Graph Theory and Modal Logic

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Aug. 5, 2013

BLAST 2013 at Chapman University
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1. Graphs = Kripke frames
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3. The hybrid logic $G$ for all graphs
4. Hybrid formulas characterizing some properties of graphs
1. Graphs = Kripke frames.
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2. Completeness for the basic hybrid logic $H$. 

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3. The hybrid logic $G$ for all graphs.
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3. The hybrid logic $G$ for all graphs.

4. Hybrid formulas characterizing some properties of graphs.
Why symmetric frames?

= My research history =
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Quantum Logic = a logic of quantum mechanics
Why symmetric frames?

= My research history =

Quantum Logic = a logic of quantum mechanics

\[\downarrow\]

Orthologic / orthomodular logic
Why symmetric frames?

= My research history =

Quantum Logic = a logic of quantum mechanics

↓

Orthologic /orthomodular logic

↓

Modal logic KTB and its extension

⋅⋅⋅ complete for reflexive and symmetric frames.
Undirected Graphs = Symmetric Kripke frames
Undirected Graphs = Symmetric Kripke frames

Every point (node) in an undirected graph must be treated as an **irreflexive point**!
To characterize irreflexivity

Proposition

There is NO formula in propositional modal logic that characterizes the class of irreflexive frames. We have to enrich our language. Employ a kind of hybrid language (nominals)
To characterize irreflexivity

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\[ \rightarrow \text{We have to enrich our language.} \]
Proposition

There is NO formula in propositional modal logic that characterizes the class of irreflexive frames.

\[\rightarrow\] We have to enrich our language.

Employ a kind of hybrid language (NOMINALS)
A Hybrid Language

› 2 sorts of variables:

• $\Phi := \{p, q, r, \ldots\}$ · · · the set of prop. variables

• $\Omega := \{i, j, k, \ldots\}$ · · · the set of nominals

where $\Phi \cap \Omega = \emptyset$.

Nominals are used to distinguish points (states) in a frame from one another.

› Our language $\mathcal{L}$ (the set of formulas) consists of

$A ::= p \mid i \mid \bot \mid \neg A \mid A \land B \mid \Box A$

… No satisfaction operator ($@i$)
A normal hybrid logic $\mathbf{L}$ over $\mathcal{L}$ is a set of formulas in $\mathcal{L}$ that contains:

1. All classical tautologies
2. $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$
3. $(i \land p) \rightarrow \Box^n (i \rightarrow p)$ for all $n \in \omega$: (nominality axiom)

and closed under the following rules:

4. Modus Ponens
   
   $$
   \begin{array}{c}
   A, A \rightarrow B \\
   \hline
   B
   \end{array}
   $$

5. Necessitation
   
   $$
   \begin{array}{c}
   A \\
   \hline
   \Box A
   \end{array}
   $$
Normal hybrid logic(2)

(6) Sorted substitution

\[
\frac{A}{A[B/p]} , \frac{A}{A[j/i]}
\]

\(p: \) prop. variable, \(i, j: \) nominals

(7) Naming

\[
\frac{i \rightarrow A}{A}
\]

\(i: \) not occurring in \(A\)

(8) Pasting

\[
\frac{(i \land \Diamond (j \land A)) \rightarrow B}{(i \land \Diamond A) \rightarrow B}
\]

\(j \neq i, \) \(j: \) not occurring in \(A\) or \(B.\)
**Normal hybrid logic**

\( H \): the smallest normal hybrid logic over \( \mathcal{L} \)

For \( \Gamma \subseteq \mathcal{L} \),

\( H \oplus \Gamma \): the smallest normal hybrid extension containing \( \Gamma \)
Semantics

\( \mathcal{F} := \langle W, R \rangle \): a (Kripke) frame

\( \mathcal{M} := \langle \mathcal{F}, V \rangle \): a model,

where, \( V : \Phi \cup \Omega \rightarrow 2^W \) such that:

For \( p \in \Phi \), \( V(p) \): a subset of \( W \),
for \( i \in \Omega \), \( V(i) \): a \textit{singleton} of \( W \).

