Aronszajn trees and the successors of a singular cardinal

Spencer Unger

UCLA

August 9, 2013

Outline

A classical theorem

Outline

Outline

A classical theorem

Definitions

The tree property

Outline

Outline

Outline

Theorem (König Infinity Lemma)

Every infinite finitely branching tree has an infinite path.

Definitions

- A tree is set T together with an ordering $<_T$ which is wellfounded, transitive, irreflexive and such that for all $t \in T$ the set $\{x \in T \mid x <_T t\}$ is linearly ordered by $<_T$.

Definitions

- A tree is set T together with an ordering $<_T$ which is wellfounded, transitive, irreflexive and such that for all $t \in T$ the set $\{x \in T \mid x <_T t\}$ is linearly ordered by $<_T$.
- The height of an element t is the order-type of the collection of the predecessors of t under $<_T$. That is, the unique ordinal α such that $(\alpha, \in) \simeq (\{x \in T \mid x <_T t\}, <_T)$.

Definitions

- A tree is set T together with an ordering $<_T$ which is wellfounded, transitive, irreflexive and such that for all $t \in T$ the set $\{x \in T \mid x <_T t\}$ is linearly ordered by $<_T$.
- The height of an element t is the order-type of the collection of the predecessors of t under $<_T$. That is, the unique ordinal α such that $(\alpha, \in) \simeq (\{x \in T \mid x <_T t\}, <_T)$.
- The α^{th} level of the tree is the collection of nodes of height α.

Definitions

- A tree is set T together with an ordering $<_T$ which is wellfounded, transitive, irreflexive and such that for all $t \in T$ the set $\{x \in T \mid x <_T t\}$ is linearly ordered by $<_T$.
- The height of an element t is the order-type of the collection of the predecessors of t under $<_T$. That is, the unique ordinal α such that $(\alpha, \in) \simeq (\{x \in T \mid x <_T t\}, <_T)$.
- The α^{th} level of the tree is the collection of nodes of height α.
- The height of a tree T is the least ordinal β such that there are no nodes of height β.

Definitions

- A tree is set T together with an ordering $<_T$ which is wellfounded, transitive, irreflexive and such that for all $t \in T$ the set $\{x \in T \mid x <_T t\}$ is linearly ordered by $<_T$.
- The height of an element t is the order-type of the collection of the predecessors of t under $<_T$. That is, the unique ordinal α such that $(\alpha, \in) \simeq (\{x \in T \mid x <_T t\}, <_T)$.
- The α^{th} level of the tree is the collection of nodes of height α.
- The height of a tree T is the least ordinal β such that there are no nodes of height β.
- A set b is a *cofinal branch* through T if $b \subseteq T$ and $(b, <_T)$ is a linear order whose order-type is the height of the tree.

The tree property

Theorem (König Infinity Lemma)

Every tree of height ω *with finite levels has a cofinal branch*

The tree property

Theorem (König Infinity Lemma)

Every tree of height ω with finite levels has a cofinal branch

Let κ be a regular cardinal.

Definition

A κ-tree is a tree of height κ with levels of size less than κ.

The tree property

Theorem (König Infinity Lemma)

Every tree of height ω with finite levels has a cofinal branch

Let κ be a regular cardinal.

Definition

A κ-tree is a tree of height κ with levels of size less than κ.

Definition

A cardinal κ has the tree property if every κ-tree has a cofinal branch. A counterexample to the tree property at κ is called a κ-Aronszajn tree.

When do Aronszajn trees exist?

Theorem (Aronszajn)

There is a tree of height ω_1 all of whose levels are countable, which has no cofinal branch.

When do Aronszajn trees exist?

Theorem (Aronszajn)

There is a tree of height ω_1 all of whose levels are countable, which has no cofinal branch.

Theorem (Specker)

If $\kappa^{<\kappa} = \kappa$, then there is a κ^+-Aronszajn tree. In particular CH implies that there is an ω_2-Aronszajn tree.

When do Aronszajn trees exist?

Theorem (Aronszajn)

There is a tree of height ω_1 all of whose levels are countable, which has no cofinal branch.

Theorem (Specker)

If $\kappa^{<\kappa} = \kappa$, then there is a κ^+-Aronszajn tree. In particular CH implies that there is an ω_2-Aronszajn tree.

