Ideal extension of semigroups and their applications

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Abstract.

Let $S$ and $T$ be disjoint semigroups, $S$ having an identity $1_S$ and $T$ having a zero element $0$. A semigroup $\Omega$ is called an ideal extension of $S$ by $T$ if it contains $S$ as an ideal and if the Rees factor semigroup $\Omega/S$ is isomorphic to $T$, i.e. $\Omega/S \cong T$.

Ideal extension for topological semigroup as subdirect product of $S \times T$ was studied by Christoph in 1970. In this talk we introduce ideal extension for topological semigroups using a new method, then we investigate the compactification spaces of these structures. As a consequence, we use this result to characterize compactification spaces for Brandt $\lambda$-extension of topological semigroups.
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Structure of ideal extension of semigroups for discrete case.

A mapping $A \mapsto \overline{A}$ of $T^* = T - \{0\}$ into $S$ is called partial homomorphism if $A B = A B$, whenever $A B \neq 0$.

It is known that a partial homomorphism $A \mapsto A$ of the semigroup $T^*$ into $S$ determines an extension $\Omega$ of $S$ by $T$ as follows:

For $A, B \in T$ and $s, t \in S$,

1. $(P_1)$ $A o B = \begin{cases} AB & \text{if } AB \neq 0 \\ A B & \text{if } AB = 0 \end{cases}$
2. $(P_2)$ $A o s = A s$
3. $(P_3)$ $s o A = s A$
4. $(P_4)$ $s o t = st$

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and every extension can be so constructed.
The following theorem provides a general solution for the existence of topological extension of topological semigroups.

Theorem

Let $S$ and $T$ be disjoint topological semigroups such that $T$ has a zero. Let \( \theta : T^* = T - \{0\} \to S \) be continuous partial homomorphism. Then $\Omega = S \cup T^*$ with multiplication \((P_1, P_2, P_3, P_4)\) is a topological extension of $S$ by $T$. Conversely, every topological extension of topological semigroup $S$ by topological semigroup $T$ can be so constructed.
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Proof.

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- Clearly, $\Omega$ is an extension of $S$ by $T$. 
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  $$\mathcal{U} = \{ v \subseteq \Omega \mid v \cap T \text{ and } v \cap S \text{ is open in } T \text{ and } S \text{ respectively} \}$$
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  \[ \tau = \{(u, su') \mid s \in S, u, u' \in \Omega\} \]
- $\rho_\Omega = \{(x, y) \in \Omega \times \Omega \mid (uxv, uyv) \in \tau, \text{ for all } u, v \in \Omega\}$. $\rho_\Omega$ is the largest congruence on $\Omega \times \Omega$ contained in $\tau$, and $\frac{\Omega}{\rho_\Omega} \simeq \frac{\Omega}{S} \simeq T$. 

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Let $(\psi, \mathcal{X})$ be a topological semigroup compactification of $\Omega$ and $\tau_\mathcal{X}$ be the equivalence relation generated by $\{(x, \psi(s)y) \mid x, y \in \mathcal{X}, s \in S\}$ and $\rho_\mathcal{X}$ be the closure of the largest congruence on $\mathcal{X} \times \mathcal{X}$ contained in $\tau_\mathcal{X}$.
Theorem

Let $S$ and $T$ be disjoint topological semigroups such that $T$ has a zero and $\Omega$ be a topological extension of $S$ by $T$. Let $(\psi, X)$ be a topological semigroup compactification of $\Omega$. Then $\frac{X}{\rho_X}$ is a topological semigroup compactification of $\frac{\Omega}{S} \simeq T$. 
**Theorem**

Let $S$ and $T$ be disjoint topological semigroups such that $T$ has a zero and $\Omega$ be a topological extension of $S$ by $T$. Let $(\varepsilon_T, T^\mathcal{P})$ and $(\varepsilon_\Omega, \Omega^\mathcal{P})$ be the universal $\mathcal{P}$-compactifications of $T$ and $\Omega$ respectively. Then $T^\mathcal{P} \simeq \frac{\Omega^\mathcal{P}}{\rho_{\Omega^\mathcal{P}}}$ if

i) $\mathcal{P}$ is invariant under homomorphism,

ii) universal $\mathcal{P}$-compactification is a topological semigroup.
Corollary

