Selective for $\mathcal{R}$ but not Ramsey for $\mathcal{R}$

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Outline

1. Background
   - Notation
   - Selective ultrafilters on ω
2. Topological Ramsey Theory
   - Definition of a topological Ramsey space
   - The topological Ramsey space $\mathcal{R}_1$.
   - The topological Ramsey space $\mathcal{R}^\star$
3. Selective but not Ramsey ultrafilters
   - $\mathcal{R}_1$
   - $\mathcal{R}_n$

Definition

For each $S \subseteq \omega^{<\omega}$,

$$[S] = \{s \in S : \forall t \in S, s \sqsubseteq t \Rightarrow s = t\}$$

$$cl(S) = \{t \in \omega^{<\omega} : t \sqsubseteq s \in S\}$$

$$\pi_0(S) = \{s_0 : s \in S\}$$

S is a **Tree on** ω, if $cl(S) = S$.
For $s, t \in \omega^{<\omega}$,

$$s \leq t \Leftrightarrow (s \sqsubseteq t \text{ or } |s| = |t| \ \& \ s \leq_{\text{lex}} t)$$

If S and T are trees on ω then

$$\binom{T}{S} = \{U \subseteq T : U \cong S\}.$$

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Theorem (Kunen, [1])

Let $\mathcal{U}$ be an ultrafilter on ω.

$\mathcal{U}$ is selective if and only if $\mathcal{U}$ is Ramsey.

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Example (The Ellentuck Space, $([\omega]^{\omega}, \subseteq, r)$)

$$r_n(\{a_0, a_1, a_2, \dots\}) = \{a_0, \dots, a_{n-1}\}$$

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The Ellentuck Theorem (Ellentuck, [3])

The Ellentuck space $([\omega]^{\omega}, \subseteq, r)$ is a topological Ramsey space.

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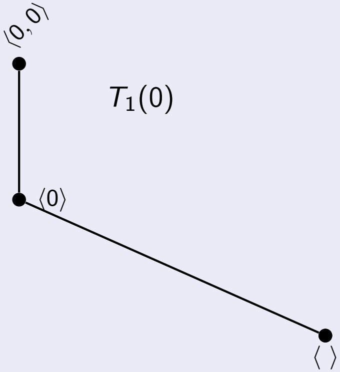

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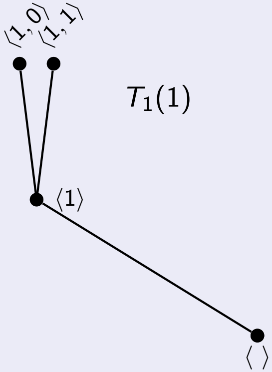

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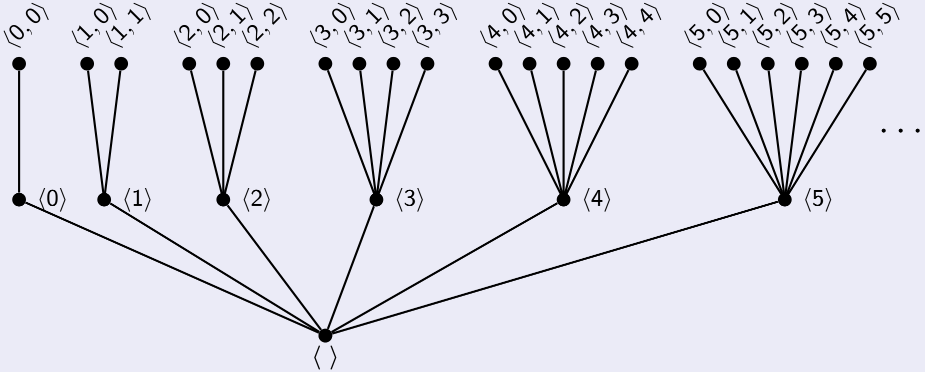

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Theorem (Dobrinen, Todorcevic, [2])

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Is Ramsey for $\mathcal{R}_1$ equivalent to selective for $\mathcal{R}_1$?

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Lemma (Follows from work of Laflamme, [4])

If $\mathcal{U}$ is Ramsey for $\mathcal{R}_1$ then $\mathcal{U}$ is weakly-Ramsey.

