Forcing, Equivalence Relations and Marker Structures

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Basic objects of study are Borel equivalence relations $E$ on Polish spaces $X$. We frequently regard $X$ as a standard Borel space.

The notion of complexity is provided be the concept of \textit{reduction}.

\textbf{Definition}

- We say $E$ is \textbf{reducible} to $F$, $E \leq F$, if there is a Borel function $f : X \to Y$ such that $x E y \iff f(x) F f(y)$.
- We say $E$ is bi-reducible with $F$, $E \sim F$, if $E \leq F$ and $F \leq E$.
- We say $E$ is emdeddable into $F$, $E \sqsubseteq F$, if in addition $f$ is one-to-one.

Note that a reduction gives a definable injection from $X/E$ to $Y/F$ so reduction can be viewed as a notion of definable cardinality for these quotient spaces.
We say $E$ is a countable (Borel) equivalence relation if all classes of $E$ are countable.

If $G$ is a Polish group and $G$ acts on $X$, then the orbit equivalence relation $E_G$ is defined by

$$xE_G y \iff \exists g \in G \ (g \cdot x = y).$$

The Feldman-Moore theorem says that every countable Borel equivalence relation is given by the Borel action of a countable group $G$. The case $G = \mathbb{Z}$ is the classical case of discrete-time dynamics.

So, we can study the equivalence relations $E_G$ group by group.
The simplest equivalence relations are the smooth or tame ones.

**Definition**

$E$ is smooth if there is a Borel reduction of $E$ to equality relation on a Polish space.

So, for a smooth $E$, $X/E$ can be regarded as a subset of a standard Borel space.

For countable Borel $E$, smooth is the same as saying there is a Borel selector for $E$. 
Definition
$E_0$ is the equivalence relation on $2^\omega$ given by

$$xE_0 y \iff \exists n \forall m \geq n (x(m) = y(m)).$$

The Harrington-Kechris-Louveau theorem says that if $E$ is a Borel equivalence relation then either $E$ is smooth or $E_0 \sqsubseteq E$.

So, there is no complexity class of equivalence relation strictly between the smooth relation $E_\equiv$ and $E_0$. 
If $G$ is a Polish group, $G$ acts of $F(G)$ by the shift action

$$g \cdot F = \{gf : f \in F\}$$

We can view this action as being on $2^G$ by

$$g \cdot x(h) = x(g^{-1}h)$$

We call this the Bernoulli (left) shift action of $G$ on $2^G$. When $G$ is countable, $2^G$ is a compact Polish space in the natural product topology.
Countable Equivalence Relations

We let $E(2^G)$ denote the shift action of $G$ on $2^G$, and $F(2^G)$ denote the free part of $2^G$ with the shift action.

**Theorem (Dougherty-J-Kechris)**

The shift action of $F_2$ on $2^{F_2}$ is a universal countable Borel equivalence relation, that is, $E \leq E(2^{F_2})$ for any countable Borel $E$.

In general, the shift action is more or less universal for actions of $G$:

**Fact**

Let $E$ the the orbit equivalence relation for a Borel action of the countable group $G$ on a Polish space $X$. Then

$$E \leq E((2^\omega)^G) \leq E(2^G \times \mathbb{Z}).$$
Definition
A countable Borel equivalence relation $E$ is hyperfinite if $E$ is the increasing union of relations $E_n$ with finite classes.

Theorem (Slaman-Steel)
The following are equivalent:

- $E$ is hyperfinite.
- $E = E_G$ where $G = \mathbb{Z}$.
- The classes of $E$ can be uniformly Borel ordered in type $\mathbb{Z}$ (or are finite).
Markers

Definition

Let $E$ be a Borel equivalence relation. A marker set $M$ is a Borel set $M \subseteq X$ such that $M \cap [x] \neq \emptyset$, $M^c \cap [x] \neq \emptyset$ for every $x \in X$.

Usually we require some additional properties on $M$, related to the structure of $G$.

Many argument in dynamics/ergodic theory and descriptive dynamics use markers sets with certain properties (e.g., Rochlin’s lemma, Ornstein’s theorem, Slaman-Steel theorem).

Hyperfiniteness proofs also typically use marker arguments.
Theorem (Weiss)

Every Borel action by $\mathbb{Z}^n$ is hyperfinite.

Theorem (Gao-J)

Every Borel action by a countable abelian group is hyperfinite.

