$P_3$-Isomorphisms for Graphs

R. E. L. Aldred*

1 DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF OTAGO
P.O. BOX 56
DUNEDIN, NEW ZEALAND
E-mail: raldred@maths.otago.ac.nz

M. N. Ellingham†

2 DEPARTMENT OF MATHEMATICS
1326 STEVENSON CENTER
VANDERBILT UNIVERSITY
NASHVILLE, TENNESSEE 37240, U.S.A.
E-mail: mne@math.vanderbilt.edu

R. L. Hemminger‡

3 DEPARTMENT OF MATHEMATICS
1326 STEVENSON CENTER
VANDERBILT UNIVERSITY
NASHVILLE, TENNESSEE 37240, U.S.A.
E-mail: hemminrl@math.vanderbilt.edu

P. Jipsen

4 DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS
UNIVERSITY OF CAPE TOWN
PRIVATE BAG
RONDEBOSCH 7700, SOUTH AFRICA
E-mail: pjipsen@maths.uct.ac.za

Received June 16, 1996

*Acknowledges partial support by PGSF scheme of New Zealand FRST, and hospitality of Vanderbilt University.
†Acknowledges partial support by PGSF scheme of New Zealand FRST, and hospitality of University of Otago.
‡Acknowledges partial support by NSF Grant # INT-9221418, and hospitality of University of Otago.

© 1997 John Wiley & Sons, Inc. CCC 0364-9024/97/010035-17
Abstract: The $P_3$-graph of a finite simple graph $G$ is the graph whose vertices are the 3-vertex paths of $G$, with adjacency between two such paths whenever their union is a 4-vertex path or a 3-cycle. In this paper we show that connected finite simple graphs $G$ and $H$ with isomorphic $P_3$-graphs are either isomorphic or part of three exceptional families. We also characterize all isomorphisms between $P_3$-graphs in terms of the original graphs. © 1997 John Wiley & Sons, Inc. J Graph Theory 26: 35–51, 1997

Keywords: isomorphism, path graph, line graph, $P_3$-graph

1. INTRODUCTION

All graphs in this paper are simple and finite, and any notation not found here may be found in Bondy and Murty [1]. In [2], Broersma and Hoede generalized the idea of line graphs to path graphs by defining adjacency as follows. Let $P_k$ and $C_k$ denote respectively a path and a cycle with $k$ vertices. Let $\Pi_k(G)$ be the set of all $P_k$'s in $G$, and let $P_k(G)$, the $P_k$-graph of $G$, be the graph on vertex set $\Pi_k(G)$ in which two $P_k$'s are adjacent when their union is a $P_{k+1}$ or $C_k$. Note that $P_2(G)$ is just the line graph $L(G)$. For a given $k$, there are two natural questions. First, which graphs are $P_k$-graphs? Second, if $P_k(G)$ is isomorphic to $P_k(H)$, are $G$ and $H$ necessarily isomorphic, and if not, how are they related? For $k = 2$, i.e., line graphs, the first problem, of characterization, was solved in different ways by several people, starting with Krausz [4] in 1943—see [3] for details. The second problem, of determination, was solved by Whitney [8] in 1932; he showed that, with four small exceptions, an edge isomorphism between connected graphs is induced by an isomorphism. (See also [3] for a simpler proof of this and for related material.)

In [2] Broersma and Hoede focused on the case $k = 3$ and characterized the graphs that are $P_3$-graphs; a problem with their characterization was resolved by Li and Lin [5]. Broersma and Hoede posed two questions about the hamiltonian properties of $P_3$-graphs that were answered by Yu [9], and also by Yuan and Lin [10]. Broersma and Hoede also showed that the answer to the determination problem for $k = 3$ was not as simple as for $k = 2$, by giving infinite families of nonisomorphic connected graphs with minimum degree 1 which are $P_3$-isomorphic but not isomorphic. However, it seemed possible that $P_3(G)$ might determine $G$ if some kind of minimum degree condition were imposed. This idea was pursued by Li, who showed in [6] that if $G$ and $H$ are connected graphs that both have minimum degree at least 4, or have minimum degree at least 3 and satisfy some extra conditions, then every $P_3$-isomorphism from $G$ to $H$ is induced by an isomorphism. In [7] he extended this, with one exception, to all connected graphs of minimum degree at least 3. He also conjectured that it is true if $G$ and $H$ both have minimum degree at least 2.

In this paper we completely solve the determination problem for $k = 3$ by characterizing all $P_3$-isomorphisms from $G$ to $H$, with no degree or connectedness constraints on $G$ and $H$. In Section 2, we discuss some basic constructions for noninduced $P_3$-isomorphisms and the situation for disconnected graphs. Each of Sections 3, 4 and 5 discusses one of the three nontrivial families of noninduced $P_3$-isomorphisms, and we conclude in Section 6 with our Main Theorem giving a characterization of all $P_3$-isomorphisms between two graphs, at least one of which is connected. Our results show that Li's conjecture for connected graphs of minimum degree 2 is false, and even the weaker conjecture that two $P_3$-isomorphic graphs with minimum degree 2 must be isomorphic is false.
2. THORNS, DIAMONDS, SWAPS AND DISCONNECTED GRAPHS

In this section we establish much of the necessary terminology for our work. We discuss four simple ways of constructing $P_3$-isomorphisms which are not induced by isomorphisms. All depend on the presence of vertices of low degree (1 or 2). We also discuss the situation for disconnected graphs, and conclude that in examining $P_3$-isomorphisms from $G$ to $H$ we can restrict our attention to cases in which at least one of $G$ or $H$ is connected.

As usual we write $G \cong H$ to mean that $G$ is isomorphic to $H$. A $P_k$-isomorphism from $G$ to $H$ is an isomorphism from $P_k(G)$ to $P_k(H)$. A $P_2$-isomorphism is also known as an edge isomorphism. If $\sigma$ is an isomorphism from $G$ to $H$, then $\sigma$ induces a $P_k$-isomorphism $\sigma^*$ from $G$ to $H$, where $\sigma^*(a_1a_2 \cdots a_k) = \sigma(a_1)\sigma(a_2) \cdots \sigma(a_k)$ for all $a_1a_2 \cdots a_k \in \Pi_k(G)$. A $P_k$-isomorphism $\tau$ is induced if $\tau = \sigma^*$ for some isomorphism $\sigma$. If $\tau_i$ is a $P_k$-isomorphism from $G_i$ to $H_i$, $i = 1$ and 2, then we say that $\tau_1$ and $\tau_2$ are equivalent if there are isomorphisms $\sigma$ and $\rho$ from $G_1$ to $G_2$ and $H_1$ to $H_2$, respectively, such that $\tau_1 = (\rho^{-1}) \circ \tau_2 \circ \sigma^*$. Loosely, two $P_k$-isomorphisms are equivalent if they are the same up to isomorphisms of the graphs concerned.

Now we focus on $k = 3$. For any graph $G$ and any vertex $a$ in $G$, let $N(a)$ denote the neighborhood of $a$ in $G$ and let $\deg(a)$ denote the degree of $a$, that is $|N(a)|$. We write $a \sim b$ if $a$ and $b$ are adjacent in $G$, and $a \nsim b$ otherwise. For $\alpha$ in $\Pi_3(G)$, define $N(\alpha)$, $\deg(\alpha)$, $\alpha \sim \beta$ and $\alpha \nsim \beta$ in $P_3(G)$ similarly. Let $m(\alpha)$ denote the middle vertex of $\alpha$, let $S(\alpha)$ be the set of all $P_3$'s with middle vertex $a$, and define $a \perp b$ to be the set of all $P_3$'s in $S(\alpha)$ with an end at $b$, where $a \perp b$ is empty if $a \nsim b$. We call $S(\alpha)$ the star of $G$ at $a$. The set $a \perp b$, if nonempty, is called a bundle with $a$ as its middle and $b$ as its base. If $R \subseteq \Pi_3(G)$ then the $P_3$-isomorphism $\tau$ is said to disperse $R$ if $m(\tau(\alpha)) \neq m(\tau(\beta))$ for some $\alpha, \beta \in R$.