Remark

The tree constructed is special in the sense that there is a function from T to κ such that $f(s) \neq f(t)$ whenever $s <_T t$.

The tree property and large cardinals

Definition

A uncountable cardinal κ is inaccessible if it is a regular limit cardinal and for all $\mu < \kappa$, $2^\mu < \kappa$.

The tree property and large cardinals

Definition

A uncountable cardinal κ is inaccessible if it is a regular limit cardinal and for all $\mu < \kappa$, $2^\mu < \kappa$.

Definition

A cardinal κ is weakly compact if κ is uncountable and for all $f : [\kappa]^2 \to 2$, there is $H \subseteq \kappa$ of size κ such that f is constant on $[H]^2$.

The tree property and large cardinals

Definition

A uncountable cardinal κ is inaccessible if it is a regular limit cardinal and for all $\mu < \kappa$, $2^{\mu} < \kappa$.

Definition

A cardinal κ is weakly compact if κ is uncountable and for all $f : [\kappa]^2 \to 2$, there is $H \subseteq \kappa$ of size κ such that f is constant on $[H]^2$.

Theorem (Tarski and Keisler)

κ is weakly compact if and only if it is inaccessible and has the tree property.

What about the tree property at non-inaccessible cardinals?

Theorem (Mitchell)

The theory ZFC + 'there is a weakly compact cardinal' is consistent if and only if the theory ZFC + 'ω_2 has the tree property' is consistent.

What about the tree property at non-inaccessible cardinals?

Theorem (Mitchell)

The theory ZFC + 'there is a weakly compact cardinal' is consistent if and only if the theory ZFC + 'ω_2 has the tree property' is consistent.

- The reverse direction of the theorem uses Gödel's constructible universe L.

What about the tree property at non-inaccessible cardinals?

Theorem (Mitchell)

The theory ZFC + 'there is a weakly compact cardinal' is consistent if and only if the theory ZFC + 'ω_2 has the tree property' is consistent.

- The reverse direction of the theorem uses Gödel's constructible universe L.
- The forward direction is an application of Cohen's method of forcing.

What about the tree property at non-inaccessible cardinals?

Theorem (Mitchell)

The theory ZFC + 'there is a weakly compact cardinal' is consistent if and only if the theory ZFC + 'ω_2 has the tree property' is consistent.

- The reverse direction of the theorem uses Gödel's constructible universe L.
- The forward direction is an application of Cohen's method of forcing.
- We focus on generalizations of the forcing direction of Mitchell's theorem, since further questions about the tree property seem to require very large cardinals.

Measurable Cardinals

Definition

A cardinal κ is measurable if there is a transitive class N and an elementary embedding $j : V \to N$ with critical point κ.

Measurable Cardinals

Definition

A cardinal κ is measurable if there is a transitive class N and an elementary embedding $j : V \to N$ with critical point κ.

Fact

κ is measurable implies κ has the tree property.

Proof.

- Let T be a κ-tree and assume that the underlying set of T is κ.

Measurable Cardinals

Definition

A cardinal κ is measurable if there is a transitive class N and an elementary embedding $j : V \to N$ with critical point κ.

Fact

κ is measurable implies κ has the tree property.

Proof.

- Let T be a κ-tree and assume that the underlying set of T is κ.
- Let j witness that κ is measurable.

Measurable Cardinals

Definition

A cardinal κ is measurable if there is a transitive class N and an elementary embedding $j : V \to N$ with critical point κ.

Fact

κ is measurable implies κ has the tree property.

Proof.

- Let T be a κ-tree and assume that the underlying set of T is κ.
- Let j witness that κ is measurable.
- $j(T)$ is a tree of height $j(\kappa)$ and $j(T) \restriction \kappa = T$.

Measurable Cardinals

Definition

A cardinal κ is measurable if there is a transitive class N and an elementary embedding $j : V \to N$ with critical point κ.

Fact

κ is measurable implies κ has the tree property.

Proof.

- Let T be a κ-tree and assume that the underlying set of T is κ.
- Let j witness that κ is measurable.
- $j(T)$ is a tree of height $j(\kappa)$ and $j(T) \restriction \kappa = T$.
- In N choose a point on level κ of $j(T)$.