Weiss’ proof (and several other proofs of this result) use a basic marker lemma:

Lemma

For each $m$, there is a relatively clopen $M_m \subseteq F(2^{\mathbb{Z}^n})$ such that

1. $\forall x \neq y \in M_m \ [\rho(x, y) > m]$

2. $\forall x \in F(2^{\mathbb{Z}^n}) \ \exists y \in M_m \ [\rho(x, y) \leq m]$

For the abelian result, we need markers with more regularity.
By a set of **marker regions** we mean a Borel equivalence relation $\mathcal{R} \subseteq E$ with $\text{dom}(\mathcal{R})$ a complete section and all classes of $\mathcal{R}$ finite.

We say $\mathcal{R}$ is clopen if for each $g \in G$ the set $\{x \in X : x \mathcal{R} g \cdot x\}$ is relatively clopen in $\text{dom}(E)$.

We say the marker regions form a **tiling** if $\text{dom}(\mathcal{R}) = \text{dom}(E)$.

**Lemma**

For each $n$, there is a clopen set of markers $\mathcal{R}_n$ for $F(2^{\mathbb{Z}_m})$ which form a tiling and such that each $\mathcal{R}$ class is a rectangle with each side length in $\{n, n + 1\}$.

We call this a clopen, almost square tiling.
The following question arises in several problems.

**Question**
Can we get a (Borel or clopen) rectangular tiling of $F(2^\mathbb{Z}_m)$ which is “almost lined-up”?

![Diagram of a rectangular tiling](image)
Note that a (Borel or clopen) almost lined-up tiling would have the following consequences:

- There would be a (Borel or clopen) “lining” of $F(2^\mathbb{Z}\times\mathbb{Z})$.
- There would be a (Borel or continuous) proper action of $\mathbb{Z}\times\mathbb{Z}$ on each class of $F(2^\mathbb{Z}\times\mathbb{Z})$.

The existence of a lining seems to be related to the (Borel, continuous) chromatic number problem for $F(2^{\mathbb{Z}^m})$.

**Theorem (Kechris-Soleci-Todorcevic)**

$$3 \leq \chi_b(m) \leq m + 1.$$  

**Theorem (Gao-J)**

$$3 \leq \chi_b(m) \leq \chi_c(m) \leq 4.$$
Definition
A 2-coloring of a group $G$ is an $x: G \to \{0, 1\}$ satisfying the following: for every $s \neq 1_G$, there is a finite $T = T(s) \subseteq G$ such that:

$$\forall g \in G \exists t \in T \ (x(gt) \neq x(gst)).$$

The notion of a 2-coloring was formulated independently by Pestov, and Glassner-Uspensky independently showed many groups admit 2-colorings.

Fact
$x \in 2^G$ is a 2-coloring iff $[x] \subseteq F(2^G)$. 
**Definition**

$x \in 2^G$ is **minimal** if $[x]$ is a minimal closed invariant set (subflow), that is, $\forall y \in [x] ([y] = [x])$.

Being minimal has a combinatorial reformulation.

**Fact**

$x \in 2^G$ is minimal iff for every $A \in G^{< \omega}$ there is a $T \in G^{< \omega}$ such that

$$\forall g \in G \ \exists t \in T \ \forall a \in A \ (x(gta) = x(a)).$$

**Remark**

Minimal $x$ exist in any subflow of any $2^G$ (don’t need AC in fact).
Theorem (Gao-J-Seward)

Every countable group $G$ has a 2-coloring.

So, there is a compact invariant set $[x] \subseteq F(2^G)$.

An early consequence of this was the following. Recall (Slaman-Steel) that for any countable equivalence relation there are Borel complete sections $B_n$ such that $\bigcap_n B_n = \emptyset$.

Corollary

Let $B_n \subseteq F(2^G)$ be relatively clopen complete sections. Then $\bigcap_n B_n \neq \emptyset$. 
Theorem (GJS; minimal 2-coloring forcing)

For any countable group $\Gamma$ there is separative forcing notion $\mathbb{P}_{mc}$ on which $\Gamma$ acts by automorphisms and such that

$$\emptyset \vdash (x_G \text{ is a minimal 2-coloring of } \Gamma).$$

The forcing can be described directly, or an instance of orbit-forcing.

Definition

Let $x \in F(2^\Gamma)$. $\mathbb{P}_x$ is the forcing notion

$$\mathbb{P}_x = \{ p \in 2^{<\Gamma} : \exists g \in \Gamma \ (p = g \cdot x \upharpoonright \text{dom}(p)) \}$$
A generic $G$ for $\mathbb{P}_x$ produces an $x_G \in [x]$.

If $x$ is a minimal 2-coloring, then $x_G$ will also be a minimal 2-coloring.

- Varying $x$ can produce different forcing effects.
- The forcings can also be described directly by (usually) finitary $\hat{p} \in 2^{\prec G}$ with extra side-conditions.

