Natural extension of median algebras

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Back to the roots: canonical extension

Canonical extension $L^\delta$ of a bounded DL $L$ with topologies $\iota$ and $\delta$:

- $L^\delta$ is doubly algebraic.
- $L \hookrightarrow L^\delta$.
- $L$ is dense in $L^\iota$.
- $L$ is dense and discrete in $L^\delta$.

A tool to extend maps in a canonical way comes with the topology $\delta$

$f^\delta$ can be defined by order and continuity properties.

Leads to canonical extension of lattice-based algebras.

Tool used to obtain canonicity of logics. JÓNSSON (1994).
A tool to extend maps in a canonical way comes with the topology $\delta$

$f^\delta$ can be defined by order and continuity properties.

Leads to canonical extension of lattice-based algebras.

Tool used to obtain canonicity of logics. JÓNSSON (1994).
It is possible to generalize canonical extension to non lattice-based algebras

<table>
<thead>
<tr>
<th>Step 1</th>
<th>Step 2</th>
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</thead>
<tbody>
<tr>
<td>Define the natural extension of an <strong>algebra</strong></td>
<td>Define the natural extension of a <strong>map</strong></td>
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</table>

**Davey, Gouveia, Haviar** and **Priestley** (2011)  
A partial solution
We adopt the settings of natural dualities

A finite algebra $\mathbb{M}$

A discrete alter-ego topological structure $\mathcal{M}$

We assume that $\mathcal{M}$ yields a duality for $\mathcal{A}$. We focus on objects.

<table>
<thead>
<tr>
<th>Algebra</th>
<th>Topology</th>
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<tbody>
<tr>
<td>$\mathbb{M}$</td>
<td>$\mathcal{M}$</td>
</tr>
<tr>
<td>$\mathcal{A} = \text{ISP}(\mathbb{M})$</td>
<td>$\mathcal{X} = \text{IS}_{\mathbb{P}}^{c}(\mathcal{M})$</td>
</tr>
<tr>
<td>$\mathcal{A} \leq_c \mathcal{M}^\mathcal{A}$</td>
<td>$\mathcal{X} \leq \mathbb{M}^\mathcal{X}$</td>
</tr>
<tr>
<td>$\mathcal{X} \ast = \mathcal{A}(\mathcal{X}, \mathbb{M}) \leq \mathcal{M}^\mathcal{X}$</td>
<td></td>
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$(\mathcal{A}^\ast)_\ast \simeq \mathcal{A}$
Natural extension of an algebra can be constructed from its dual

<table>
<thead>
<tr>
<th>Canonical extension</th>
<th>Natural extension</th>
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<tr>
<td>$L^\delta$ is the algebra of order-preserving maps from $L^*$ to $\mathcal{2}$.</td>
<td>$A^\delta$ is the algebra of structure preserving maps from $A^*$ to $M$.</td>
</tr>
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Davey, Gouveia, Haviar and Priestley (2011)
The variety of median algebras will perfectly illustrate the construction

Median algebras are the \((\cdot,\cdot,\cdot)\)-subalgebras of the distributive lattices where

\[(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z).\]
The variety of median algebras will perfectly illustrate the construction

Median algebras are the \((\cdot, \cdot, \cdot)\)-subalgebras of the distributive lattices where

\[(x, y, z) = (x \land y) \lor (x \land z) \lor (y \land z).\]

\[B\] Boolean algebras \[\text{ISP}(2)\ 2 = \langle\{0, 1\}, \lor, \land, \neg, 0, 1\rangle\]

\[D\] Bounded DL \[\text{ISP}(2)\ 2 = \langle\{0, 1\}, \lor, \land, 0, 1\rangle\)

\[A\] Median algebra \[\text{ISP}(2)\ 2 = \langle\{0, 1\}, (\cdot, \cdot, \cdot)\rangle\]

On \(\{0, 1\}\), operation \((\cdot, \cdot, \cdot)\) is the majority function.
There is a natural duality for median algebras

\[ \mathcal{Z} := \langle \{0, 1\}, \leq, \cdot, 0, 1, \iota \rangle. \]

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<tr>
<td>( \mathcal{A} = \text{ISP}(2) ) is the variety of median algebras</td>
<td>( \mathcal{X} = \text{ISC}_{P^+}(2) ) is the category of bounded strongly complemented Priestley spaces</td>
</tr>
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</table>

Proposition (Isbell (1980), Werner (1981))

1. **Structure** \( \mathcal{Z} \) yields a logarithmic duality for median algebras.

2. **Operation** \( \cdot \) is an involutive order reversing homeomorphism such that \( 0^\bullet = 1 \) and \( x \leq x^\bullet \rightarrow x = 0 \).
We may associate orders to a median algebra

Let \( a \in A = \langle A, (\cdot, \cdot, \cdot) \rangle \). Define \( \leq_a \) on \( A \) by

\[
b \leq_a c \quad \text{if} \quad (a, b, c) = b.
\]

Then \( \leq_a \) is a \( \land \)-semilattice order on \( A \) with \( b \land_a c = (a, b, c) \).