Interpretation of a nominal:

\( (\mathcal{M}, a) \models i \) if and only if \( V(i) = \{a\} \)

In this sense, \( i \) is a \textit{name} for the point \( a \) in this model \( \mathcal{M} \)!
Soundness for $H$

For a frame $\mathcal{F}$,

$$\mathcal{F} \models A \iff \text{def} \Rightarrow \forall V \text{ on } \mathcal{F}, \forall a \in W, (\langle \mathcal{F}, V \rangle, a) \models A$$
For a frame $\mathcal{F}$,

$$\mathcal{F} \models A \iff \text{def} \Rightarrow \forall V \text{ on } \mathcal{F}, \forall a \in W, ((\langle \mathcal{F}, V \rangle, a) \models A)$$

**Theorem (Soundness for the logic $\textbf{H}$)**

*For $A \in \mathcal{L}$, $A \in \textbf{H}$ implies $\mathcal{F} \models A$ for any frame $\mathcal{F}$.***
Completeness for $H$

For $\Gamma \subseteq L$, $A \in L$, $H : \Gamma \vdash A$

$\iff$ def $\Rightarrow \exists B_1, B_2, \ldots, B_n \in \Gamma (H \vdash (B_1 \land B_2 \land \cdots \land B_n) \rightarrow A)$
For $\Gamma \subseteq \mathcal{L}$, $A \in \mathcal{L}$,

$$\mathbf{H} : \Gamma \vdash A$$

$\iff$ def $\Rightarrow \exists B_1, B_2, \ldots, B_n \in \Gamma (\mathbf{H} \vdash (B_1 \land B_2 \land \cdots \land B_n) \rightarrow A)$

**Theorem (Strong completeness for the logic $\mathbf{H}$)**

For $\Gamma \subseteq \mathcal{L}$, $A \in \mathcal{L}$, suppose that $\mathbf{H} : \Gamma \nvdash A$.

Then there exists a model $\mathfrak{M}$ and a point $a$ such that:

1. $(\mathfrak{M}, a) \models B$ for all $B \in \Gamma$,
2. $(\mathfrak{M}, a) \nvdash A$
Theorem

(1) $H$ admits filtration, and so, it has the finite model property.

(2) $H$ is decidable.
Axiom for Irreflexivity

Proposition

For any frame $\mathcal{F} = \langle W, R \rangle$, $\mathcal{F} \models i \rightarrow \lozenge \neg i$ if and only if $\mathcal{F} \models \forall x \in W (\text{Not}(xRx))$. 
### Axiom for Irreflexivity

**Proposition**

For any frame $\mathcal{F} = \langle W, R \rangle$, $\mathcal{F} \models i \rightarrow \Box \neg i$ if and only if $\mathcal{F} \models \forall x \in W (\text{Not}(xRx))$.

**Proof.**

$(\Rightarrow:) \text{ Suppose that there is a point } a \in W \text{ s.t. } aRa. \text{ Define a valuation } V \text{ as: } V(i) := \{a\}. \text{ Then } a \not\models i \rightarrow \Box \neg i$

$(\Leftarrow:) \text{ Suppose } \mathcal{F} \not\models i \rightarrow \Box \neg i. \text{ Then, there exists } a \in W, \text{ s.t. } a \models i, \text{ but } a \not\models \Box \neg i, \text{ which is equivalent to } a \models \Diamond i. \text{ The latter means that there is } b \in W \text{ s.t. } aRb \text{ and } b \models i. \text{ Then, } V(i) = \{a\} = \{b\}. \text{ Thus } a = b \text{ and that } aRa \square$
The logic $G$ for undirected graphs

$$G := H \oplus (p \rightarrow \Box \Diamond p) \oplus (i \rightarrow \Box \neg i)$$
The logic $G$ for undirected graphs

$G := H \oplus (p \to \Box \Diamond p) \oplus (i \to \Box \neg i)

Lemma

(1) For any frame $\mathcal{F}$, $\mathcal{F} \models (p \to \Box \Diamond p) \land (i \to \Box \neg i)$ if and only if $\mathcal{F}$ is an undirected graph.

(2) The canonical frame for $G$ is also irreflexive and symmetric.
The logic $G$ for undirected graphs

$$G := H \oplus (p \rightarrow \Box \Diamond p) \oplus (i \rightarrow \Box \neg i)$$

**Lemma**

1. For any frame $\mathcal{F}$, $\mathcal{F} \models (p \rightarrow \Box \Diamond p) \land (i \rightarrow \Box \neg i)$ if and only if $\mathcal{F}$ is an undirected graph.
2. The canonical frame for $G$ is also irreflexive and symmetric.

**Theorem**

The logic $G$ is strong complete for the class of all undirected graphs.
The logic $G$ for undirected graphs

$$G := H \oplus (p \rightarrow \square \diamond p) \oplus (i \rightarrow \square \neg i)$$

**Lemma**

1. For any frame $\mathcal{F}$, $\mathcal{F} \models (p \rightarrow \square \diamond p) \land (i \rightarrow \square \neg i)$ if and only if $\mathcal{F}$ is an undirected graph.
2. The canonical frame for $G$ is also irreflexive and symmetric.