To illustrate the give the direct definition of $\mathbb{P}_{mc}$ for the case $\Gamma = \mathbb{Z} \times \mathbb{Z}$. 
\( \mathbb{P}_{mc} \) consists of conditions

\[
p = (\hat{p}; s_0, \ldots, s_n; T_0, \ldots, T_n; A_0, \ldots, A_m; U_0, \ldots, U_m)
\]
satisfying the following:

1. \( \hat{p} \in 2^R \) where \( R = [a, b] \times [c, d] \subseteq \mathbb{Z} \times \mathbb{Z} \).
2. \( T_0, \ldots, T_n, U_0, \ldots, U_m \in 2^{<(\mathbb{Z} \times \mathbb{Z})} \).
3. \( A_i \in 2^{<(\mathbb{Z} \times \mathbb{Z})} \) and \( \exists h \) \( [\hat{p} \upharpoonright (h \cdot (\text{dom}(A_i))) = A_i] \).
4. \( \forall g \in \text{dom}(\hat{p}) \forall i \leq n \exists t \in T_i \ [gt, gst \in \text{dom}(\hat{p}) \land \hat{p}(gt) \neq \hat{p}(gst)] \)
5. \( \forall g \in \text{dom}(\hat{p}) \forall i \leq m \exists t \in U_i \ [\hat{p} \upharpoonright (gt \cdot (\text{dom}(A_i))) = A_i] \)
   and
   \( \forall g \in \text{dom}(\hat{p}) \forall i \leq m \exists t \in U_i \ [\hat{p} \upharpoonright (gt \cdot (\text{dom}(A_i))) = 1 - A_i] \)
We have the following facts about $\mathbb{P}_{mc}$.

**Lemma**

For any $g \in \mathbb{Z} \times \mathbb{Z}$, $D_g = \{p: g \in \text{dom}(\hat{p}) \text{ is dense}\}$.

**Lemma**

For each $s \neq (0, 0)$ in $\mathbb{Z} \times \mathbb{Z}$, $D_s = \{p: \exists i \ (s = s_i)\}$ is dense.

**Lemma**

$\forall p \in \mathbb{P}_{mc} \ \forall A \subseteq \hat{p} \ \exists i \leq m_q \ A \subseteq A_i(q)$ is dense below $p$. 

$D_{p,A} = \{q: \exists i \leq m_q \ A \subseteq A_i(q)\}$.
Let $G$ be a generic for $\mathbb{P}_{mc}$, and let $x_G = \bigcup\{\hat{p}: p \in G\}$. So, $x_G \in 2^{(\mathbb{Z} \times \mathbb{Z})}$.

The first lemma shows that $x_G = 2^{\mathbb{Z} \times \mathbb{Z}}$, the second lemma shows that $x_G$ is a 2-coloring, and the third lemma shows that $x_G$ is minimal.

For example, to show second lemma, copy the domain $R$ of $\hat{p}$ to a larger rectangular domain using copies of $\hat{p}$ and $1 - \hat{p}$ in such a way that we block the shift $s$. 

![Diagram showing a 2-coloring of a grid]

\[
\begin{array}{ccc}
\hat{p} & & \bullet \text{gs} \\
\cdot & g & \\
\cdot & & 1 - \hat{p}
\end{array}
\]
The following two theorems are proved using $\mathbb{P}_{mc}$.

**Theorem (GJS)**

Let $G$ be a countable group and $E_G$ the equivalence relation generated by the shift action of $G$ on $F(2^G)$. Let $B_n \subseteq X$ be Borel complete sections, and let $f : \omega \to \omega$ with $\limsup f = \infty$. There there an $x \in F(2^G)$ such that $\exists \infty n \rho(x, B_n) < f(n)$.

**Remark**
The Slaman-Steel markers are Borel complete sections $B_n \subseteq F(2^\mathbb{Z})$ with $\bigcap_n B_n = \emptyset$.

**Remark**
There does exists a sequence $B_n \subseteq F(2^{\mathbb{Z}^n})$ of relatively clopen complete sections such that for all $x \in F(2^{\mathbb{Z}^n})$ we have $\rho(x, B_n) \to \infty$. 
Theorem (GJS)
Let $G$ be a countable group and $E_G$ the equivalence relation generated by the shift action of $G$ on $F(2^G)$. Let $f : (F(2^G), E_G) \rightarrow (Y, F)$ be a Borel invariant map (i.e., $F$ is a factor of $E_G$). Then $F$ has a recurrent point.