Given two $P_3$'s $\alpha$ and $\beta$ in $G$, the permutation of $\Pi_3(G)$ which swaps $\alpha$ and $\beta$ and fixes everything else is a $P_3$-isomorphism if $N(\alpha) = N(\beta)$. Below we examine four common situations in which this swap is not an induced $P_3$-isomorphism.

The first three situations arise from $P_3$'s with one or both ends of degree 1. A vertex of degree 1 is also called terminal, and an edge is terminal if it has a terminal end. Suppose we are working with a graph $G$. Define an $i$-thorn to be a $P_3$ with exactly $i$ ($i = 1$ or 2) terminal ends in $G$, and a thorn to be a 1- or 2-thorn. A $P_3$ in $G$ is called terminal if it has degree 1 in $P_3(G)$; a terminal $P_3$ must be a 1-thorn with nonterminal end of degree 2. Let $T_i(G)$ be the set of $i$-thorns in $G$.

Our first swap involves 2-thorns. After defining it, we discuss the role of 2-thorns in more detail.

(1) The 2-thorns in $G$ are precisely the isolated vertices of $P_3(G)$. Any swap of two 2-thorns is a $P_3$-isomorphism, which we call a 2-thorn swap, or T-swap for short. It is induced when each 2-thorn is itself a component of $G$, or when both 2-thorns are subgraphs of a single component of $G$ isomorphic to $K_{1,3}$; otherwise, it is not induced. For example, in Figure 1(a) swapping $d_1dd_2$ and $c_1cc_2$ is a noninduced T-swap.

In deriving structural relationships between two graphs based on a $P_3$-isomorphism between them, 2-thorns are almost no help. All $P_3$'s of a connected graph $G$ are 2-thorns if and only if the component is a star, i.e., $K_{1,n}$ for some $n \geq 0$. (When $n = 0$ or 1, the $P_3$-graphs of the stars $K_{1,0} = K_1$ and $K_{1,1} = K_2$ are the empty graph.) In characterizing $P_3$-isomorphisms, we therefore wish to ignore 2-thorns and star components as much as possible. We say that two $P_3$-isomorphisms $\tau_i$ from $G_i$ to $H_i$, $i = 1$ and 2, are T-related if (i) $G_1$ and $G_2$ differ only in their star components, as do $H_1$ and $H_2$; (ii) $|T_2(G_1)| = |T_2(G_2)|$; and (iii) $\tau_1(\alpha) = \tau_2(\alpha)$ for every $\alpha \in \Pi_3(G_1)$, $T_2(G_1) = \Pi_3(G_2)$. Loosely, T-related $P_3$-isomorphisms act on graphs with the same number of 2-thorns and behave identically for non-2-thorns. As a special
case, two $P_3$-isomorphisms $\tau_1$ and $\tau_2$, both from $G$ to $H$, are $T$-related precisely when $\tau_2^{-1} \circ \tau_1$ is the identity or a composition of $T$-swaps. However, the definition is more general than this, as one or both of the pairs $G_1, G_2$ and $H_1, H_2$ may be nonisomorphic.

Our second and third types of swap involve 1-thorns.

(2) Consider two 1-thorns $abc$ and $abd$ where $\deg(a) = 2$ and $\deg(c) = \deg(d) = 1$. Since $N(abc) = N(abd) = a \ldots b$, swapping $abc$ and $abd$ gives a $P_3$-isomorphism, which we call a bundle 1-thorn swap, or $B$-swap for short. It is induced when $a, c$ and $d$ are the only neighbors of $b$; otherwise it is not induced. For example, in Figure 1(a) swapping $g_1dd_1$ and $g_1dd_2$ is a noninduced $B$-swap.

(3) Suppose $abcde$ is a $P_5$ in $G$ with both $abc$ and $cde$ terminal 1-thorns, i.e., $\deg(a) = \deg(e) = 1$ and $\deg(c) = 2$. We call $abc$ and $cde$ a split 1-thorn pair. Since $N(abc) = N(cde) = \{bcd\}$, swapping $abc$ and $cde$ gives a $P_3$-isomorphism, which we call a split 1-thorn swap, or $S$-swap for short. It is induced if $G \cong P_5$; otherwise it is not induced. For example, in Figure 1(b) swapping $v_1vt_1$ and $t_1wv_1$ is an $S$-swap.

The last type of swap arises from certain $P_3$’s with both ends of degree 2. For distinct $a, b \in V(G)$ let $D_{a,b}$ denote the subgraph of $G$ consisting of the union of all $P_3$’s with ends $a$ and $b$ and with middle vertex of degree 2 in $G$. If $D_{a,b}$ is nonempty we call it the diamond with ends $a$ and $b$ and say that two diamonds are adjacent if they have a common end, but nothing else in common. We usually write $V(D_{a,b}) = \{a, b\} \cup \{c_1, c_2, \ldots, c_k\}$ and say that $D_{a,b}$ is a trivial diamond if $k = 1$, and nontrivial otherwise; we call $k$ the width of $D_{a,b}$, and refer to $D_{a,b}$ as a $k$-diamond. Note that if $a \sim b$, the edge $ab$ is not included in $D_{a,b}$. To distinguish the two possibilities, we say that the diamond $D_{a,b}$ is braced if $a \sim b$ and unbraced otherwise.

Note that two diamonds can share an edge, but only if they are determined by a $P_4$ with internal vertices of degree 2 in $G$ or by a $C_3$ with two vertices of degree 2 in $G$. In either case both diamonds are trivial, and the $C_3$ case is characterized by the fact that the overlapping diamonds are $D_{a,b}$ and $D_{a,d}$ for distinct vertices $a, b$ and $d$. 

FIGURE 1. Illustrating swaps, diamond inflations and Whitney type $P_3$-isomorphisms.
Suppose that \(D_{a,b}\) is a nontrivial diamond with vertices labelled as above. For \(1 \leq i < j \leq k\), the \(P_3\)'s \(ac_i\) and \(bc_j\) are called \textit{diamond paths} while the pair of \(P_3\)'s \(c_iac_j\) and \(c_ibc_j\) is called the \textit{diamond pair} associated with \(ac_i\) and \(bc_j\). A \(P_3\) of the form \(cad\) where \(\deg(c) = \deg(d) = 2\) that is not one of a diamond pair (thus \(c\) and \(d\) are in different diamonds) is called a \textit{diamond connector}.

Two diamonds are therefore adjacent if and only if they each contain one (but not the same) edge of the same diamond connector.

Our fourth type of swap is now as follows.

(4) Suppose \(c_iac_j\) and \(c_ibc_j\) are a diamond pair. Since \(N(c_iac_j) = N(c_ibc_j) = \{ac_i,b,ac_j\}\), swapping \(c_iac_j\) and \(c_ibc_j\) gives a \(P_3\)-isomorphism, which we call a \textit{diamond pair swap}, or \textit{D-swap} for short. It is induced if \(G \cong C_4\) and not induced otherwise. For example, in Figure 1(a) swapping \(g_1ag_2\) and \(g_1dg_2\) is a D-swap.

Note that diamond pair swaps provide a multitude of counterexamples to Li’s conjecture [7] that all \(P_3\)-isomorphisms between graphs with minimum degree at least 2 are induced.

