Boolean topological graphs of semigroups

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universal Horn classes

**uH-sentences** look like

\[(\forall \bar{x}) [\varphi_1(\bar{x}) \land \cdots \land \varphi_n(\bar{x}) \rightarrow \varphi(\bar{x})],\]

or like

\[(\forall \bar{x}) [\neg \varphi_1(\bar{x}) \lor \cdots \lor \neg \varphi_n(\bar{x})]\]

where \(\varphi_i(\bar{x}), \varphi(\bar{x})\) are atomic formulas.

**uH-classes** look like \(\text{Mod}(\text{uH-sentences})\).

The uH-class generated by a class \(\mathcal{K}\) equals \(\text{SP}^+\text{P}_U(\mathcal{K})\).

uH-class \(\mathcal{H}\) is finitely axiomatizable (finitely based) if \(\mathcal{H} = \text{Mod}(\Sigma)\) for some finite set \(\Sigma\) of uH-sentences.
The graph of a semigroup $S = (S, \cdot)$ is NOT a graph. It is the relational structure
\[ G(S) = (S, R), \]
where
\[(a, b, c) \in R \quad \text{iff} \quad a \cdot b = c. \]

For a class $C$ of semigroups let $G(C) = \{ G(S) \mid S \in C \}$.

**Theorem (Gornostaev, S)**
Let $C$ be a class of semigroups possessing a nontrivial member with a neutral element. Then $SP^+PUG(C)$ is not finitely axiomatizable.
Fact
Let $\mathcal{H}$ be a finitely axiomatizable uH-class of relational structures. Then there is a finite $n$ such that for each relational structure $M$ we have

$$M \in \mathcal{H} \iff (\forall N \leq M) \left[ |N| \leq n \rightarrow N \in \mathcal{H} \right].$$

Thus it is enough to construct for each $n$ a structure $M_n$ such that

- $M_n \notin SG(\text{Semigroups})$,
- if $N \leq M_n$ and $|N| \leq n$, then $N \in SPG(\mathcal{C})$. 
construction of $M_n$

<table>
<thead>
<tr>
<th>Elements of $M_k$</th>
<th>Elements of $Z_2^{n+6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_0$</td>
<td>1100 000···000···000 0</td>
</tr>
<tr>
<td>$a_1$</td>
<td>0011 000···000···000 0</td>
</tr>
<tr>
<td>$a_0$ →</td>
<td>1010 000···000···000 0</td>
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<tr>
<td>$a_1$</td>
<td>0101 000···000···000 0</td>
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<tr>
<td>$b$ →</td>
<td>1111 000···000···000 0</td>
</tr>
<tr>
<td>$c_0$</td>
<td>0000 100···000···000 0</td>
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<tr>
<td>$c_1$</td>
<td>0000 010···000···000 0</td>
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<td>...</td>
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<tr>
<td>$c_k$ →</td>
<td>0000 000···100···000 0</td>
</tr>
<tr>
<td>$c_{k+1}$</td>
<td>0000 000···001···000 0</td>
</tr>
<tr>
<td>...</td>
<td>.................................................</td>
</tr>
<tr>
<td>$c_n$</td>
<td>0000 000···000···001 0</td>
</tr>
<tr>
<td>$d_0$</td>
<td>0011 100···000···000 0</td>
</tr>
<tr>
<td>$d_1$</td>
<td>0011 110···000···000 0</td>
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<td>...</td>
<td>.................................................</td>
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<tr>
<td>$d_k$ →</td>
<td>0011 111···100···000 0</td>
</tr>
<tr>
<td>$d_k$</td>
<td>0011 111···110···000 1</td>
</tr>
<tr>
<td>$d_{k+1}$</td>
<td>0011 111···111···000 1</td>
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<td>...</td>
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<tr>
<td>$d_n$</td>
<td>0011 111···111···111 1</td>
</tr>
<tr>
<td>$d_0$</td>
<td>0101 100···000···000 0</td>
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<tr>
<td>$d_1$</td>
<td>0101 110···000···000 0</td>
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<tr>
<td>$d_k$ →</td>
<td>0101 111···100···000 0</td>
</tr>
<tr>
<td>$d_k$</td>
<td>0101 111···110···000 0</td>
</tr>
<tr>
<td>$d_{k+1}$</td>
<td>0101 111···111···000 0</td>
</tr>
<tr>
<td>...</td>
<td>.................................................</td>
</tr>
<tr>
<td>$d_n$</td>
<td>0101 111···111···111 0</td>
</tr>
<tr>
<td>$e$ →</td>
<td>1111 111···111···111 1</td>
</tr>
</tbody>
</table>

Table: The mapping $j_k$. Elements of $Z_2^{n+6}$ are represented as words over $Z_2$. For the sake of clarity we divided these words into 3 segments of length 4, $n+1$ and 1 respectively. In the second segment ($k-1$)th, $k$th and ($k+1$)th digits are placed between dots.
Fact
Let $\mathcal{H}$ be a finitely axiomatizable uH-class of relational structures. Then there is a finite $n$ such that for each relational structure $\mathbf{M}$ we have

\[ \mathbf{M} \in \mathcal{H} \iff (\forall \mathbf{N} \leq \mathbf{M}) [|\mathbf{N}| \leq n \rightarrow \mathbf{N} \in \mathcal{H}]. \]

Thus it is enough to construct for each $n$ a structure $\mathbf{M}_n$ such that

- $\mathbf{M}_n \not\in \text{uHG(Semigroups)}$,
- if $\mathbf{N} \leq \mathbf{M}_n$ and $|\mathbf{N}| \leq n$, then $\mathbf{N} \in \text{uHG}(\mathcal{C})$.

