Duality for sheaves of distributive-lattice-ordered algebras over stably compact spaces

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(joint work with Mai Gehrke)

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This talk in a picture

\[ F(Y) = D \]

\[ X = D_* \]
Let \( \mathcal{V} \) be a variety of abstract algebras, \((Y, \rho)\) a topological space.

Let \((A_y)_{y \in Y}\) be a \(Y\)-indexed family of \(\mathcal{V}\)-algebras.

Let \(E := \bigsqcup_{y \in Y} A_y\), with \(p : E \to Y\) the natural surjection.

Suppose \(\tau\) is a topology on \(E\) such that \(p : (E, \tau) \to (Y, \rho)\) is a local homeomorphism: any point has an open neighbourhood on which \(p\) has a right inverse.

\(p : (E, \tau) \to (Y, \rho)\) is called an \textit{étale space} of \(\mathcal{V}\)-algebras.
Sheaf from an étale space

- Let $p : (E, \tau) \to (Y, \rho)$ be an étale space of $\mathcal{V}$-algebras.
- For any $U \in \rho$, write $F(U)$ for the set of local sections over $U$:

  $$F(U) := \{s : U \to E \text{ continuous s.t. } p \circ s = \text{id}_U\}.$$ 

- Note: $F(U)$ is a $\mathcal{V}$-algebra (being a subalgebra of $\prod_{y \in U} A_y$).
- If $U \subseteq V$, there is a natural restriction map $F(V) \to F(U)$.
- $F$ is called the sheaf associated with $p$. 
Definition of sheaf

- In general, a sheaf $F$ on $Y$ consists of the data:
  - For each open $U$, a $\mathcal{V}$-algebra $F(U)$ ("local sections");
  - For each open $U \subseteq V$, a $\mathcal{V}$-homomorphism
    $$(\cdot)|_U : F(V) \to F(U)$$ ("restriction maps");

  such that the appropriate diagrams commute, satisfying the following patching property:
  - For any open cover $(U_i)_{i \in I}$ of an open set $U$, $(s_i)_{i \in I}$ a 
    "compatible family" of local sections, i.e., $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$
    for all $i, j \in I$.
  - there exists a unique $s \in F(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

- $F(Y)$ is called the algebra of global sections of the sheaf $F$. 
Sheaves vs. étale spaces

Fact

Any sheaf arises from an étale space, and vice versa.
Let $A$ be an abstract algebra.

A **Boolean product representation** of $A$ is a sheaf $F$ on a Boolean space $Y$ such that $A$ is isomorphic to the algebra of global sections of $F$.

Equivalent: a subdirect embedding $A \hookrightarrow \prod_{y \in Y} A_y$ satisfying:

1. **(Open equalizers)** For any $a, b \in A$, the equalizer $\parallel a = b \parallel := \{ y \in Y \mid a_y = b_y \}$ is open;
2. **(Patch)** For $K$ clopen in $Y$, $a, b \in A$, there exists $c \in A$ such that $a|_K = c|_K$ and $b|_{K^c} = c|_{K^c}$. 
Boolean product, pictorially
Boolean product, pictorially

\[ F \]

\[ y \]

\[ Y \]
Boolean product, pictorially

\[ A/y \]

\[ F \]

\[ y \]

\[ Y \]
Boolean product, pictorially

\[ A \setminus y \]

\[ F \]

\[ y \]

\[ y' \]
Boolean product, pictorially
Boolean product, pictorially
Boolean product, pictorially

\[
\begin{align*}
A/y & \quad \quad A/y' \\
\end{align*}
\]
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Boolean product, pictorially

\[
\begin{align*}
A/y & \quad A/y' \\
\text{Legend:} & \\
\text{Red:} & \quad b \\
\text{Green:} & \quad a, c \\
\text{Blue:} & \quad F
\end{align*}
\]
Lattices of congruences

Theorem (Comer 1971, Burris & Werner 1980)

Boolean product representations of $A$ are in a natural one-to-one correspondence with relatively complemented distributive lattices of permuting congruences on $A$. 
Boolean sum decompositions

- Let $D$ be a distributive lattice.

**Theorem (Gehrke 1991)**

Boolean product representations $D \hookrightarrow \prod_{y \in Y} D_y$ are in a natural one-to-one correspondence with **Boolean sum decompositions** of the Stone dual space $X$ of $D$ into the Stone dual spaces $(X_y)_{y \in Y}$ of the lattices $(D_y)_{y \in Y}$.