In characterizing \(P_3\)-isomorphisms, we need only do this up to T-relation, which takes T-swaps (and more) into account. We also need to take the other three kinds of swap into account. Suppose \(\tau_1\) and \(\tau_2\) are \(P_3\)-isomorphisms from \(G\) to \(H\). We say \(\tau_1\) and \(\tau_2\) are \textit{B-related} if \(\tau_2^{-1} \circ \tau_1\) is the identity or a composition of B-swaps; S-related and D-related are defined similarly. We use joins of these four equivalence relations: for example, two \(P_3\)-isomorphisms are TBSD-related if we can get from one to the other by a chain of zero or more T-, B-, S- and/or D-relations. Other joins will be denoted by analogous notation.

It is natural at this point to ask if all \(P_3\)-isomorphisms are TBSD-related to an induced one. Unfortunately, the answer is no. We have not even accounted for the examples of Broersma and Hoede [2] of nonisomorphic connected graphs with isomorphic \(P_3\)-graphs. If there is a \(P_3\)-isomorphism from \(G\) to \(H\) which is TBSD-related to an induced one, then \(G\) and \(H\) have the same nonstar components, which is not true for the Broersma and Hoede examples.

Before proceeding to examine \(P_3\)-isomorphisms that are not TBSD-related to induced ones, we deal with the case of disconnected graphs. Star components produce only isolated vertices in a \(P_3\)-graph. For the nonstar components, we have the following lemma, whose proof is straightforward.

\textbf{Lemma [Component].} Let \(\alpha\) and \(\beta\) be two \(P_3\)'s in a graph \(G\), neither of which is a 2-thorn, and whose middle vertices are connected by a path in \(G\). Then \(\alpha\) and \(\beta\) are connected by a path in \(P_3(G)\).

Thus, there is a one-to-one correspondence between the nonstar components of \(G\) and the nontrivial components of \(P_3(G)\), and a \(P_3\)-isomorphism \(\tau\) from \(G\) to \(H\) induces a one-to-one correspondence between the nonstar components of \(G\) and \(H\). Suppose \(G_i\) is a nonstar component of \(G\), with counterpart \(H_i\) in \(H\). Then \(\tau\) restricts to a bijection \(\tau_i\) from \(\Pi_3(G_i)\) to \(\Pi_3(H_i)\). If necessary, add one or more stars (copies of \(K_{1,2} = P_3\) always work) to one but not both of \(G_i\) or \(H_i\) to obtain graphs \(G_i^*\) and \(H_i^*\) with the same number of 2-thorns. By putting the 2-thorns of \(G_i^*\) and \(H_i^*\) into correspondence in an arbitrary way, we extend \(\tau_i\) to a \(P_3\)-isomorphism \(\tau_i^*\) from \(G_i^*\) to \(H_i^*\). Understanding each \(\tau_i^*\) is the key to understanding \(\tau\), and so we now make the following assumption.

\textbf{Standing Assumption.} Throughout the rest of the paper we let \(\tau\) be a \(P_3\)-isomorphism from \(G\) to \(H\) where at least one of \(G\) or \(H\) is connected and neither is a star. Thus, each has one nonstar component (which we call \(G_0\) and \(H_0\), respectively), while all other components, if any, are stars. For \(\alpha \in \Pi_3(G)\), we let \(\alpha'\) denote \(\tau(\alpha)\).
3. $K_{3,3}$ AND RELATED GRAPHS

Besides the relations discussed in Section 2, there are three families of graphs which provide noninduced $P_3$-isomorphisms. One is a family of small graphs related to $K_{3,3}$, the second is an infinite family derived from Whitney’s examples of noninduced edge isomorphisms, and the third is obtained from bipartite graphs. In this section we discuss the first family, and show that they correspond to one particular type of noninduced $P_3$-isomorphism.

The following construction shows that even $P_3$-isomorphisms between graphs of minimum degree 3 may not be induced.

**Construction on $K_{3,3}$**. Let vertex sets $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ be a bipartition of $K_{3,3}$ and define $\tau_0 : \Pi_3(K_{3,3}) \to \Pi_3(K_{3,3})$ by $\tau_0(u_i v_j u_k) = u_k v_k u_j$, $\tau_0(v_i u_j v_k) = v_k u_k v_j$, $\tau_0(u_i v_j u_k) = u_j v_j u_k$, and $\tau_0(v_i u_j v_k) = v_k u_k v_j$ for each $i, j, k$ with $\{i, j, k\} = \{1, 2, 3\}$. It is not difficult to check that $\tau_0$ is a noninduced $P_3$-isomorphism of $K_{3,3}$ to itself. Since $K_{3,3}$ has no degree 1 or 2 vertices, $\tau_0$ is TBSD-related only to itself, so it is not TBSD-related to an induced $P_3$-isomorphism, either.

The important property of $\tau = \tau_0$ is the following.

**Situation 1.** Either $\tau$ or $\tau^{-1}$ disperses some bundle with base of degree 3 or more.

For example, $\tau_0(u_1 v_1 u_2)$ and $\tau_0(u_1 v_1 u_3)$ have different middle vertices. For minimum degree at least 3, it turns out that Situation 1 characterizes $\tau_0$. Before verifying this, we give two lemmas with some basic properties that will be useful here and later.

**Lemma [End].** If $\alpha$ and $\beta$ are $P_3$’s in $G$ with a common end (possibly both ends in common) but no common terminal edge, then $\alpha'$ and $\beta'$ have a common end.

**Proof.** Two $P_3$’s have a common end but no common terminal edge if and only if there is a third $P_3$ adjacent to both. Now $\tau$ preserves the latter property and hence the former.

**Lemma [Cycle].** Let $\tau$ be a $P_3$-isomorphism from $G$ to $H$ with at least one of $G$ or $H$ connected.

(i) If $\alpha \beta \gamma \alpha$ is a 3-cycle in $P_3(G)$, then $G$ contains a 3-cycle $C$ with $\Pi_3(C) = \{\alpha, \beta, \gamma\}$.

(ii) If $\alpha \beta \gamma \delta \alpha$ is a 4-cycle in $P_3(G)$, then either $G$ contains a 4-cycle $C$ with $\Pi_3(C) = \{\alpha, \beta, \gamma, \delta\}$, or else $G$ contains an edge ab with $\alpha, \gamma \in a \vdash b$ and $\beta, \delta \in b \vdash a$. Loosely, $\alpha \beta \gamma \delta \alpha$ comes either from a 4-cycle in $G$ or from pairs of $P_3$’s at opposite ends of an edge.

Note that the first possibility is distinguished from the second in that $m(\alpha) \neq m(\gamma)$ in the first.

**Proof.** Obvious.

**Theorem [K3,3].** Let $G$ and $H$ be connected graphs of minimum degree at least 2 and let $\tau$ be a $P_3$-isomorphism from $G$ to $H$. If $\tau$ disperses $a \vdash b$ where $\deg(b) \geq 3$, then $G \cong H \cong K_{3,3}$ and $\tau$ is equivalent to $\tau_0$ as in the Construction on $K_{3,3}$.

**Proof.** Suppose that $m(\alpha'_1) \neq m(\alpha'_2)$ where $\alpha_1 = cab$ and $\alpha_2 = dab$ with $c \neq d$. Let $\beta_1 = abc$ and $\beta_2 = abf$ with $c \neq f$. Note that $m(\alpha'_1) = m(\alpha'_2)$ if and only if $m(\beta'_1) = m(\beta'_2)$ by (ii) of the Cycle Lemma, so anything we prove about $\alpha_1$ and $\alpha_2$ is also true for $\beta_1$ and $\beta_2$. By (ii) of the Cycle Lemma, there is a 4-cycle $u_1 v_1 u_2 v_2 u_1$ in $H$ with $\alpha'_i = v_1 u_i v_2$ and $\beta'_i = u_1 v_i u_2$ for $i = 1, 2$.