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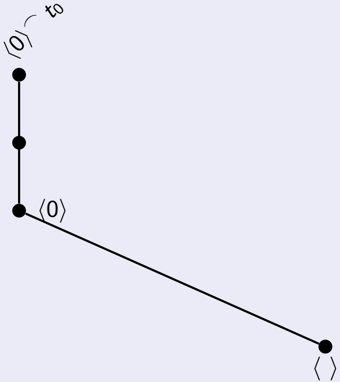

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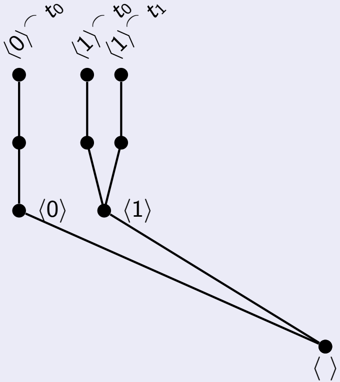

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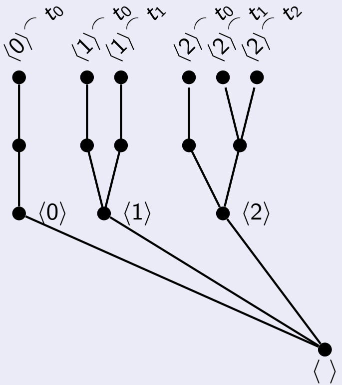

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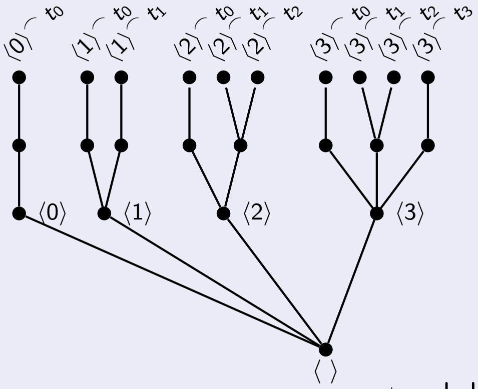

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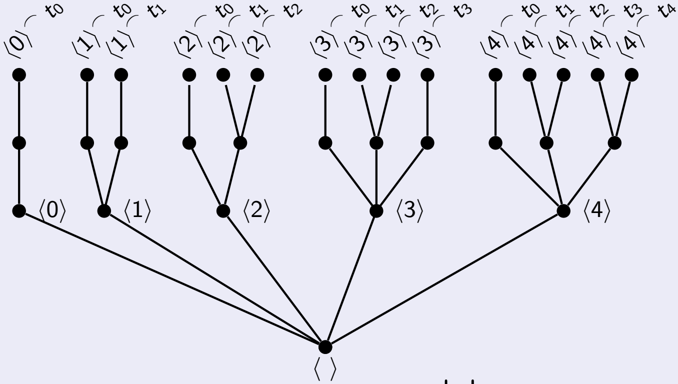

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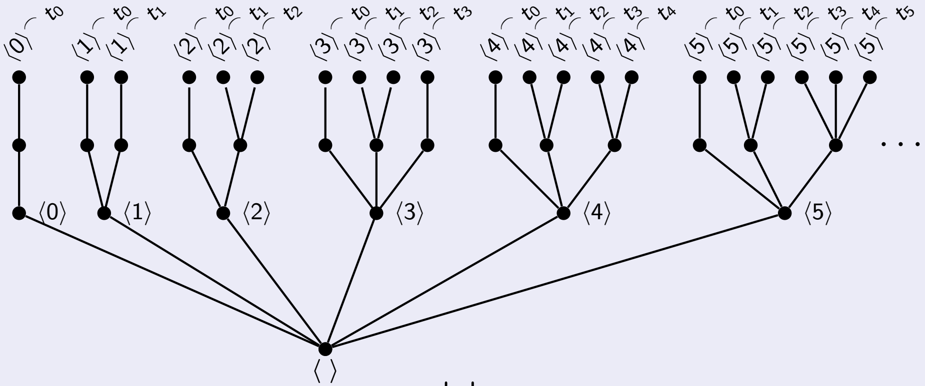

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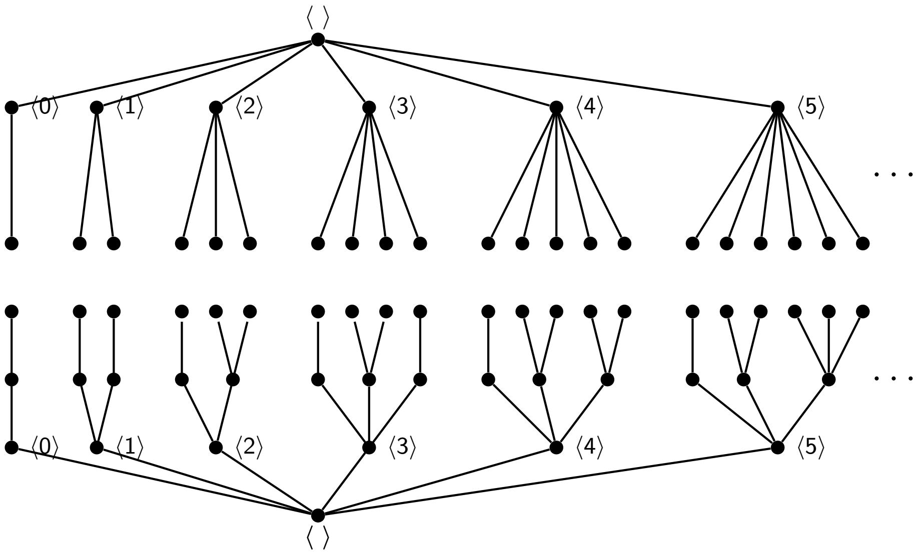