- Also see [Hansoul & Vrancken-Mawet 1984] for a version for the Priestley dual spaces.
Dual characterization, pictorially

\[ F(Y) = D \]
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Question

- What if $Y$ is no longer a Boolean space?
Motivation

- Many interesting sheaf representations use a base space which is spectral or compact Hausdorff.
- **Stably compact spaces** form a common generalization of these two classes.
Stably compact spaces

“Generalisation of compact Hausdorff to $T_0$-setting”

Definition

Stably compact space =

- $T_0$, 
- Sober,
- Locally compact,
- Intersection of compact saturated is compact.
De Groot dual and patch topology

- For any topological space \((Y, \rho)\), define its de Groot dual

\[
\rho^\partial := \langle U \subseteq Y \mid Y \setminus U \text{ is compact saturated in } \rho \rangle_{\text{top}}
\]
De Groot dual and patch topology

- For any topological space \((Y, \rho)\), define its **de Groot dual**

  \[
  \rho^\partial := \langle U \subseteq Y \mid Y \setminus U \text{ is compact saturated in } \rho \rangle_{\text{top}}
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- **Fact:** If \((Y, \rho)\) is stably compact, then so is \(Y^\partial := (Y, \rho^\partial)\).
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- **Fact:** If \((Y, \rho)\) is stably compact, then so is \(Y^\partial := (Y, \rho^\partial)\).

- Define \(\rho^p := \rho \vee \rho^\partial\), the patch topology.
De Groot dual and patch topology

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Define \(\rho^p := \rho \lor \rho^\partial\), the patch topology.

Fact: \((Y, \rho^p)\) is a compact Hausdorff space.
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- **Fact:** \((Y, \rho^p)\) is a compact Hausdorff space.

- Let \(y \leq y' \iff y' \in \overline{\{y\}}\), the specialization order of \(\rho\).
De Groot dual and patch topology

- For any topological space \((Y, \rho)\), define its De Groot dual

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- **Fact:** If \((Y, \rho)\) is stably compact, then so is \(Y^\partial := (Y, \rho^\partial)\).
- Define \(\rho^p := \rho \lor \rho^\partial\), the **patch** topology.
- **Fact:** \((Y, \rho^p)\) is a compact Hausdorff space.
- Let \(y \leq y' \iff y' \in \overline{\{y\}}\), the **specialization order** of \(\rho\).
- **Fact:** \(\leq\) is a closed subspace of \((Y \times Y, \rho^p \times \rho^p)\).
De Groot dual and patch topology

- For any topological space \((Y, \rho)\), define its de Groot dual

\[ \rho^\partial := \langle U \subseteq Y \mid Y \setminus U \text{ is compact saturated in } \rho \rangle_{\text{top}} \]

- **Fact:** If \((Y, \rho)\) is stably compact, then so is \(Y^\partial := (Y, \rho^\partial)\).

- Define \(\rho^p := \rho \lor \rho^\partial\), the patch topology.

- **Fact:** \((Y, \rho^p)\) is a compact Hausdorff space.

- Let \(y \leq y' \iff y' \in \overline{\{y\}}\), the specialization order of \(\rho\).

- **Fact:** \(\leq\) is a closed subspace of \((Y \times Y, \rho^p \times \rho^p)\).

- So \((Y, \rho^p, \leq)\) is a compact ordered space (Nachbin 1965).
Conversely, given a compact ordered space \((Y, \pi, \leq)\), let \(\pi^\downarrow\) the topology of open down-sets.

Then \((Y, \pi^\downarrow)\) is a stably compact space, and \((\pi^\downarrow)^\partial = \pi^\uparrow\).
Conversely, given a compact ordered space \((Y, \pi, \leq)\), let \(\pi^\downarrow\)
the topology of open down-sets.

Then \((Y, \pi^\downarrow)\) is a stably compact space, and \((\pi^\downarrow)^\partial = \pi^\uparrow\).