Next we note that none of $\alpha_i$, $\alpha'_i$, $\beta_i$, or $\beta'_i$, $i = 1, 2$, is in a 3-cycle in $P_3(G)$ or $P_3(H)$. For, 3-cycles in $P_3(G)$ correspond to 3-cycles in $P_3(H)$ and so, by symmetry, we only need to consider
\( \alpha_1 \). If \( \alpha_1 \) is in a 3-cycle, then we may choose \( \beta_1 \) to be the other member of that 3-cycle that also contains the edge \( ab \). Thus, by (i) of the Cycle Lemma, \( \alpha'_1 \) and \( \beta'_1 \) are in a common 3-cycle in \( P_3(H) \), a contradiction of the above. Consequently, \( a, b, c, d, e \) and \( f \) are all distinct while \( u_1 \neq u_2 \) and \( v_1 \neq v_2 \). Letting \( \alpha_3 = cad \) and \( \beta_3 = ebf \) and using symmetry and the End Lemma, we can assume that there are vertices \( u_3, v_3, x, y \in V(H) \) with \( \alpha'_3 = v_1 u_3 x, \beta'_3 = u_1 v_3 y \) and \( u_3, v_3 \notin \{ u_1, u_2, v_1, v_2 \} \).

Suppose that \( \alpha_4, \beta_4 \in \Pi_3(G) \) with \( \alpha'_4 = v_1 u_1 v_3 \) and \( \beta'_4 = u_1 v_1 u_3 \). Now \( \alpha'_4 \sim \beta'_4, \beta_4 \), so \( m(\alpha_4) = e \neq m(\alpha_1) \). Hence, since \( \alpha'_1, \alpha'_4 \in u_1 \vdash v_3 \), we conclude, as above, that \( u_3 \neq v_3 \), that \( c \sim e \), and that \( \alpha_4 = bee \) and \( \beta_4 = ace \). If \( \alpha_5, \beta_5 \in \Pi_3(G) \) with \( \alpha'_5 = v_2 u_1 v_3 \) and \( \beta'_5 = u_2 v_1 u_3 \), we have \( \beta_5 = adg \) and \( \alpha_5 = bfh \), where \( g \) and \( h \) need not be new vertices. Since \( \beta_5 \) has at least three neighbors, so does \( \beta'_5 \). Hence, \( \text{deg}(u_2) \geq 3 \) or \( \text{deg}(u_3) \geq 3 \). In the first case, we must have a new member of \( \Pi_3(H) \) that is adjacent to both \( \beta'_4 \) and \( \beta'_5 \), while the second case requires one adjacent to both \( \beta'_4 \) and \( \beta'_5 \); in either case, we conclude that \( g = e \in G \). And, by symmetry, we get \( h = c \).

Now, let \( \alpha_6 = ced \), \( \alpha_7 = bed \), \( \beta_6 = ecf \) and \( \beta_7 = acf \). The adjacencies of \( \alpha_7 \) and \( \beta_7 \) with the previously found \( P_3 \)'s require that \( y = u_2 \) with \( \alpha'_6 = v_1 u_2 v_3 \) and \( x = v_2 \) with \( \beta'_6 = v_1 u_2 v_3 \), while the combined adjacencies of \( \alpha_6 \) and \( \beta_6 \) require that \( u_3 \sim v_3 \) with \( \alpha'_6 = v_2 u_2 v_3 \) and \( \beta'_6 = u_2 v_2 u_3 \). Similarly, supposing that \( \alpha_8, \alpha_9, \beta_8, \beta_9 \in \Pi_3(G) \) with \( \alpha'_8 = v_2 u_2 v_3, \alpha'_9 = v_2 u_3 v_3, \beta'_8 = u_2 v_2 u_3 \) and \( \beta'_9 = u_2 v_3 u_3 \) requires \( d \sim f \) with \( \alpha_8 = bfd \), \( \alpha_9 = cfd \), \( \beta_8 = adf \) and \( \beta_9 = edf \).

To finish the proof, we note that \( \text{deg}(u) = 3 \); otherwise, we would have a third \( P_3 \) in \( P_3(H) \) that is adjacent to both \( \beta'_4 \) and \( \beta'_5 \), an impossibility. So, by the total symmetry in \( G \) and \( H \) at this stage, we conclude that \( G \cong H \cong K_{3,3} \). Moreover, if we let \( \sigma \) be the isomorphism from \( G \) to \( H \) given by \( \sigma(a) = u_3, \sigma(b) = v_3, \sigma(c) = v_2, \sigma(d) = v_1, \sigma(e) = u_2 \) and \( \sigma(f) = u_1 \), then one easily verifies that \( \tau = \tau_0 \circ \sigma^* \).

**Construction of the Generalized \( K_{3,3} \) Pairs.** If we relax the minimum degree restriction of the above theorem, and allow terminal vertices in \( G \) and \( H \), then we obtain six more \( P_3 \)-isomorphisms \( \tau \) that disperse bundles. All seven are listed below, using the following notation. We write \( (c, d)ab(e, f) \rightarrow vwux \) if \( G \) contains edges \( ab, ac, ad, be, bf, H \) contains the \( C_4 \) \( uvwux \), and \( \tau \) maps \( cab \mapsto xuw, dab \mapsto vwx, abc \mapsto uvw \) and \( abf \mapsto wux \). We also write \( abc(d, c) \rightarrow uvwx \) if \( G \) contains edges \( ab, bc, cd, ce, H \) contains the \( P_5 \) \( uvwxy \), and \( \tau \) maps \( abc \mapsto vwx, bed \mapsto uvw \) and \( bce \mapsto wxy \). This notation will be reversed (e.g., \( abcd \mapsto (w, x)uv(y, z) \)) as needed. Each description below contains one or more such items which together completely specify all edges of \( G \) and \( H \) except those in star components, and all \( P_3 \)'s except 2-thorns. Some edges and \( P_3 \)'s occur in more than one item. The vertex names mostly follow those in the above proof.

(i) \((c, d)ab(e, f) \rightarrow u_1 v_1 u_2 v_2 u_1, \) and \( cad \) and \( ebf \) map to \( P_3 \) components of \( H \).

(ii) \((c, d)ab(e, f) \rightarrow u_1 v_1 u_2 v_2 u_1, kebfh \rightarrow yv_3 u_1 (v_1, v_2), \) and \( cad \) maps to a \( P_3 \) component.

(iii) \((c, d)ab(e, f) \rightarrow u_1 v_1 u_2 v_2 u_1, (k, l)eb(a, f) \rightarrow u_1 v_1 u_2 v_3 u_1, (h, i)fb(a, e) \rightarrow u_1 v_2 u_2 v_3 u_1, \) and \( cad, kel \) and \( hfi \) map to \( P_3 \) components.

(iv) \((c, d)ab(e, f) \rightarrow u_1 v_1 u_2 v_2 u_1, ecadg \rightarrow xuv_3 u_1 (u_1, u_2), \) and \( ebfh \rightarrow yv_3 u_1 (v_1, v_2). \) Note that \( G \) and \( H \) are connected and isomorphic.

(v) \((c, d)ab(e, f) \rightarrow u_1 v_1 u_2 v_2 u_1, ebfhe \rightarrow (v_1, v_2)u_1 v_3 (y, z), \) and \( cad \) maps to \( yv_3 z. \) Again \( G \) and \( H \) are connected and isomorphic.