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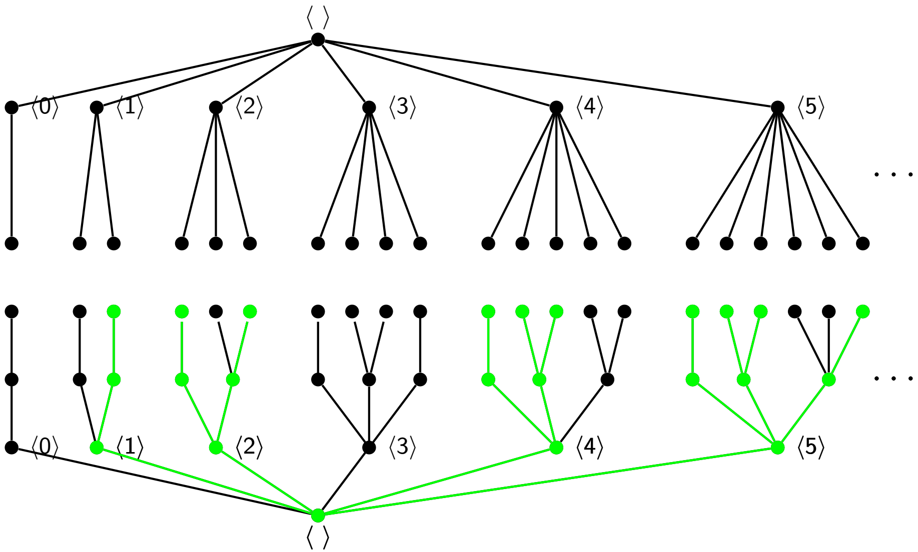
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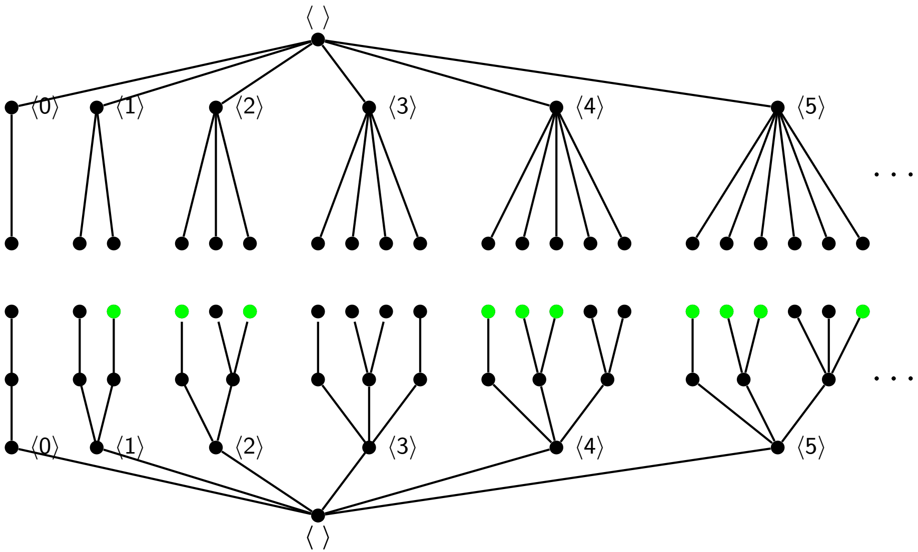
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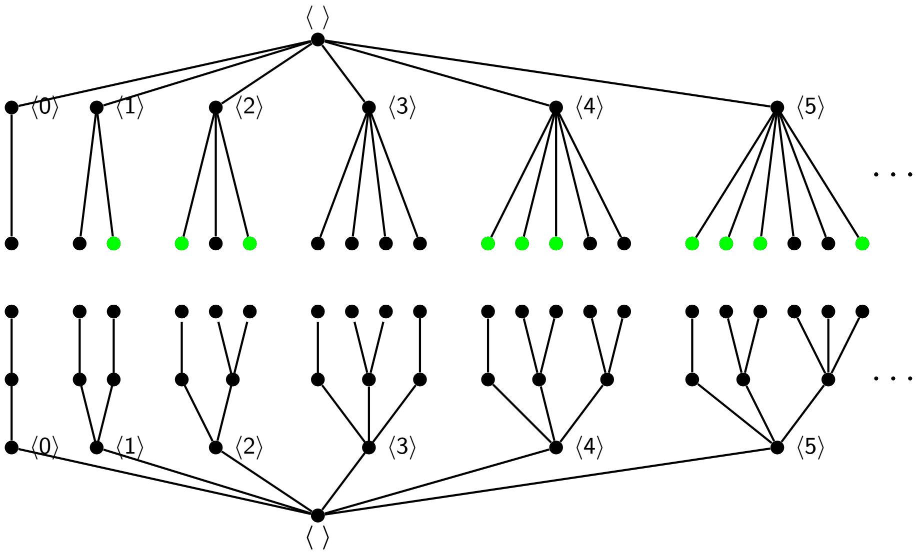
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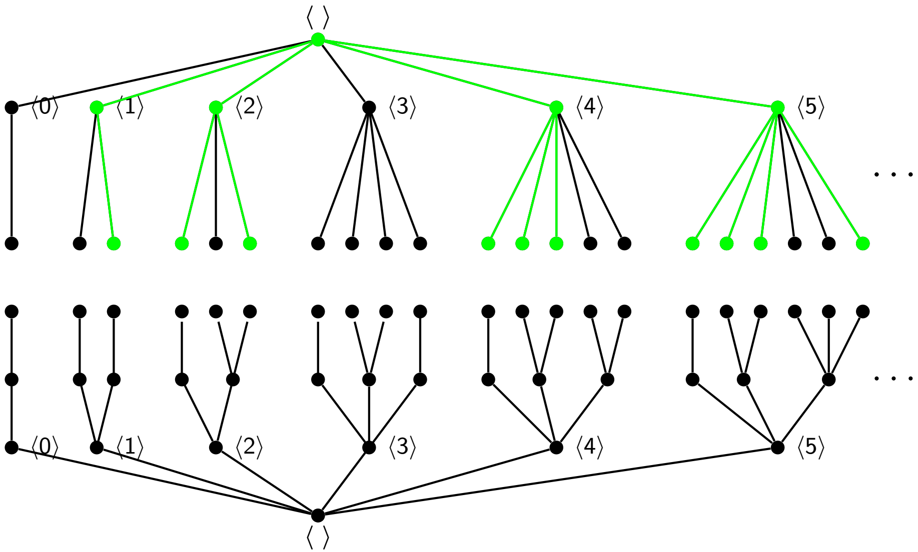
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Theorem (T., [7])