Measurable Cardinals

Definition

A cardinal κ is measurable if there is a transitive class N and an elementary embedding $j : V \to N$ with critical point κ.

Fact

κ is measurable implies κ has the tree property.

Proof.

- Let T be a κ-tree and assume that the underlying set of T is κ.
- Let j witness that κ is measurable.
- $j(T)$ is a tree of height $j(\kappa)$ and $j(T) \restriction \kappa = T$.
- In N choose a point on level κ of $j(T)$.
- This point determines a branch through T.

□

Mitchell's forcing

Let κ be a measurable cardinal.

Ideas of the construction:

Mitchell's forcing

Let κ be a measurable cardinal.

Ideas of the construction:

- Avoid CH.

Mitchell's forcing

Let κ be a measurable cardinal.

Ideas of the construction:

- Avoid CH.
- End up with $2^\omega = \kappa = \omega_2$.

Mitchell's forcing

Let κ be a measurable cardinal.

Ideas of the construction:

- Avoid CH.
- End up with $2^\omega = \kappa = \omega_2$.
- Somehow prove that the tree property holds.

Mitchell's forcing

Let κ be a measurable cardinal.

Ideas of the construction:

- Avoid CH.
- End up with $2^\omega = \kappa = \omega_2$.
- Somehow prove that the tree property holds.

Vague definition of the forcing:

Mitchell's forcing

Let κ be a measurable cardinal.

Ideas of the construction:

- Avoid CH.
- End up with $2^\omega = \kappa = \omega_2$.
- Somehow prove that the tree property holds.

Vague definition of the forcing: We call the forcing $\mathbb{M}$.
Conditions are pairs (p, q) such that

- p is in $\mathbb{P} = \mathrm{Add}(\omega, \kappa)$

Mitchell's forcing

Let κ be a measurable cardinal.

Ideas of the construction:

- Avoid CH.
- End up with $2^\omega = \kappa = \omega_2$.
- Somehow prove that the tree property holds.

Vague definition of the forcing: We call the forcing $\mathbb{M}$.
Conditions are pairs (p, q) such that

- p is in $\mathbb{P} = \mathrm{Add}(\omega, \kappa)$
- q is a partial function whose domain is a subset of κ

Mitchell's forcing

Let κ be a measurable cardinal.

Ideas of the construction:

- Avoid CH.
- End up with $2^\omega = \kappa = \omega_2$.
- Somehow prove that the tree property holds.

Vague definition of the forcing: We call the forcing $\mathbb{M}$.
Conditions are pairs (p, q) such that

- p is in $\mathbb{P} = \mathrm{Add}(\omega, \kappa)$
- q is a partial function whose domain is a subset of κ
- for each α in the domain of q, $q(\alpha)$ is a $\mathbb{P} \restriction \alpha$-name for an element of a forcing which collapses $2^\omega = \alpha$.

Mitchell's forcing

Let κ be a measurable cardinal.

Ideas of the construction:

- Avoid CH.
- End up with $2^\omega = \kappa = \omega_2$.
- Somehow prove that the tree property holds.

Vague definition of the forcing: We call the forcing $\mathbb{M}$.
Conditions are pairs (p, q) such that

- p is in $\mathbb{P} = \mathrm{Add}(\omega, \kappa)$
- q is a partial function whose domain is a subset of κ
- for each α in the domain of q, $q(\alpha)$ is a $\mathbb{P} \restriction \alpha$-name for an element of a forcing which collapses $2^\omega = \alpha$.

We let $(p_1, q_1) \leq (p_2, q_2)$ if

- $p_1 \leq p_2$,

Mitchell's forcing

Let κ be a measurable cardinal.

Ideas of the construction:

- Avoid CH.
- End up with $2^\omega = \kappa = \omega_2$.
- Somehow prove that the tree property holds.

Vague definition of the forcing: We call the forcing $\mathbb{M}$.
Conditions are pairs (p, q) such that

- p is in $\mathbb{P} = \mathrm{Add}(\omega, \kappa)$
- q is a partial function whose domain is a subset of κ
- for each α in the domain of q, $q(\alpha)$ is a $\mathbb{P} \restriction \alpha$-name for an element of a forcing which collapses $2^\omega = \alpha$.