(vi) \((c, d)ab(e, f) \rightarrow u_1 v_1 u_2 v_2 u_1, (c, d)eb(a, f) \rightarrow u_1 v_1 u_2 v_3 u_1, (h, i)fb(a, e) \rightarrow u_1 u_2 u_2 v_3 u_1, accda \rightarrow (w, x)u_3 v_1 (u_1, u_2), \) and \( hfi \) maps to \( wu_3 x. \) Again \( G \) and \( H \) are connected and isomorphic.
Either \( \tau \) or \( \tau^{-1} \) as in cases (i) through (vii) above, or any equivalent \( P_3 \)-isomorphism, is said to be of generalized \( K_{3,3} \) type.

**Corollary [Generalized \( K_{3,3} \) Type].** Situation 1 occurs if and only if \( \tau \) is \( T \)-related to a \( P_3 \)-isomorphism of generalized \( K_{3,3} \)-type.

**Proof.** The ‘if’ part is trivial, so consider the ‘only if’ part. Under our standing assumption, suppose \( \tau \) disperses \( a \vdash b \), where \( \deg(b) \geq 3 \). As in the proof of the \( K_{3,3} \) Theorem, we know that (i) occurs inside \( G \) and \( H \). The remaining cases come about by the addition of more edges. We only sketch this since the details are just as in the proof of the \( K_{3,3} \) Theorem. As there, we have that \( a, b, c, d, e, f \) are all distinct, \( u_1 \neq u_2 \) and \( v_1 \neq v_2 \). Likewise, any vertices of degree at least 3 are of degree exactly 3.

So we can assume by symmetry that there is an edge \( v_3 u_1 \) in \( H \). Constructing the \( P_5 \)'s in \( G \) corresponding to \( v_1 u_1 v_3 \) and \( v_2 u_1 v_3 \) forces us to (ii).

There are now four nonsymmetric ways of adding further edges to \( H \) in (ii): at \( u_2 \) which gives (iii), at \( v_1 \) which gives (iv), at \( v_3 \) which gives (v), and at \( y \) which once again gives (iii). Moreover, following any one of those three additions by another gives (vi). Finally, adding anything to \( G \) or \( H \) in (vi) gives (vii).

Note that in the statement of the above corollary we need \( T \)-relation to take 2-thorns into account in cases (i), (ii) and (iii). There are no split 1-thorn pairs, so we do not need \( S \)-relation. We do not need \( B \)- or \( D \)-relation, despite the presence of diamond pairs and 1-thorns in the same bundle in cases (i), (ii), (iii), (v) and (vi), because each diamond pair swap in \( G \) corresponds to an induced bundle 1-thorn swap in \( H \), and vice versa. This means that composing with a diamond pair swap or a bundle 1-thorn swap just takes each \( P_3 \)-isomorphism in these cases to an equivalent one.

### 4. Examples from Whitney’s Edge Isomorphisms

In this section we construct an infinite family of noninduced \( P_3 \)-isomorphisms from Whitney’s examples of noninduced edge isomorphisms. We begin with a rather general idea which will be used here and in the next section.

**Diamond Inflation.** Suppose \( F \) is a graph. A diamond inflation of \( F \) is a graph obtained by replacing each \( ab \in E(F) \) by an unbraced \( s_{ab} \)-diamond \( D_{a,b} \) \( (s_{ab} \geq 1) \), and adding \( t_a \) terminal edges incident with each \( a \in V(F) \) \( (t_a \geq 0) \). For example, in Figure 1(a) we have a diamond inflation of a \( K_{1,3} \) with vertices \( a, b, c, d \) where \( s_{ab} = 2, s_{ac} = 1, s_{ad} = 3, t_a = 0, t_b = 1, t_c = 2 \) and \( t_d = 3 \). Figure 1(b) is also a diamond inflation, of a \( K_3 \) with vertices \( u, v, w \).

Now suppose \( \varphi \) is an edge isomorphism between graphs \( F \) and \( F' \). Suppose \( I \) and \( I' \) are diamond inflations of \( F \) and \( F' \) respectively, with the following property: for every \( ab \in E(F) \), if \( \varphi(ab) = uv \) then (i) \( s_{uv} = s_{ab} \) and (ii) \( t_u + t_v = t_a + t_b \). Obtain \( G \) and \( H \) from \( I \) and \( I' \) respectively by adding star components to one of them (if necessary) to make the number of 2-thorns equal. Then we can define a \( P_3 \)-isomorphism \( \tau \) from \( G \) to \( H \), as follows. Suppose \( ab \in E(F) \) and \( \varphi(ab) = uv \). Let \( \tau \Pi_3(D_{a,b}) \) be induced by any isomorphism from \( D_{a,b} \) to \( D_{u,v} \): the two diamonds are the same size by (i). For any diamond path \( \alpha \) of \( D_{a,b} \), the \( t_a + t_b \) terminal 1-thorns adjacent to \( \alpha \) can be mapped arbitrarily to the \( t_u + t_v \) terminal 1-thorns adjacent to \( \tau(\alpha) \), since the numbers are equal by (ii). The 2-thorns of \( G \) can be mapped arbitrarily to the
2-thorns of $H$. This only leaves the diamond connectors: the image of each diamond connector is uniquely determined by the images of the two diamond paths which are its neighbors. We say $\tau$ is a diamond inflation of $\varphi$. Note that anything TBS-related to $\tau$ is also a diamond inflation of $\varphi$.

It turns out there are two situations in which diamond inflation yields $P_3$-isomorphisms not TBSD-related to induced ones. The first situation comes from Whitney's exceptional edge isomorphisms.

**Theorem** [Whitney [8], also see [3]]. Suppose that $\varphi$ is an edge isomorphism from $G$ to $H$ where $G$ and $H$ are both connected. If $\varphi$ is not induced, then $i = |E(G)| = |E(H)| \in \{3, 4, 5, 6\}$, $G$ and $H$ are isomorphic to $W_i$ and $W'_i$ in some order, and $\varphi$ is equivalent to $\varphi_i$ or $\varphi_i^{-1}$, where

1. $W_6 \cong W'_6 \cong K_4$, with $V(W_6) = \{a, b, c, d\}$, $V(W'_6) = \{u, v, w, x\}$, and $\varphi_6$ maps $ab \mapsto uw$, $ac \mapsto uw$, $ad \mapsto vw$, $bc \mapsto wx$, $bd \mapsto wx$ and $cd \mapsto wx$;
2. $W_5 = W_6 - cd, W'_5 = W'_6 - wx$ and $\varphi_5 = \varphi_6|E(W_5)$;
3. $W_4 = W_6 - \{bd, cd\}, W'_4 = W'_6 - \{vx, wx\}$ and $\varphi_4 = \varphi_6|E(W_4)$; and
4. $W_3 = W_6 - \{bc, bd, cd\} \cong K_{1, 3}, W'_3 = W'_6 - x \cong K_3$, and $\varphi_3 = \varphi_6|E(W_3)$.

**Construction on the Whitney graphs.** A $P_3$-isomorphism $\tau$ is said to be of Whitney type $i$ if $\tau$ or $\tau^{-1}$ is equivalent to a diamond inflation of $\varphi_i$ as above, $i = 3, 4, 5, 6$. As an example, the two graphs of Figure 1 are related by a Whitney type 3 $P_3$-isomorphism, with vertices labelled exactly as above.

