}$ such that $m \geq n \Rightarrow s_m(a) \neq 0$.

(The filter base of tails of the sequence is convergent; s is eventually non-zero on a.)

Proposition

If a sequence s converges in L then it has a "limit" $t : L \to 2$ given by $t(a) = 1 \Leftrightarrow \exists n \in \mathbb{N}\ \forall m \geq n,\ s_m(a) \neq 0$.

One can proceed with natural definitions of subsequence, clustering, sequential closure, sequential compactness and establish initial results relating these concepts. Inevitably, however, the notion is inadequate.

Definition

A (generalised) *sequence* on a frame L is a collection of frame homomorphisms $s_n : L \to T_n$ indexed by $\mathbb{N}$.

Definition

1. A sequence (s_n) on L is *convergent* if for any cover C of L there exists $a \in C$ and $n \in \mathbb{N}$ such that $m \geq n \Rightarrow s_m(a) \neq 0$.
2. A sequence (s_n) on L *clusters* if for any cover C of L there exists $a \in C$ such that for all $n \in \mathbb{N}$ there exists $m \geq n$ with $s_m(a) \neq 0$.

Proposition

If a sequence (s_n) has a convergent subsequence then (s_n) clusters.

Definition

1. A sublocale $L \stackrel{h}{\twoheadrightarrow} M$ is *sequentially closed* if for any sequence (s_n) on M, if $(s_n h)$ is convergent then so is (s_n).
2. A frame L is *sequentially compact* if any sequence on L has a convergent subsequence.

Proposition

1. *If L is sequentially compact and $h : K \to L$ injective, then K is sequentially compact.*
2. *If L is sequentially compact and $L \stackrel{h}{\twoheadrightarrow} M$ a sequentially closed sublocale, then M is sequentially compact.*

Lemma

If a sequence (s_n) on a frame L does not cluster, then there is a countable cover $\{b_n\}$ of L with $b_n \leq b_{n+1}$ for each $n \in \mathbb{N}$ and $s_m(b_n) = 0$ for any $m \geq n$ in $\mathbb{N}$.

Theorem

A frame L is countably compact iff every sequence on L clusters.

Corollary

L is sequentially compact $\Rightarrow$ L countably compact.

Definition

1. A generalised filter on a frame L is a $(0, \wedge, 1)$-homomorphism $\varphi : L \to T$.
2. A generalised filter $\varphi : L \to T$ is strongly convergent if there is a frame homomorphism $h : L \to T$ with $h \leq \varphi$.
3. A sublocale $L \overset{h}{\twoheadrightarrow} M$ is strongly convergence closed if for any generalised filter φ on M, φh strongly convergent $\Rightarrow \varphi$ strongly convergent.

Proposition

A sublocale $L \overset{h}{\twoheadrightarrow} M$ is closed iff it is strongly convergence closed.

Definition

1. A sublocale $L \overset{h}{\twoheadrightarrow} M$ is *extension closed* if for every cover C of M there is a cover D of L such that $h[D] = C$.
2. A sublocale $L \overset{h}{\twoheadrightarrow} M$ is *nearly closed* if for every cover C of M there is a cover D of L such that for each $d \in D$ there is a finite $A \subseteq C$ with $h(d) \leq \bigvee A$.

Remark

1. $L \overset{h}{\twoheadrightarrow} M$ is extension closed iff for every cover C of M, $h_*[C]$ covers L.
2. $L \overset{h}{\twoheadrightarrow} M$ is nearly closed iff for every directed cover C of M, $h_*[C]$ covers L.

Definition

1. $L \overset{h}{\twoheadrightarrow} M$ is *(countably) extension closed* if for every (countable) cover C of M, $h_*[C]$ covers L.
2. $L \overset{h}{\twoheadrightarrow} M$ is *(countably) nearly closed* if for every (countable) directed cover C of M, $h_*[C]$ covers L.
3. An up-set F in L is *A-convergent* if any A-cover of L meets F, where A $\in$ {countable, directed, countable directed}.

Proposition

$L \overset{h}{\twoheadrightarrow} M$ is $\langle$appropriate notion$\rangle$ *closed iff for every up-set F on L, $h^{-1}(F)$* $\langle$obvious$\rangle$*-convergent $\Rightarrow$ F* $\langle$obvious$\rangle$*-convergent.*

Proposition

1. *If L is countably compact and $L \overset{h}{\twoheadrightarrow} M$ countably nearly closed, then M is countably compact.*
2. *If M is countably compact then any $L \overset{h}{\twoheadrightarrow} M$ is countably nearly closed.*
3. *For a sublocale, the following closure properties relate:*

$$
\begin{tikzcd}
\textit{Closed} \arrow[r, Rightarrow] & \textit{Extension closed} \arrow[r, Rightarrow] \arrow[rr, Rightarrow, bend left] \arrow[d, Rightarrow] & \textit{Nearly closed} \arrow[d, Rightarrow] & \textit{Sequentially closed} \arrow[d, Rightarrow] \\
 & \textit{Countably extension closed} \arrow[r, Rightarrow] & \textit{Countably nearly closed} \arrow[r, Rightarrow] & \textit{Cluster closed}
\end{tikzcd}
$$