We let $(p_1, q_1) \leq (p_2, q_2)$ if

- $p_1 \leq p_2$,
- $\mathrm{dom}(q_1) \supseteq \mathrm{dom}(q_2)$ and

Mitchell's forcing

Let κ be a measurable cardinal.

Ideas of the construction:

- Avoid CH.
- End up with $2^\omega = \kappa = \omega_2$.
- Somehow prove that the tree property holds.

Vague definition of the forcing: We call the forcing $\mathbb{M}$.
Conditions are pairs (p, q) such that

- p is in $\mathbb{P} = \mathrm{Add}(\omega, \kappa)$
- q is a partial function whose domain is a subset of κ
- for each α in the domain of q, $q(\alpha)$ is a $\mathbb{P} \restriction \alpha$-name for an element of a forcing which collapses $2^\omega = \alpha$.

We let $(p_1, q_1) \leq (p_2, q_2)$ if

- $p_1 \leq p_2$,
- $\mathrm{dom}(q_1) \supseteq \mathrm{dom}(q_2)$ and
- if $\alpha \in \mathrm{dom}(q_2)$, then $p_1 \restriction \alpha \Vdash q_1(\alpha) \leq q_2(\alpha)$.

The tree property at ω_2

We just sketch the proof.

The tree property at ω_2

We just sketch the proof.

1. Let T be an ω_2-tree in $V[\mathbb{M}]$.

The tree property at ω_2

We just sketch the proof.

1. Let T be an ω_2-tree in $V[\mathbb{M}]$.
2. Let $j : V \to N$ witness that κ is measurable.

The tree property at ω_2

We just sketch the proof.

1. Let T be an ω_2-tree in $V[\mathbb{M}]$.
2. Let $j : V \to N$ witness that κ is measurable.
3. By using a similar argument to the one given above, T has a cofinal branch in the model $N[j(\mathbb{M})]$.

The tree property at ω_2

We just sketch the proof.

1. Let T be an ω_2-tree in $V[\mathbb{M}]$.
2. Let $j : V \to N$ witness that κ is measurable.
3. By using a similar argument to the one given above, T has a cofinal branch in the model $N[j(\mathbb{M})]$.
4. The tree T is a member of $N[\mathbb{M}]$, but the forcing $j(\mathbb{M})/\mathbb{M}$ which takes us from $N[\mathbb{M}]$ up to $N[j(\mathbb{M})]$ could not have added the cofinal branch.

The tree property at ω_2

We just sketch the proof.

1. Let T be an ω_2-tree in $V[\mathbb{M}]$.
2. Let $j : V \to N$ witness that κ is measurable.
3. By using a similar argument to the one given above, T has a cofinal branch in the model $N[j(\mathbb{M})]$.
4. The tree T is a member of $N[\mathbb{M}]$, but the forcing $j(\mathbb{M})/\mathbb{M}$ which takes us from $N[\mathbb{M}]$ up to $N[j(\mathbb{M})]$ could not have added the cofinal branch.
5. So the tree property holds at ω_2 in $V[\mathbb{M}]$.

Questions

Question

Is it consistent that all regular cardinals greater than $\aleph_1$ *have the tree property?*

Questions

Question

Is it consistent that all regular cardinals greater than $\aleph_1$ have the tree property?

This question is too hard. So a better question is:

Questions

Question

Is it consistent that all regular cardinals greater than $\aleph_1$ have the tree property?

This question is too hard. So a better question is:

Question

What is the largest initial segment of regular cardinals which can have the tree property?

Successive cardinals with the tree property

Theorem (Abraham)

If there is a supercompact cardinal with a weakly compact cardinal above it, then it is consistent that $\aleph_2$ *and* $\aleph_3$ *have the tree property simultaneously.*

Successive cardinals with the tree property

Theorem (Abraham)

If there is a supercompact cardinal with a weakly compact cardinal above it, then it is consistent that $\aleph_2$ and $\aleph_3$ have the tree property simultaneously.

Theorem (Cummings and Foreman)

If there are infinitely many supercompact cardinals, then it is consistent that simultaneously for all $n \geq 2$, $\aleph_n$ has the tree property.

Successive cardinals with the tree property

Theorem (Abraham)

If there is a supercompact cardinal with a weakly compact cardinal above it, then it is consistent that $\aleph_2$ and $\aleph_3$ have the tree property simultaneously.