For Whitney type $P_3$-isomorphisms, condition (i) of Diamond Inflation just requires that both of a pair of corresponding edges, for example $ad$ and $vw$ in all four types, are replaced by diamonds of the same width. Letting $t_z$ denote the number of terminal edges incident with $z$ for $z$ in $\{a, b, c, d\}$ or $\{u, v, w, x\}$, we see that condition (ii) of Diamond Inflation gives one equation from each pair of corresponding edges of the original Whitney graphs: for example, in Whitney type 4, 5 or 6 the corresponding edges $bc$ and $ux$ give that $t_b + t_c = t_u + t_x$. Solving for $t_u, t_v, t_w, t_x$ in terms of $t_a, t_b, t_c, t_d$ we find the same solutions for all four types:

$$
t_u = \frac{1}{2}(t_a + t_b + t_c - t_d), \quad t_w = \frac{1}{2}(t_a - t_b + t_c + t_d),
$$

$$
t_v = \frac{1}{2}(t_a + t_b - t_c + t_d), \quad t_x = \frac{1}{2}(-t_a + t_b + t_c + t_d) \text{ (except for type 3)}.
$$

These equations impose obvious conditions on $t_a, t_b, t_c, t_d$ to make $t_u, t_v, t_w$ and (except for type 3) $t_x$ integral and nonnegative.

Note that the case where no terminal edges are added ($t_a = t_b = t_c = t_d = 0$) includes an infinite family of pairs of $P_3$-isomorphic but not isomorphic graphs of minimum degree at least 2. Thus, even the weak form of Li’s conjecture for minimum degree 2 is false.

**The Connected Case for the Whitney Types.** If we want both $G$ and $H$ to be connected, then both must have the same number of 2-thorns in their nonstar components. For a Whitney pair of type 3, this requires that we have

$$
\left(\frac{t_a}{2}\right) + \left(\frac{t_b}{2}\right) + \left(\frac{t_c}{2}\right) + \left(\frac{t_d}{2}\right) = \left(\frac{t_u}{2}\right) + \left(\frac{t_v}{2}\right) + \left(\frac{t_w}{2}\right).
$$
After some computations that use the equations above, we can reduce this to a quadratic equation whose solutions are \( t_a = t_b + t_c + t_d \) and \( t_a = t_b + t_c + t_d - 2 \).

Somewhat surprisingly, for Whitney types 4, 5 and 6 the graphs turn out to be already connected, since the conditions given in their construction can be shown to imply that

\[
\left( \frac{t_a}{2} \right) + \left( \frac{t_b}{2} \right) + \left( \frac{t_c}{2} \right) + \left( \frac{t_d}{2} \right) = \left( \frac{t_u}{2} \right) + \left( \frac{t_v}{2} \right) + \left( \frac{t_w}{2} \right) + \left( \frac{t_x}{2} \right).
\]

The Special Whitney Type. There is one special example of a noninduced \( P_3 \)-isomorphism which is closely related to the Whitney types, and so we also introduce it here. Let \( SW \) be the graph obtained by subdividing each edge of \( K_{1,3} \) exactly once; then \( P_3(SW) \cong C_6 \). Rotation of this \( C_6 \) by one step is a noninduced \( P_3 \)-isomorphism from \( SW \) to itself; we say this or any equivalent \( P_3 \)-isomorphism is of special Whitney type.

The important property of the Whitney types (including the special Whitney type) is the following.

Situation 2. Situation 1 does not occur, but either \( \tau \) or \( \tau^{-1} \) disperses two \( P_3 \)'s in the same bundle, neither of which is a thorn or one of a diamond pair.

Below we show that this characterizes the Whitney types; first we need some basic properties of diamonds.

Lemma [Diamond]. Suppose Situation 1 does not occur.

(i) \( \alpha, \beta \) is a diamond pair if and only if \( \alpha', \beta' \) is, in which case \( N(\alpha) = N(\beta) \) and consists solely of the two diamond paths associated with \( \alpha \) and \( \beta \).

(ii) If \( D_{a,b} \) is a nontrivial diamond in \( G \), then there exist a unique and nontrivial diamond \( D_{u,v} \) in \( H \) and an isomorphism \( \varphi \) from \( D_{a,b} \) to \( D_{u,v} \) such that \( \varphi(a) = u, \varphi(b) = v \) and \( \varphi^* \) is \( D \)-related to \( \tau | \Pi_3(D_{a,b}) \). Moreover, \( D_{a,b} \) is braced if and only if \( D_{u,v} \) is. We call \( D_{u,v} \) the image, under \( \tau \), of \( D_{a,b} \).

\textbf{Proof.} (i) If \( \alpha, \beta \) is a diamond pair but \( \alpha', \beta' \) is not, then, by (ii) of the Cycle Lemma, Situation 1 occurs for \( \alpha' \) and \( \beta' \).

(ii) The first claim follows from (i); for if \( \delta_1 = ac_1b \) is a diamond path in \( D_{a,b} \), then \( \delta \) is a diamond path in \( D_{a,b} \) if and only if there is a diamond pair \( \alpha, \beta \) with \( N(\alpha) = \{ \delta_1, \delta \} \). Thus \( \Pi_3(D_{a,b}) \) maps into \( \Pi_3(D_{u,v}) \) for some nontrivial diamond \( D_{u,v} \). Applying the same argument to \( \tau^{-1} \) and \( D_{u,v} \) we see that \( \tau \) maps \( \Pi_3(D_{a,b}) \) onto \( \Pi_3(D_{u,v}) \). The second claim is immediate from (i) of the Cycle Lemma.

Note that the simplest pairs (that is, using 1-diamonds and no terminal edges in the construction) of Whitney types 3, 4, and 5 show the necessity of requiring a nontrivial diamond in part (ii) of the Diamond Lemma.

Theorem [Whitney Type]. Situation 2 occurs if and only if \( \tau \) is of special Whitney type or \( D \)-related to a \( P_3 \)-isomorphism of Whitney type.

\textbf{Proof.} The ‘if’ part is trivial, so consider the ‘only if’ part. We suppose that \( \alpha_1 = c_1ac_2, \alpha_2 = c_2ac_3, \alpha_3 = c_3ac_1 \) with neither \( \alpha_1 \) nor \( \alpha_2 \) a thorn or one of a diamond pair and \( m(\alpha_1') \neq m(\alpha_2') \). By symmetry we assume that \( m(\alpha_1') \neq m(\alpha_3') \). Since \( \deg(c_i) \geq 2 \) for each \( i = 1, 2, 3 \), we have \( P_3 \)’s \( \beta_i = abc_i, i = 1, 2, 3 \). In fact, \( \deg(c_1) = \deg(c_2) = 2 \) since Situation 2 implies that Situation 1 does not occur, and so \( \beta_1 \) and \( \beta_2 \) are unique and \( \alpha_1 \) is a diamond connector. Also,
Thus, we need \( D \) different diamonds and \( b \) fourth diamond at \( v \); otherwise, we would have another edge added, where \( \{a, b, c\} \) are diamond connectors in \( H \); that is, \( \text{deg}(\alpha_1') = 2 \), \( i = 1, 2, 3 \) and so the middle vertices of \( \alpha_1', \alpha_2', \alpha_2', \alpha_3', \beta_1', \beta_1', \beta_1' \) form a 6-cycle in \( H \). Hence \( \beta_1', \beta_2' \) and \( \beta_3' \) are all in different diamonds and \( b_1, b_2 \) and \( b_3 \) are pairwise distinct.