By a recurrent point $y \in Y$ we mean that for every non-empty open set $U \subseteq Y$ there is a $A \in G^{<\omega}$ such that
\[ \forall z \in [y] \exists g \in A \ g \cdot y \in U. \]

In fact, for any non-empty Borel set $B \subseteq Y$, there is a $y \in Y$ which is recurrent for $B$. 
We specialize to the groups $G = \mathbb{Z}^n$.

Some of these results are related to the coloring problem for $\mathbb{Z}^n$.

**Question (Kechris-Solecki-Todorcevic)**

What the Borel/clopen chromatic number of $F(2^{\mathbb{Z}^n})$?

It is known (Gao-Jackson) that

$$3 \leq \chi_b(F(2^{\mathbb{Z}^n})) \leq \chi_c(F(2^{\mathbb{Z}^n})) \leq 4$$
Theorem

There does not exist a Borel coloring $c : F(2\mathbb{Z}^n) \to k$ such that for every $x \in F(2\mathbb{Z}^n)$ there are arbitrarily large regions in $[x]$ which are 2-colored by $c$.

To prove this we need a variation of the minimal 2-coloring forcing which we call the odd minimal 2-coloring forcing.

Conditions in this forcing $\mathbb{P}_o$ are just like those of $\mathbb{P}$ (the minimal 2-coloring forcing) except we require that the domain of $\hat{p}$ have odd side lengths.

Previous density lemmas go through just as before.
Suppose $c : F(2^{\mathbb{Z}^n}) \to k$ is Borel. Let $x = x_G$ where $G$ is generic for $\mathbb{P}_\circ$.

Suppose $p = (\hat{p}; \cdots) \in G$ and $p \models c(x_G) = 0$, say.

Let $q \leq p$, $q \in G$, be such that $\hat{p} \subseteq A_i$ for some $A_i \in \tilde{A}(q)$.

Let $r \leq q$, $r \in G$ be such that there are copies of $\hat{q}$ an odd distance apart in $\hat{r}$ (such sets are dense).

Let $g \in \mathbb{Z}^n$ be such that $g \cdot x \mid R$ is 2-colored by $c$, where $R$ is sufficiently large (say twice the size of $R$).

For some $h \in \mathbb{Z}^n$ we have $hg \cdot x \mid \text{dom}(r) = \hat{r}$ and $hg(\text{dom}(r)) \subseteq R$. This is a contradiction as $gh \cdot x$ is still generic.
A Ramsey-type result

Theorem

Let \( B \subseteq F(2^\mathbb{Z}^n) \) be Borel. Then there is an \( x \in F(2^\mathbb{Z}^n) \) and a rectangular lattice \( L \subseteq [x] \) such that either \( L \subseteq B \) or \( L \subseteq B^c \). If \( B \) is a complete section, then we have \( L \subseteq B \).

We use another variation of the minimal 2-coloring forcing. We use a forcing which builds a minimal 2-coloring but all conditions have a periodicity requirement.

Conditions of the form

\[ p = (R, \Delta, \{a, b\}, c, \Lambda) \]
- $R \subseteq \mathbb{Z} \times \mathbb{Z}$ is a rectangle.
- $\Delta$ is a translate of a rectangular lattice $L$ and $\mathbb{Z}^2$ is the disjoint union of $\delta R$ for $\delta \in \Delta$.
- $\{a, b\} \subseteq R$
- $c: (\bigcup_{\delta \in \Delta} \delta(R - \{a, b\})) \rightarrow \{0, 1\}$.
- $\Lambda \subseteq L$ is a rectangular lattice and $c$ has period $\Lambda$.
- (local recognizability) If $x \in \Delta$, $y \notin \Delta$, then there is a $g \in R$ such that $c(gx) \neq c(gy)$ and both are defined.

**Remark**
The local recognizability condition is not necessary as it will hold generically.
Figure: a condition in the forcing
Figure: the extension relation
Using variations of minimal 2-colorings we have the following.

**Theorem**
There is no continuous “lining” of $F(2^\mathbb{Z} \times \mathbb{Z})$.

**Corollary**
This is no clopen, almost lined up rectangular marker regions for $F(2^\mathbb{Z} \times \mathbb{Z})$.

Extending (and simplifying) these arguments Ed Krohne has shown:

**Theorem**
There is no continuous 3-coloring of $F(2^\mathbb{Z} \times \mathbb{Z})$. 
So we have:

\[ \chi_c(F(2^{\mathbb{Z}^n})) = \begin{cases} 
3 & \text{if } n = 1 \\
4 & \text{if } n \geq 2
\end{cases} \]

For \( n \geq 2 \) we still don’t know \( \chi_b(F(2^{\mathbb{Z}^n})) \).