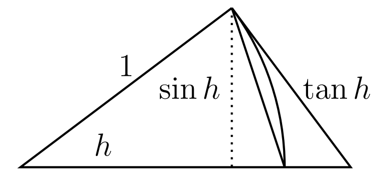
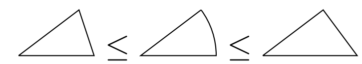
Derivative of $\sin x$

Since $\sin(x+y) = \sin x \cos y + \cos x \sin y$ we have

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \left(\frac{\cos x \sin h}{h} + \frac{\sin x \cos h - \sin x}{h} \right) \\
&= \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h} + \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} \\
&= \cos x \cdot 1 + \sin x \cdot 0 \\
&= \cos x
\end{aligned}$$





$$\tfrac{1}{2}\sin h \le \tfrac{1}{2}h \le \tfrac{1}{2}\tan h$$

$$\Rightarrow 1 \le \frac{h}{\sin h} \le \frac{1}{\cos h}$$

$$\Rightarrow \cos h \le \frac{\sin h}{h} \le 1$$

$$\lim_{h\to 0} \cos h = 1 \Rightarrow \boxed{\lim_{h\to 0} \frac{\sin h}{h} = 1}$$

$$\begin{aligned}
\lim_{h\to 0} \frac{\cos h - 1}{h} &= \lim_{h\to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)} \\
&= \lim_{h\to 0} \frac{-\sin^2 h}{h(\cos h + 1)} \\
&= \lim_{h\to 0} \frac{\sin h}{h} \cdot \frac{-\sin h}{\cos h + 1} \\
&= 1 \cdot \frac{-0}{1+1} = 0
\end{aligned}$$

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