So we have distinct diamonds \( D_{a,b_1}, D_{a,b_2} \) and \( D_{a,b_3} \), pairwise adjacent at \( a \), whose images in \( H \) form a triangle of diamonds, that is, we can choose the notation so that their images are of the form \( D_{u_1,u_2}, D_{u_2,u_3} \) and \( D_{u_3,u_1} \), respectively, with \( u_i = m(\alpha_i') \), \( i = 1, 2, 3 \). Note that these diamonds are all unbraced; in fact, all edges at \( a \) other than terminal ones are accounted for; otherwise, we would have another \( \alpha \in a \rightarrow c \), and so just as in the preceding, we would get a fourth diamond at \( a \) whose image, along with those of two of the previous ones, would form yet another triangle of diamonds in \( H \), an impossibility.

If this accounts for all of \( G \) and \( H \), other than for some terminal edges, then \( \tau \) is D-related to a \( P_3 \)-isomorphism of Whitney type 3. Otherwise, we can assume that there is an unaccounted-for vertex of degree at least 2 that is adjacent to \( u_1 \). Applying the same reasoning as above, but for \( \tau^{-1} \) at \( u_1 \), we see that there is a third unbraced diamond \( D_{u_1,u_4} \) at \( u_1 \) that is the image of \( D_{b_1,b_2} \) and that there can be no more nonterminal edges at \( u_1 \). If this is all of \( G \) and \( H \), except for some terminal edges, then \( \tau \) is D-related to a \( P_3 \)-isomorphism of Whitney type 4. If this still doesn’t account for all of \( G \), then continuing in the same way, we see that \( \tau \) must be D-related to a \( P_3 \)-isomorphism of Whitney type 5 or 6, and we cannot continue beyond this.

5. EXAMPLES FROM BIPARTITE GRAPHS

So far we have seen two families of noninduced \( P_3 \)-isomorphisms. However, we still have not captured Broersma and Hoede’s examples! In this section we construct noninduced \( P_3 \)-isomorphisms starting from an arbitrary bipartite graph, using the idea of diamond inflation introduced in the previous section. This construction accounts for the Broersma and Hoede examples, and produces many others.

**Construction on a Bipartite Graph.** Start with a positive integer \( k \) and an arbitrary bipartite graph \( F \) with at least one edge and with bipartition \( (A, B) \). Let \( I \) and \( I' \) be different diamond inflations of \( F \), where each edge \( e \) is inflated to a diamond of the same width \( s_e \), both times, but in producing \( I \) each vertex \( v \) has \( t_v \) terminal edges added, while in producing \( I' \) it has \( t'_v \) terminal edges added, where

\[
   t'_v = \begin{cases} 
   t_v - k & \text{if } v \in A, \\
   t_v + k & \text{if } v \in B.
   \end{cases}
\]

(Thus, we need \( t_v \geq k \) for all \( v \in A \).) Let \( \varphi \) be the identity edge isomorphism from \( F \) to itself. Clearly \( \varphi \), \( I \) and \( I' \) satisfy condition (i) of Diamond Inflation, and condition (ii) is satisfied because each edge of \( F \) has the form \( ab \) with \( a \in A \) and \( b \in B \), so that \( t'_a + t'_b = (t_a-k) + (t_b+k) = t_a + t_b \). We can therefore obtain a \( P_3 \)-isomorphism \( \tau \) by diamond inflation; \( \tau \) is in general not induced. We say \( \tau \) and \( \tau^{-1} \), or any equivalent \( P_3 \)-isomorphisms, are of *bipartite type*. 

Some Examples. We note that the Broersma and Hoede examples [2, Figs. 2 and 3] are fairly simple, albeit quite illustrative, cases of this construction. With a slight change in labelling, the graphs of their Figure 2 appear in our Figure 2. They may be obtained by starting with a tree $F$ of diameter 3, whose vertices are the solid vertices of the figure. To obtain $I = T_{m,l}$ from $F$, replace all edges with unbraced 1-diamonds (that is, subdivide the edges of $F$), add a single terminal edge at each vertex of $A = \{x, r_1, r_2, \ldots, r_l\}$, and add nothing at each vertex of $B = \{y, u_1, u_2, \ldots, u_m\}$. To obtain $I' = T'_{m,l}$, we take $k = 1$ in our construction. The added terminal edges for $I$ and $I'$ are emboldened in the figure.

The graphs of Broersma and Hoede's Figure 3 are equally simple, differing only in that the starting graph is $K_{1,n+1}$ and one of the replacing diamonds is a 2-diamond, again an unbraced one. If we do the same thing to $P_3$, but using only unbraced 1-diamonds, then we get the other example they noted, namely, $P_7$ and the graph that results from deleting a terminal vertex from a subdivision of $K_{1,3}$. Both of these graphs have $P_5$ as their $P_3$-graphs; on the other hand, it is not hard to show that if $G$ is connected and $P_3(G) \cong P_{n}, n \neq 5$, then $G \cong P_{n+2}$.

The simplest pairs obtained by the Construction on a Bipartite Graph are the ones that start by subdividing a $K_2$ with the resulting pair of graphs having one nonthorn $P_3$ with an arbitrary, but same total number of terminal edges attached at each end; the smallest nonisomorphic pair obtained this way is $P_5 \cup P_3$ and the graph $Y$ that results from deleting two terminal vertices from a subdivision of $K_{1,3}$. Note that the relationship between $P_5$ and $Y$ occurs inside several of the

![Diagram](image-url)
Generalized \( K_{3,3} \) examples, and was used in describing them, but we do not consider it one of them because Situation 1 does not occur.

**The Connected Case for the Construction on a Bipartite Graph.** Using \( t_z \) for the number of terminal edges incident with \( z \) in \( G \) and requiring that both \( G \) and \( H \) be connected implies that

\[
\sum_{a \in A} \left( \frac{t_a}{2} \right) + \sum_{b \in B} \left( \frac{t_b}{2} \right) = \sum_{a \in A} \left( \frac{t_a - k}{2} \right) + \sum_{b \in B} \left( \frac{t_b + k}{2} \right)
\]

which reduces to

\[
\sum_{a \in A} (2t_a - k - 1) = \sum_{b \in B} (2t_b + k - 1).
\]

As with the Generalized \( K_{3,3} \) and Whitney types, we will characterize the \( P_3 \)-isomorphisms of bipartite type in terms of things that are not preserved. First we must investigate the consequences of the following assumption.

**Additional Standing Assumption.** For the rest of this section we assume that Situations 1 and 2 do not occur.

Under this assumption, we know that in many cases two \( P_3 \)'s with the same middle vertex will not be dispersed by \( \tau \), so that their images will have the same middle vertex. This allows us to ‘bind’ certain vertices of \( G \) to corresponding vertices in \( H \). A \( P_3 \) is said to be nonbinding if it is one of a diamond pair, a terminal 1-thorn, or a 2-thorn; otherwise it is binding. Let \( B(a) \) denote the set of binding \( P_3 \)'s in \( S(a) \). A vertex \( a \) is strongly bound if \( B(a) \) is nonempty. It is not difficult to see that there are only three (overlapping) types of vertices that are not strongly bound: first, terminal vertices; second, the central vertex of a star component; and third, a vertex \( a \) all of whose nonterminal neighbors belong to a single unbraced diamond \( D_{a,b} \). In the third case, if there is some diamond path \( acb \) with both \( b \) and \( c \) strongly bound, we say that \( a \) is weakly bound. A vertex is bound if it is either strongly or weakly bound, and unbound otherwise. Since all diamond paths ending at a weakly bound vertex \( a \) are similar, all are binding; thus, every neighbor of \( a \) is either an unbound terminal vertex or a strongly bound vertex of degree 2, and the unique vertex \( b \) at distance 2 from \( a \) is strongly bound. Note that a weakly bound vertex may be a terminal vertex.