**Fact**

_The categories of stably compact spaces and compact ordered spaces are isomorphic._
Representing stably compact spaces

Example (Open basis presentation)

- $X$ stably compact space
- $D$ lattice-basis of open sets for $X$
- Define $U \preceq R \preceq V$ iff there exists compact saturated $K \subseteq X$ such that $U \subseteq K \subseteq V$
- **Fact:** $X$ can be recovered as the space of “round prime ideals” of $R$. 
Fact (Johnstone, 1982)

A topological space $X$ is stably compact iff
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A topological space $X$ is stably compact iff there exists a spectral space $Y$.
Fact (Johnstone, 1982)

A topological space $X$ is stably compact iff there exists a spectral space $Y$ and a continuous retraction of $Y$ onto $X$. 
Duality for spectral spaces with continuous maps

- **Fact:** $DL_j \cong^{op} SpecSp_c$
- Here, $SpecSp_c$: spectral spaces with **continuous** maps,
- and $DL_j$: distributive lattices with $j$-morphisms:

**Definition**

A relation $H \subseteq D \times E$ between distributive lattices $D$ and $E$ is called a **$j$-morphism** iff:

- $\geq \circ H \circ \geq = H$
- $a H \lor B \iff \forall b \in B \ aHb$
- $\forall A H b \iff \forall a \in A \ aHb$
- If $\lor A H b$ then $\exists B \subseteq \omega H[A]$ such that $b \leq \lor B$. 
Duality for stably compact spaces

Definition

A join-strong proximity lattice is a pair \((D, R)\) where \(D\) is a distributive lattice, \(R^{-1}: D \to D\) is a j-morphism, and \(R \circ R = R\).
Duality for stably compact spaces

Definition
A join-strong proximity lattice is a pair \((D, R)\) where \(D\) is a distributive lattice, \(R^{-1} : D \to D\) is a j-morphism, and \(R \circ R = R\).

Fact
The categories of stably compact spaces and join-strong proximity lattices are equivalent.
Duality for stably compact spaces

**Definition**

A **join-strong proximity lattice** is a pair \((D, R)\) where \(D\) is a distributive lattice, \(R^{-1} : D \rightarrow D\) is a j-morphism, and \(R \circ R = R\).

**Fact**

*The categories of stably compact spaces and join-strong proximity lattices are equivalent.*

**Proof.**

- Stably compact spaces are retracts of spectral spaces.
Duality for stably compact spaces

**Definition**

A **join-strong proximity lattice** is a pair \((D, R)\) where \(D\) is a distributive lattice, \(R^{-1} : D \rightarrow D\) is a j-morphism, and \(R \circ R = R\).

**Fact**

*The categories of stably compact spaces and join-strong proximity lattices are equivalent.*

**Proof.**

- Stably compact spaces are retracts of spectral spaces.
- Therefore, duals of stably compact spaces are retracts of distributive lattices in the category \(DL_j\).
The case of MV-algebras

Theorem (Gehrke, Marra, vG 2012)

The Priestley dual space $X$ of the distributive lattice underlying an MV-algebra $A$ decomposes as a stably compact sum over the base space $Y$ of prime MV ideals of $A$. 
Stably compact sum decompositions

**Definition**

A **stably compact sum decomposition** of a Priestley space $X$ is a continuous surjection $q : X \to Y^\partial$, with $Y$ stably compact, satisfying the following dual patching property:

**P** Let $(U_i)_{i=1}^n$ be any finite cover of $Y$ by $\rho^\partial$-open sets, and let $(\hat{a}_i)_{i=1}^n$ be any finite collection of clopen downsets of $X$ such that

$$\hat{a}_i \cap q^{-1}(U_i \cap U_j) = \hat{a}_j \cap q^{-1}(U_i \cap U_j)$$

holds for any $i, j \in \{1, \ldots, n\}$. Then the set

$$\bigcup_{i=1}^n (\hat{a}_i \cap q^{-1}(U_i))$$

is a clopen downset in $X$. 
Property (P), pictorially
Property (P), pictorially

\[
X \quad Y^\partial \quad U_1
\]

\[
\hat{a}_2 \quad \hat{a}_1 \quad U_1 \quad U_2 \quad y \quad X \quad y \quad \bigcup_{i=1} \left( \hat{a}_i \cap q^{i-1} \left( U_i \right) \right)
\]
Property (P), pictorially
Property (P), pictorially

\[ X \bigcup_{n=i=1}^{\infty} (\hat{a_i} \cap q^{i-1}(U_i)) \]
Property (P), pictorially
Property (P), pictorially

\[ \bigcup_{i=1}^{n} (\hat{a}_i \cap q^{-1}(U_i)) \]
Spectral sum yields sheaf

Theorem (Gehrke, Marra, vG 2012)

If $X$ is the Priestley space of a distributive lattice $A$, then any stably compact sum decomposition $q : X \to Y^\partial$ yields a sheaf representation of $A$ over $Y$.