Semilattices obtained in this way are the median semilattices.

**Proposition**

In a median semilattice, principal ideals are distributive lattices.

Natural extension completes everything it can complete

\( A^\delta \equiv \) the algebra of \( \{\leq, 0, 1, \cdot\} \)-preserving maps from \( A^* \) to \( \mathcal{Z} \).

**Proposition**

*Let \( a, b \in A \)*

1. \( \langle A^\delta, \land_a \rangle \) is a bounded complete \( \land_a \)-semilattice which is an extension of \( \langle A, \land_a \rangle \).

2. \( (b]\langle A^\delta, \land_a \rangle \) is a canonical extension of \( (b]\langle A, \land_a \rangle \).
We can define $\mathbf{A}$ in $\mathbf{A}^\delta$ in a purely topological language

$$\chi_p(\mathbf{A}^*, 2) := \bigcup \{ \chi(F, 2) \mid F \leq_c \mathbf{A}^\ast \}.$$ 

Consider the topology $\delta$ generated by the family $\Delta$ of the

$$O_f = \{ x \in \mathbf{A}^\delta \mid x \supseteq f \}, \quad f \in \chi_p(\mathbf{A}^*, 2).$$

**Lemma**

1. $\Delta$ is a topological basis of $\delta$.
2. $\mathbf{A}$ is dense and discrete in $\mathbf{A}^\delta$.

The lemma generalizes to any logarithmic dualities.
We use the topology $\delta$ to canonically extend maps to multimaps

$$f : A \rightarrow B$$
We use the topology $\delta$ to canonically extend maps to multimaps

$$\tilde{f} : A^\delta \to \Gamma(B^\delta)$$

$$f : A \to B$$
We define the multi-extension of $f : A \to B$

\[
\bar{f} : A \to \Gamma(B^\delta_l) : a \mapsto \{f(a)\}.
\]

$A$ is dense in $A^\delta_\delta$ and $\Gamma(B^\delta_l)$ is a complete lattice.

**Definition**
The *multi-extension* $\tilde{f}$ of $f$ is defined by

\[
\tilde{f} : A^\delta_\delta \to \Gamma(B^\delta_l) : x \mapsto \limsup_\delta \bar{f}(x),
\]

In other words, for any $F \subseteq B^*$,

\[
\tilde{f}(x)^F = \bigcap\{\{f(a)^F \mid a \in V\} \mid V \in \delta_x\},
\]

where the closure is computed in $B^\delta_l$. 
The multi-extension is a continuous map

We say that $f$ is smooth if $\#\tilde{f}(x) = 1$ for any $x \in A^\delta$.

Let $\sigma \downarrow$ be the co-Scott topology on $\Gamma(B^\delta_\iota)$.

Proposition (Generalizes to logarithmic dualities)

1. For any $a \in A$, $\tilde{f}(a) = \{f(a)\}$.
2. The map $\tilde{f} : A^\delta \to \Gamma(B^\delta_\iota)$ is $(\delta, \sigma \downarrow)$.
3. If $f' : A^\delta \to \Gamma(B^\delta_\iota)$ satisfies 1 and 2 then $\tilde{f}(x) \subseteq f'(x)$ for every $x \in A^\delta$. 
The multi-extension is a continuous map

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Proposition (Generalizes to logarithmic dualities)

1. For any $a \in A$, $\tilde{f}(a) = \{f(a)\}$.

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3. If $f' : A^\delta \to \Gamma(B^\delta_i)$ satisfies 1 and 2 then $\tilde{f}(x) \subseteq f'(x)$ for every $x \in A^\delta$.

4. $f$ is smooth if and only if it admits an $(\delta, \iota)$-continuous extension, namely $f^\delta : A^\delta \to B^\delta : x \mapsto f^\delta(x) \in \tilde{f}(x)$.

5. If $f$ is not smooth, there is no extension $f' : A^\delta \to B^\delta$ of $f$ and that is $(\delta, \iota)$-continuous that satisfies $f'(x) \in \tilde{f}(x)$. 

We can use $\leq_a$ to turn the multi-extension into an extension

**Definition**

Let $b \in B$. The map $f_b^\delta : A^\delta \to B^\delta$ is defined by

$$f_b^\delta(x) = \bigwedge_b \tilde{f}(x).$$
Proposition

1. *The map* \( f^\delta_b \) *is* \((\delta, \nu_b \uparrow)\)-*continuous.*

2. *If* \( f : A \to A \) *respects* \( \wedge_a \) *on finite subsets then* \( f^\delta_b \) *respects* \( \wedge_a \) *on any set.*

3. *For a median algebra, being a bounded DL is a property preserved by natural extension.*

4. *For a median algebra, being a Boolean algebra is a property preserved by natural extension.*
Among open questions/further work

- How to canonically extend maps if the duality fails to be logarithmic?
- Use continuity properties to study preservation of equations.
- Determine the links with profinite extension.
- Do something clever with that.