Proof of L'Hospital's Rule

Theorem: Suppose $f'(x), g'(x)$ exist and $g'(x) \neq 0$ for all x in an interval $(a, b]$.

$$\text{If } \lim_{x\to a^+} f(x) = 0 = \lim_{x\to a^+} g(x) \text{ and } \lim_{x\to a^+} \frac{f'(x)}{g'(x)} \text{ exists then } \lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}.$$

Proof: We may assume that $f(a) = 0 = g(a)$ (since the limit is not affected by the value of the function at a). Also $g(b) \neq 0$, else $g'(x) = 0$ at some $x \in (a, b)$ by Rolle's Theorem.

Define $h(x) = f(x) - \frac{f(b)}{g(b)} g(x)$, then $h(a) = 0 = h(b)$, and h is continuous on $[a, b]$ and $h'(x) = f'(x) - \frac{f(b)}{g(b)} g'(x)$ exists on (a, b).

By Rolle's Theorem there exist $x \in (a, b)$ such that $h'(x) = 0$, hence $\frac{f'(x)}{g'(x)} = \frac{f(b)}{g(b)}$.

Since $a < x < b$, it follows that $\lim_{b\to a^+} \frac{f(b)}{g(b)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}$. □

The theorem can be adapted for $x \to a^-$, $x \to a$, $x \to \pm\infty$, or $\lim f(x) = \pm\infty = \lim g(x)$.

It does **not** apply e.g. if $x \to 0^+$, $f(x) = 1/x + \sin(1/x)$ and $g(x) = 1/x$ (can you see why?)