If $\mathcal{C}$ is a $(\mathcal{R}_1^\star, \leq^*)$-generic filter then $\Gamma''\mathcal{C}$ generates an ultrafilter on $[T_1]$ which is selective for $\mathcal{R}_1$ but not Ramsey for $\mathcal{R}_1$.

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Definition

Let n be a positive integer. Suppose that $T_1, T_2, \ldots, T_n$ have been defined.

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$$T_{n+1}(i) = \left\{ \langle i \rangle^\frown s : s \in T_n(j) \ \& \ \frac{i(i+1)}{2} \leq j \leq \frac{(i+1)(i+2)}{2} \right\}$$

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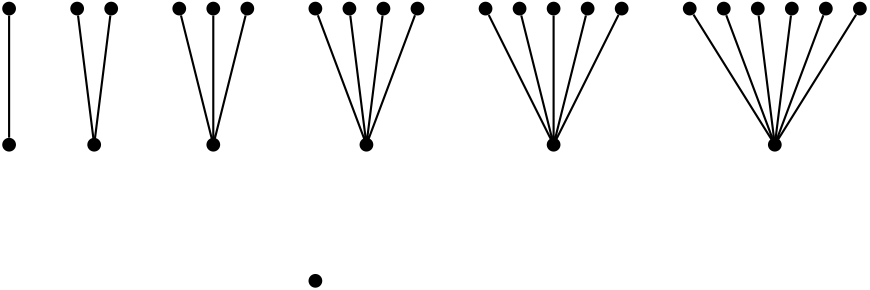

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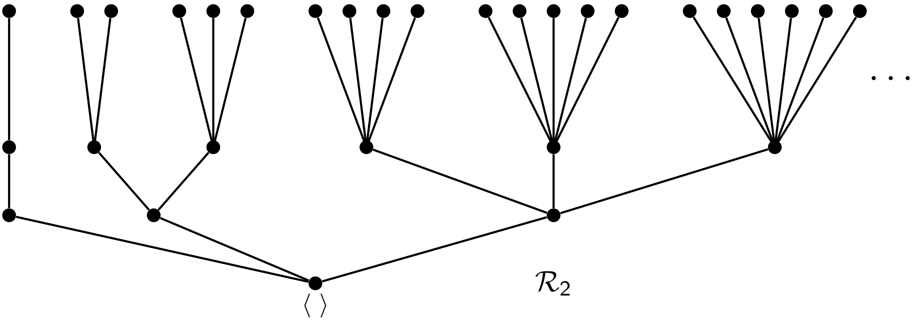

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