Example

For an MV-algebra $A$, there are two natural stably compact sum decompositions of the dual space $X$, each of which yields a sheaf representation of $A$: one over its prime, the other over its maximal spectrum.
Fitted sheaves

- Question: which sheaves can be captured by such a decomposition?
Fitted sheaves

- Question: **which** sheaves can be captured by such a decomposition?
- Let $B$ a basis for the base space $Y$.
- Call a sheaf $F$ **fitted for $B$** if, for each $U \in B$, the restriction map $F(Y) \to F(U)$ is surjective.
- ("Fitted for $\mathcal{O}(Y)$" = “flabby” or “flasque”...)
Lattices of congruences, revisited

- Let $F$ be a sheaf of distributive lattices on a topological space $Y$ which is fitted for a lattice basis $B$ for $Y$ with $A := F(Y)$.
- For $U \in B$, define $\theta_F(U) := \ker(F(Y) \twoheadrightarrow F(U))$.

**Proposition**

The function $\theta_F : B^{\text{op}} \rightarrow \text{Con}_{\text{DL}}(A)$ is a lattice homomorphism, and any two congruences in the image of $\theta_F$ permute.
Given a sheaf $F$ fitted for $B$, lift this lattice homomorphism

$$\theta_F : B^{\text{op}} \to \text{Con}_{\text{DL}}(A)$$

to $\overline{\theta}_F : \mathcal{O}(Y^{\partial}) \to \text{Con}_{\text{DL}}(A)$.

Note that $\text{Con}_{\text{DL}}(A) \cong \mathcal{O}(X)$, where $X$ is the Priestley dual space of the distributive lattice $A$.

By pointless duality, we obtain a continuous map $q : X \to Y^{\partial}$. 
Lemma (Lifting)

Suppose that $B$ is a lattice basis for the open sets of a stably compact space $Y$ and that $h : B^{\text{op}} \to F$ is a lattice homomorphism from $B^{\text{op}}$ into a frame $F$. Then the function $\overline{h} : \mathcal{O}(Y^\partial) \to F$ defined by

$$\overline{h}(W) := \bigvee \{ h(U) \mid U \in B, \; U^c \subseteq W \}$$

is a frame homomorphism.

- Proof based on strong proximity lattice of $(O, K)$-pairs by Jung & Sünderhauf (1996).
Proof of lifting lemma

- To show: $\overline{h}(W) := \bigvee \{ h(U) \mid U \in B, \; U^c \subseteq W \}$ preserves $\bigvee$. 
Proof of lifting lemma

- To show: \( \overline{h}(W) := \bigvee \{ h(U) \mid U \in B, \ U^c \subseteq W \} \) preserves \( \bigvee \).
- Enough: \( \overline{h}(\bigcup_{i \in I} W_i) \leq \bigvee_{i \in I} \overline{h}(W_i) \).
Proof of lifting lemma

- To show: $\overline{h}(W) := \bigvee\{h(U) \mid U \in B, \ U^c \subseteq W\}$ preserves $\bigvee$.
- Enough: $\overline{h}(\bigcup_{i \in I} W_i) \leq \bigvee_{i \in I} \overline{h}(W_i)$.
- From the fact that $B$ is a basis, deduce that
  $W_i = \bigcup\{V \in O(Y^\partial) \mid \exists U \in B : V \subseteq U^c \subseteq W_i\}$. 
Proof of lifting lemma

- To show: $\overline{h}(W) := \bigvee \{ h(U) \mid U \in B, \ U^c \subseteq W \}$ preserves $\bigvee$.
- Enough: $\overline{h}(\bigcup_{i \in I} W_i) \leq \bigvee_{i \in I} \overline{h}(W_i)$.
- From the fact that $B$ is a basis, deduce that
  \[ W_i = \bigcup \{ V \in \mathcal{O}(Y^\partial) \mid \exists U \in B : V \subseteq U^c \subseteq W_i \}. \]
- So, if $U \in B$ and $U^c \subseteq \bigcup_{i \in I} W_i$, by compactness pick finite
  cover $\mathcal{F} \subseteq \{ V \in \mathcal{O}(Y^\partial) \mid \exists i \in I, U \in B : V \subseteq U^c \subseteq W_i \}$. 
Proof of lifting lemma