Theorem (Cummings and Foreman)

If there are infinitely many supercompact cardinals, then it is consistent that simultaneously for all $n \geq 2$, $\aleph_n$ has the tree property.

Theorem (Neeman)

Assuming that there are ω supercompact cardinals it is consistent that all regular cardinals in the interval $[\aleph_2, \aleph_{\omega+1}]$ have the tree property.

Successors of a singular cardinal

Theorem (Gitik and Sharon)

Assuming the existence of a supercompact cardinal, it is consistent that there is a singular strong limit cardinal κ of cofinality ω such that $2^\kappa = \kappa^{++}$ and there are no special κ^+-trees.

Successors of a singular cardinal

Theorem (Gitik and Sharon)

Assuming the existence of a supercompact cardinal, it is consistent that there is a singular strong limit cardinal κ of cofinality ω such that $2^\kappa = \kappa^{++}$ and there are no special κ^+-trees.

Theorem (Cummings and Foreman)

Assuming that there is a supercompact cardinal with a weakly compact cardinal above it, it is consistent that there is a singular strong limit cardinal κ of cofinality ω such that κ^{++} has the tree property.

Successors of a singular cardinal

Theorem (Gitik and Sharon)

Assuming the existence of a supercompact cardinal, it is consistent that there is a singular strong limit cardinal κ of cofinality ω such that $2^\kappa = \kappa^{++}$ and there are no special κ^+-trees.

Theorem (Cummings and Foreman)

Assuming that there is a supercompact cardinal with a weakly compact cardinal above it, it is consistent that there is a singular strong limit cardinal κ of cofinality ω such that κ^{++} has the tree property.

Theorem (Neeman)

Assuming that there are ω supercompact cardinals it is consistent that there is a singular strong limit cardinal κ of cofinality ω such that $2^\kappa = \kappa^{++}$ and κ^+ has the tree property.

Successors of singulars continued

Theorem (U)

Assuming that there is a supercompact cardinal with a weakly compact cardinal above it, it is consistent that there is a singular strong limit cardinal κ of cofinality ω such that

1. $2^\kappa = \kappa^{++}$,
2. *there are no special κ^+-trees and*
3. *κ^{++} has the tree property.*

A few words on the proof

Let κ be supercompact and $\lambda > \kappa$ be weakly compact.

A few words on the proof

Let κ be supercompact and $\lambda > \kappa$ be weakly compact.

- The key idea is to replace the use of $\mathrm{Add}(\omega, \kappa)$ in Mitchell's forcing with the two step iteration of $\mathrm{Add}(\kappa, \lambda) * \mathbb{D}$ where $\mathbb{D}$ is diagonal Prikry forcing.

A few words on the proof

Let κ be supercompact and $\lambda > \kappa$ be weakly compact.

- The key idea is to replace the use of $\mathrm{Add}(\omega, \kappa)$ in Mitchell's forcing with the two step iteration of $\mathrm{Add}(\kappa, \lambda) * \mathbb{D}$ where $\mathbb{D}$ is diagonal Prikry forcing.
- The rest of the proof can be seen as working to recover analogous properties to Mitchell's original forcing.

A few words on the proof

Let κ be supercompact and $\lambda > \kappa$ be weakly compact.

- The key idea is to replace the use of $\mathrm{Add}(\omega, \kappa)$ in Mitchell's forcing with the two step iteration of $\mathrm{Add}(\kappa, \lambda) * \mathbb{D}$ where $\mathbb{D}$ is diagonal Prikry forcing.
- The rest of the proof can be seen as working to recover analogous properties to Mitchell's original forcing.
- Fortunately, much of this work is done by the paper of Cummings and Foreman.

A few words on the proof

Let κ be supercompact and $\lambda > \kappa$ be weakly compact.

- The key idea is to replace the use of $\mathrm{Add}(\omega, \kappa)$ in Mitchell's forcing with the two step iteration of $\mathrm{Add}(\kappa, \lambda) * \mathbb{D}$ where $\mathbb{D}$ is diagonal Prikry forcing.
- The rest of the proof can be seen as working to recover analogous properties to Mitchell's original forcing.
- Fortunately, much of this work is done by the paper of Cummings and Foreman.
- Unfortunately, there is also a mistake in that paper at a critical point in the argument.