Let $\Omega$ be a topological extension of topological semigroup $S$ by topological semigroup $T$. Let $(\varepsilon_s, S_{sap}^s)$, $(\varepsilon_\Omega, \Omega_{sap}^s)$ [resp. $(\varepsilon_s, S_{ap}^s)$, $(\varepsilon_\Omega, \Omega_{ap}^s)$] be the strongly almost periodic compactifications [resp. almost periodic compactifications] of $S$ and $\Omega$, respectively. Then $T_{sap}^s \sim \frac{\Omega_{sap}^s}{\rho_{\Omega_{sap}^s}}$ [resp. $T_{ap}^s \sim \frac{\Omega_{ap}^s}{\rho_{\Omega_{ap}^s}}$].
Question. If \( X_S \) and \( X_T \) are topological semigroup compactifications of \( S \) and \( T \) respectively, whether topological extension of \( X_S \) and \( X_T \) exist and is semigroup compactification of extension of \( S \) by \( T \)?

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Let \( (\psi_S, X_S) \) and \( (\psi_T, X_T) \) be topological semigroup compactifications of \( S \) and \( T \) respectively such that \( X_S \cap X_T = \emptyset \). Then the following assertions hold.

a) Topological extension \( X_\Omega \) of \( X_S \) by \( X_T \) exist.

b) Topological center \( \Lambda(\Omega) \) is a topological extension of \( \Lambda(S) \) by \( \Lambda(T) \).

c) \( (\psi_\Omega, X_\Omega) \) is a topological semigroup compactification of \( \Omega \) where \( \psi_\Omega|_T = \psi_T \), \( \psi_\Omega|_S = \psi_S \).
Question. If $X_S$ and $X_T$ are topological semigroup compactifications of $S$ and $T$ respectively, whether topological extension of $X_S$ and $X_T$ exist and is semigroup compactification of extension of $S$ by $T$?
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**Theorem**

Let $S$ and $T$ be disjoint topological semigroups such that $T$ has a zero and $\Omega$ be a topological extension of $S$ by $T$. Let $(\psi_S, X_S)$ and $(\psi_T, X_T)$ be topological semigroup compactifications of $S$ and $T$ respectively such that $X_S \cap X_T = \emptyset$. Then the following assertions hold.

- **a)** Topological extension $X_\Omega$ of $X_S$ by $X_T$ exist.
- **b)** Topological center $\Lambda(\Omega)$ is a topological extension of $\Lambda(S)$ by $\Lambda(T)$.
- **c)** $(\psi_\Omega, X_\Omega)$ is a topological semigroup compactification of $\Omega$ where $\psi_\Omega|_T = \psi_T$, $\psi_\Omega|_S = \psi_S$. 

Following theorem shows that topological semigroup compactifications of \( S \) and \( T \) can be constructed by topological semigroup compactifications of their topological extension.
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**Theorem**

*Let $S$ and $T$ be disjoint topological semigroups such that $T$ has a zero and $\Omega$ be a topological extension of $S$ by $T$. Suppose $(\psi_{\Omega}, X_{\Omega})$ is a topological semigroup compactification of $\Omega$. Then there are topological semigroups compactifications $(\psi_{S}, X_{S})$, $(\psi_{T}, X_{T})$ of $S$ and $T$ respectively such that $X_{\Omega}$ is a topological extension of $X_{S}$ by $X_{T}$.***
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- In following we use topological extension technique to characterizing compactification spaces of Brandt $\lambda$-extension.

- Let $G^0 = G \cup \{0\}$ [resp. $G$] be a group with zero [resp. group], $E$ and $F$ be arbitrary nonempty sets.
Let $P$ be a $E \times F$ matrix over $G_0$ [resp. $G$]. The set $S = G \times E \times F \cup \{0\}$ [resp. $S = G \times E \times F$] is a semigroup under the composition $(i, a, j) \circ (l, b, k) = \{(i, apjlb, k) \quad \text{if } pjl \neq 0 \quad \text{o otherwise}\}$. This semigroup is denoted by $S = M(G, P, E, F)$ and is called Rees $E \times F$ matrix semigroup over $G_0$ [resp. $G$] with the sandwich matrix $P$. 
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In special case, if $P = I$ is an identity matrix, $S = G_0$ is a semigroup with zero, and $E = F = I_\lambda$ is a set of cardinality $\lambda \geq 1$.