**Theorem**

The logic $G$ is strong complete for the class of all undirected graphs.

**Question:** Does $G$ admit filtration?
Formulas charactering some graph properties

\( \mathcal{F} \): a graph (irreflexive and symmetric frame)
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(1) Degree of a graph
Every point in \( \mathcal{F} \) has at most \( n \) points that connects to it iff \( \mathcal{F} \models \text{Alt}_n \)

\[
\text{Alt}_n := \Box p_1 \lor \Box(p_1 \rightarrow p_2) \lor \cdots \lor \Box(p_1 \land \cdots \land p_n \rightarrow p_{n+1})
\]
$\mathcal{F}$: a graph (irreflexive and symmetric frame)

(1) Degree of a graph
Every point in $\mathcal{F}$ has at most $n$ points that connects to it iff $\mathcal{F} \models \text{Alt}_n$

\[
\text{Alt}_n := \Box p_1 \lor \Box(p_1 \rightarrow p_2) \lor \cdots \lor \Box(p_1 \land \cdots \land p_n \rightarrow p_{n+1})
\]

(2) Diameter of a graph
The diameter of $\mathcal{F}$ is less than $n$ iff $\mathcal{F} \models \neg \varphi_n$.

\[
\begin{cases}
\varphi_1 := p_1. \\
\varphi_{n+1} := p_{n+1} \land \neg p_n \land \cdots \land \neg p_1 \land \Diamond \neg \varphi_n.
\end{cases}
\]
(3) Hamilton cycles
\( \mathcal{F} \): a graph that has \( n \) points.
\( \mathcal{F} \) has a Hamilton cycle if and only if \( \mathcal{F} \) sat \( \psi_n \), so
\( \mathcal{F} \) does NOT have a Hamilton cycle if and only if \( \mathcal{F} \models \neg \psi_n \).

\[
\psi_n := \sigma_1 \land \Diamond (\sigma_2 \land \Diamond (\cdots \Diamond (\sigma_n \land \Diamond \sigma_1) \cdots)), \quad \text{where} \\
\sigma_k := \neg i_1 \land \neg i_2 \land \cdots \land i_k \land \cdots \land \neg i_n
\]
(3) Hamilton cycles
\[ \mathcal{F}: \text{a graph that has } n \text{ points.} \]
\[ \mathcal{F} \text{ has a Hamilton cycle iff } \mathcal{F} \text{ sat } \psi_n, \text{ so} \]
\[ \mathcal{F} \text{ does NOT have a Hamilton cycle iff } \mathcal{F} \models \neg \psi_n. \]
\[ \psi_n := \sigma_1 \land \Diamond (\sigma_2 \land \Diamond (\cdots \Diamond (\sigma_n \land \Diamond \sigma_1) \cdots)), \text{ where} \]
\[ \sigma_k := \neg i_1 \land \neg i_2 \land \cdots \land i_k \land \cdots \land \neg i_n \]

(Q:) How to characterize having Euler cycles?
(4) Coloring

$\mathcal{F}$: a graph whose diameter is at most $n$.

$\mathcal{F}$ is $k$-colorable iff $\mathcal{F} \text{ sat } \text{color}(k)$, so
$\mathcal{F}$ is NOT $k$-colorable iff $\mathcal{F} \models \neg \text{color}(k)$

\[
\text{color}(k) := \Box^{(n)} \left( \bigvee_{\ell=1}^{k} c_{\ell} \land \bigwedge_{\ell=1}^{k} (c_{\ell} \rightarrow \Box \neg c_{\ell}) \right),
\]

each $c_{\ell}$ is a prop. variable representing a color.
(4) Coloring
\( \mathcal{F} \): a graph whose diameter is at most \( n \).

\( \mathcal{F} \) is \( k \)-colorable iff \( \mathcal{F} \) \textbf{sat} \( \text{color}(k) \), so
\( \mathcal{F} \) is NOT \( k \)-colorable iff \( \mathcal{F} \models \neg \text{color}(k) \)

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\text{color}(k) := \Box^{(n)} \left( \bigvee_{\ell=1}^k c_\ell \land \bigwedge_{\ell=1}^k (c_\ell \to \Box \neg c_\ell) \right),
\]
each \( c_\ell \) is a prop. variable representing a color.

(Q:) How to characterize being planar?
Future Study

(1) What kind of graph properties are definable over the logic $G$?
Future Study

(1) What kind of graph properties are definable over the logic $G$?

(2) Can we prove theorems from graph theory by constructing a formal proof?