Belinda’s guess
Maybe it lifts to a topological setting.
Boolean core of a uH-class

Boolean core of $\mathcal{H}$ is

$$\mathcal{H}_{BC} = S_cP^+(\mathcal{H}_{fin})$$

$\mathcal{H}_{fin}$ - finite structures from $\mathcal{H}$ with the discrete topology
$P^+$ - the nontrivial product class operator
$S_c$ - the closed substructure class operator

Example

Priestley spaces $= S_cP^+(\{0, 1\}, \leq) = SP^+(\{0, 1\}, \leq)_{BC}$.

Facts

- Every member of $\mathcal{H}_{BC}$ has Boolean topology (compact, Hausdorff, totally disconnected).
- $\mathcal{H}_{BC}$ consists of all profinite structures built, as inverse limits, from finite members of $\mathcal{H}$.
General problem
Axiomatize $\mathcal{H}_{BC}$ among all structures with Boolean topology.
Theorem (Clark, Krauss)
Topological quasivarieties may be described by an extension of uH-logic imitating topological convergence.

But it is a nasty and awkward infinite logic.

Is there a better logic?
standardness

$\mathcal{H}$ is standard if $\mathcal{H}_{BC}$ consists of all Boolean topological structures with reducts in $\mathcal{H}$.

If $\mathcal{H}$ is standard, then $\mathcal{H}_{BC}$ is axiomatizable by $uH$-theory of $\mathcal{H}$.

Theorem (Numakura)
The variety of all semigroups is standard.

Theorem (Clark, Davey, Haviar, Pitkethly, Talukder)
Every variety with finitely determined syntactic congruences is standard.
Examples: all varieties of semigroups, monoids, groups, rings, varieties with definable principal congruences.

Theorem (Nešetřil, Pultr, Trotta)
Finitely generated $uH$-class of simple graphs is standard iff it is one of $\emptyset$, $SP(\bullet)$, $SP(\bullet \bullet)$, $SP(\bullet\bullet\bullet)$.
A (surjective) inverse system over $\omega$ is a collection of structures $M_n$, $n \in \omega$, together with (surjective) homomorphisms $\varphi_n: M_{n+1} \to M$. Its inverse limit is

$$\lim\leftarrow M_n = \{a \in \prod_{n \in \omega} M_n \mid (\forall n) \varphi_n(a(n+1)) = a(n)\}$$

with structure and (Boolean) topology inherited from the product $M = \lim\leftarrow M_n$ is pointwise non-separable with respect to $\mathcal{H}$ if there is a predicate $R$ and a tuple $\bar{b} \in M - R^M$ such that for every homomorphism $\psi: M_n \to N \in \mathcal{H}$ we have $\psi(\bar{b}(n)) \in R^N$.

**Theorem (Clark, Davey, Jackson, Pitkethly)**

Assume that $M = \lim\leftarrow M_n$, a surjective inverse limit of finite structures, is pointwise non-separable with respect to $\mathcal{H}$ and every $n$-element substructure of $M_n$ is in $\mathcal{H}$. Then $\mathcal{H}$ is non-standard.
Theorem \((S, T)\)

Let \(\mathcal{H} = SP^+P \cup G(\mathcal{C})\) be the \(uH\)-class generated by a class \(G(\mathcal{C})\) of graphs of semigroups possessing a nontrivial member with a neutral element. Then \(\mathcal{H}\) is non-standard - \(\mathcal{H}_{BC}\) is not definable in \(uH\)-logic.

pseudoProof

Structures \(M_n\) from non-finite axiomatization proof may be slightly modified and connected by homomorphism, thus giving a needed inverse system.
Maybe $H_{BC}$ is fo-definable?

**Example (Clark, Davey, Jackson, Pitkethly)**
Let $L$ be a finite structure with a lattice reduct. Then $S_cP(L)$ is first order definable. But there are some non-standard $S_cP(L)$.

**Example (Stralka, Clark, Davey, Jackson, Pitkethly)**
Priestley spaces form a non-fo definable class.

**pseudoProof**
Because there exists Stralka space $(C, \leq)$:
- $C$ - Cantor space
- $\leq$ - cover or equal relation

$(C, \leq)$ is a union of copies of $(\{0\}, \equiv)$ and $(\{0, 1\}, \leq)$ but it is **NOT** a Priestley space.
A topological space is a $\lambda$-space, $\lambda \in \mathbb{N}$, if it is a disjoint union of at most $\lambda$ pieces each of which is either a one point or one point compactification of a discrete topological space.

**Theorem (Clark, Davey, Jackson, Pitkethly)**

Let $\mathcal{H}$ be non-standard, witnessed by $\mathbf{M}$ ($\mathbf{M}$ has Boolean topology an the relational reduct in $\mathcal{H}$). If

- up to isomorphism, $\mathbf{M}$ has only finitely many connected components and all them are finite (1st technique)

or

- $\mathbf{M}$ has a $\lambda$-topology + some technical condition (2nd technique)

then $\mathcal{H}_{BC}$ is not fo-definable.
Theorem \((S, T)\)

Let \(\mathcal{H} = \text{SP}^+ \cup G(C)\) be the uH-class generated by a class \(G(C)\) of graphs of semigroups possessing a nontrivial member with a neutral element. Then \(\mathcal{H}_{BC}\) is not fo-definable.

pseudoProof

- If \(\{0, 1\}, \lor\) \(\in C\), then 1\(^{st}\) technique applies to a modification of Stralka space.

- If \((\mathbb{Z}_k, +) \in C\) or \((\mathbb{N}, +) \in C\), then 2\(^{nd}\) technique applies to \(M\) constructed for disproving standardness.
General problem
Axiomatize $\mathcal{H}_{BC}$ among all structures with Boolean topology.

What about monadic second order logic?
This is all Thank you!