Define the semigroup operation on the set $B_\lambda(S) = M(S, I, I_\lambda, I_\lambda)$ by

$$(i, a, j) \circ (l, b, k) = \begin{cases} (i, ab, k) & \text{if } j = l, \\ 0 & \text{if } j \neq l \text{ and } (i, a, j) \neq (i, a, 0) = 0 = (i, a, 0). \end{cases}$$

The semigroup $B_\lambda(S)$ is called Brandt $\lambda$-extension of $S$. 
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The semigroup \( B_\lambda(S) \) is called Brandt \( \lambda \)-extension of \( S \).
Now let $i \to u_i$ and $j \to v_j$ be mappings of $E$ and $F$ to $S$ such that $u_k \cdot u_k = 1_S$, for all $k \in \lambda$. Then mapping $\theta: B_\lambda(S) \to S$ by $\theta(i, s, j) = u_i s u_j$ is a partial homomorphism.
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Then mapping $\theta : B_\lambda(S)^* = B_\lambda(S) - \{0\} \rightarrow S$ by $\theta(i, s, j) = u_i su_j$ is a partial homomorphism.
Let $S$ be a topological semigroup with zero and Brandt $\lambda$-extension of $S$, $B_\lambda(S)$ be equipped with product topology then $B_\lambda(S)$ is a topological semigroup.

Now $\theta : B_\lambda(S) \ast = B_\lambda(S) - \{0\} \to S \ast = S - \{0\}$ by $\theta((i,s,j)) = u^isu^j$ is a continuous partial homomorphism.

Then there exists a topological extension $\Omega$ of $S \ast$ by $B_\lambda(S)$ and $\Omega S \ast \simeq B_\lambda(S)$. 
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Then there exists a topological extension $\Omega$ of $S^*$ by $B_\lambda(S)$ and $\frac{\Omega}{S^*} \simeq B_\lambda(S)$. 
Corollary

Let $S$ be a topological semigroup with zero and $\Omega$ be a topological extension of $S^* = S - \{0\}$ by $B_\lambda(S)$. Let $(\psi, X)$ be a topological semigroup compactification of topological semigroup $\Omega$. Then $X_\rho X$ is a topological semigroup compactification of $B_\lambda(S)$.
Corollary

Let $S$ be a topological semigroup with zero and $\Omega$ be a topological extension of $S^* = S - \{0\}$ by $B_\lambda(S)$. Let $(\psi, X)$ be a topological semigroup compactification of topological semigroup $\Omega$. Then $\frac{X}{\rho_X}$ is a topological semigroup compactification of $B_\lambda(S)$. 
Corollary

Let $S$ be a topological semigroup with zero and $\Omega$ be a topological extension of $S^* = S - \{0\}$ by $B_\lambda(S)$. Suppose $(\varepsilon_{B_\lambda(S)}, B_\lambda(S)^P)$ and $(\varepsilon_\Omega, \Omega^P)$ are the universal $P$-compactifications of $B_\lambda(S)$ and $\Omega$ respectively. Then $B_\lambda(S)^P \simeq \frac{\Omega^P}{\rho_{\Omega^P}}$, if

i) $P$ is invariant under homomorphism,

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Let $S$ be a topological semigroup with zero and $\Omega$ be a topological extension of $S^* = S - \{0\}$ by $B_\lambda(S)$. Let $(\varepsilon_{B_\lambda(S)}, B_\lambda(S)^{sap})$ [resp. $(\varepsilon_{B_\lambda(S)}, B_\lambda(S)^{ap})$] and $(\varepsilon_\Omega, \Omega^{sap})$ [resp. $(\varepsilon_\Omega, \Omega^{ap})$] be the strongly almost periodic compactifications [resp. almost periodic compactifications] of $B_\lambda(S)$ and $\Omega$ respectively. Then $B_\lambda(S)^{sap} \simeq \frac{\Omega^{sap}}{\rho_{\Omega^{sap}}}$ [ resp. $B_\lambda(S)^{ap} \simeq \frac{\Omega^{ap}}{\rho_{\Omega^{ap}}}$ ].
Thank you for your attention