- To show: \( \overline{h}(W) := \bigvee \{ h(U) \mid U \in B, \quad U^c \subseteq W \} \) preserves \( \bigvee \).
- Enough: \( \overline{h}(\bigcup_{i \in I} W_i) \leq \bigvee_{i \in I} \overline{h}(W_i) \).
- From the fact that \( B \) is a basis, deduce that
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- So, if \( U \in B \) and \( U^c \subseteq \bigcup_{i \in I} W_i \), by compactness pick finite cover \( \mathcal{F} \subseteq \{ V \in \mathcal{O}(Y^\partial) \mid \exists i \in I, U \in B : V \subseteq U^c \subseteq W_i \} \).
- For each \( V \in \mathcal{F} \), pick \( U_V \in B, \quad i_V \in I \), with
  \[ V \subseteq (U_V)^c \subseteq W_{i_V} \quad \text{and} \quad U^c \subseteq \bigcup \mathcal{F}. \]
Proof of lifting lemma

- To show: \( \overline{h}(W) := \bigvee \{ h(U) \mid U \in B, \ U^c \subseteq W \} \) preserves \( \bigvee \).
- Enough: \( \overline{h}(\bigcup_{i \in I} W_i) \leq \bigvee_{i \in I} \overline{h}(W_i) \).
- From the fact that \( B \) is a basis, deduce that \( W_i = \bigcup \{ V \in \mathcal{O}(Y^\partial) \mid \exists U \in B : V \subseteq U^c \subseteq W_i \} \).
- So, if \( U \in B \) and \( U^c \subseteq \bigcup_{i \in I} W_i \), by compactness pick finite cover \( \mathcal{F} \subseteq \{ V \in \mathcal{O}(Y^\partial) \mid \exists i \in I, U \in B : V \subseteq U^c \subseteq W_i \} \).
- For each \( V \in \mathcal{F} \), pick \( U_V \in B, i_V \in I \), with \( V \subseteq (U_V)^c \subseteq W_{i_V} \) and \( U^c \subseteq \bigcup \mathcal{F} \).
- Then \( h(U) \leq h(\bigcap_{V \in \mathcal{F}} U_V) \leq \bigvee_{V \in \mathcal{F}} h(U_V) \leq \bigvee_{i \in I} \overline{h}(W_i) \).
The decomposition map

Let $\overline{\theta_F} : \mathcal{O}(Y^\partial) \rightarrow \text{Con}_{\text{DL}}(A)$ be the frame homomorphism associated to a sheaf $F$. 

See Mai Gehrke's talk yesterday afternoon.
The decomposition map

- Let $\overline{\theta}_F : \mathcal{O}(Y^\partial) \to \text{Con}_{DL}(A)$ be the frame homomorphism associated to a sheaf $F$.
- The function $p : (X, \tau^p) \to (Y, \rho^\partial)$ dual to $\overline{\theta}_F$ is given by:

$$p(x) = \max\{y \in Y \mid x \in X_y\}.$$
The decomposition map

- Let $\overline{\theta}_F : \mathcal{O}(Y^\partial) \to \text{Con}_{DL}(A)$ be the frame homomorphism associated to a sheaf $F$.

- The function $p : (X, \tau^p) \to (Y, \rho^\partial)$ dual to $\overline{\theta}_F$ is given by:

$$p(x) = \max\{y \in Y \mid x \in X_y\}.$$ 

- This shows that an analogue of the function $k$ from MV-algebras$^1$ is available in the context of any fitted sheaf representation!

$^1$See Mai Gehrke’s talk yesterday afternoon.
Theorem (Gehrke, vG 2013)

*A fitted sheaf representation of a distributive lattice \( A \) over a stably compact space \( Y \) yields a stably compact sum decomposition of the Priestley dual space \( X \) of \( A \) over \( Y^{\partial} \).*
This talk in a picture

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\[ X = D_* \]
### Analogy with Boolean case

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Further work

- To retrieve the topology of the dual space from the topologies on the subspaces and on the base space;
- To apply these results to more general and to other classes of lattice-ordered algebras.
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