**Lemma [Strong Binding].**

(i) A \( P_3 \)-isomorphism maps binding \( P_3 \)'s to binding \( P_3 \)'s, and nonbinding \( P_3 \)'s to nonbinding \( P_3 \)'s.

(ii) For every strongly bound vertex \( a \) of \( G \) there is a unique strongly bound vertex of \( H \), which we denote \( a' \), such that \( \tau(B(a)) = B(a') \). Thus, if \( \alpha \in \Pi_3(G) \) is binding with middle vertex \( a \), then \( \alpha' \) has middle vertex \( a' \).

(iii) If \( a \) is strongly bound, \( \deg(b) \geq 2 \), and \( a \sim b \), then there exists a binding \( P_3 \) in \( a + b \).

(iv) If \( a \) is strongly bound and \( \alpha \in \Pi_3(G) \) has an end at \( a \), then \( \alpha' \) has an end at \( a' \).

**Proof.** (i) This follows from (i) of the Diamond Lemma and the fact that \( \alpha \) is a 2-thorn or terminal 1-thorn if and only if \( \deg(\alpha) \leq 1 \).

(ii) Suppose \( \alpha_1, \alpha_2 \in B(a) \). Let \( \alpha' = m(\alpha_1) \). First, suppose that one of \( \alpha_1 \) or \( \alpha_2 \), say \( \alpha_1 \), has an end \( c \) of degree 3 or more. If \( c \) is also an end of \( \alpha_2 \) then \( m(\alpha_2') = \alpha' \) since Situation 1 does not occur. If \( c \) is not an end of \( \alpha_2' \), then \( \alpha_2' \) has an end \( d \) of degree 2 or more. Let \( \gamma = cad \). Then \( m(\alpha_1') = m(\gamma') \) (since Situation 1 does not occur) and \( m(\gamma') = m(\alpha_2') \) (since Situation
1 does not occur, if \( \deg(d) \geq 3 \), or since Situation 2 does not occur, if \( \deg(d) = 2 \). Therefore \( m(\alpha'_2) = a' \).

Second, suppose all ends of both \( \alpha_1 \) and \( \alpha_2 \) have degree 2. Write \( \alpha_1 = c_1ac_2 \) and \( \alpha_2 = d_1ad_2 \). Since \( \alpha_1 \) is not one of a diamond pair, at most one of \( \gamma_1 = c_1ad_1 \) and \( \gamma_2 = c_2ad_1 \) is one of a diamond pair: suppose \( \gamma_1 \) is not one of a diamond pair. Then since Situation 2 does not occur, \( m(\alpha'_2) = m(\gamma'_1) = a' \).

So \( \tau(B(a)) \subseteq S(a') \), and by (i) above, \( \tau(B(a)) \subseteq B(a') \). Applying the same reasoning to \( \tau^{-1} \) yields \( \tau^{-1}(B(a')) \subseteq B(a) \), so \( \tau(B(a)) = B(a') \).

(iii) If \( \deg(b) \geq 3 \), then any \( P_3 \) in \( a \leftrightarrow b \) is binding, so suppose \( \deg(b) = 2 \). Since \( a \) is strongly bound, there is some neighbour \( c \) of \( a \) which is either of degree 3 or more, or in a different diamond from \( b \), and then \( bac \) is binding.

(iv) Suppose \( \alpha = abc \). By (iii) there is some binding \( dab \). Now \( \alpha' = (abc)' \sim (dab)' = ua'v \) (for some \( u, v \)), so \( a' \) is an end of \( \alpha' \).

Lemma [Weak Binding].

(i) The ends of a binding diamond path are either both unbound, both strongly bound, or one strongly bound and one weakly bound.

(ii) For every weakly bound vertex \( a \) of \( G \) there is a unique weakly bound vertex of \( H \), which we denote \( a' \), such that for every binding \( P_3 \) \( a \) with an end at \( a \), \( a' \) has an end at \( a' \).

Proof. Part (i) follows from the definition of weakly bound. For (ii), let \( D_{a,b} \) be the unique (necessarily unbraced) diamond in \( G \) with end \( a \), having diamond paths \( \alpha_i = ac_i b, i = 1, 2, \ldots, k \): these are the binding \( P_3 \)'s with an end at \( a \). For each \( i \), \( \alpha'_i \) is binding and \( b' \), \( c'_i \) are strongly bound, so \( \alpha'_i = xc'_i b' \) for some \( x \) by (ii) and (iv) of the Strong Binding Lemma. By (ii) of the Diamond Lemma, if \( k \geq 2 \) then \( x \) is the same for all \( i \). Since \( b' \) is strongly bound and \( \alpha'_i \) is binding, \( x \) is bound by (i). If \( x \) is strongly bound, then \( a \) is also strongly bound by (iv) of the Strong Binding Lemma, and so \( x \) is weakly bound. Now \( x \) has all the required properties, so we let \( a' = x \).

We may summarize the important implications of the Strong and Weak Binding Lemmas as follows.

Theorem [Binding]. Let \( \sigma \) be the map \( a \rightarrow a' \) from the bound vertices of \( G \) to the bound vertices of \( H \). Let \( a, b \) be bound vertices of \( G \).

(i) The map \( \sigma \) is a bijection which maps strongly bound vertices to strongly bound vertices, and weakly bound vertices to weakly bound vertices.

(ii) For every \( P_3 \) \( a \) with an end at \( a \), \( a' \) has an end at \( a' \).

(iii) For every binding \( P_3 \) \( a \) with middle vertex \( a \), \( a' \) has middle vertex \( a' \).

(iv) \( a \sim b \) if and only if \( a' \sim b' \).

Proof. Parts (i), (ii) and (iii) follow directly from the Strong and Weak Binding Lemmas applied to \( \tau \) and \( \tau^{-1} \)—note that (iii) is vacuously true when \( a \) is weakly bound. For (iv), if \( a \sim b \) then one of \( a \) or \( b \) is strongly bound, and by the definition of weakly bound or by (iii) of the Strong Binding Lemma there is a binding \( P_3 \) containing the edge \( ab \) to which (ii) and (iii) can be applied.

Theorem [Trivial Bipartite Type]. Suppose Situations 1 and 2 do not occur, and there is an unbound nonterminal vertex in \( G_0 \) or \( H_0 \). Then \( \tau \) is either D-related to a \( P_3 \)-isomorphism of bipartite type starting from \( F \cong K_2 \) or is TBSD-related to an induced \( P_3 \)-isomorphism.
**Proof.** Suppose $a$ is a nonterminal unbound vertex in $G_0$. Every edge incident with $a$ is terminal or an edge of a unique unbraced diamond $D = D_{a,b}$ of width $k$. Consider a diamond path $abc$: if it is one of a diamond pair, then $G \cong H \cong C_4$; if it is a terminal 1-thorn then $G \cong H \cong P_4$. In either case $\tau$ is induced, so we may suppose $acb$ is binding. Then $b$ is unbound by (i) of the Weak Binding Lemma. Now $G_0$ consists only of $D$ and possibly terminal edges incident with $a$ and/or $b$. It follows (immediately when $k = 1$, and by (ii) of the Diamond Lemma if $k \geq 2$) that $H_0$ consists of a diamond $D_{a,b}$ of width $k$ and possibly terminal edges incident with $u$ and/or $v$. If $\{\deg(a), \deg(b)\} = \{\deg(u), \deg(v)\}$ then $\tau$ is TBSD-related to an induced $P_3$-isomorphism, and otherwise $\tau$ is $D$-related to some $\tau'$ of bipartite type starting from $F \cong K_2$.

After eliminating the possibility of unbound nonterminal vertices, $\sigma : a \mapsto a'$ is very close to inducing $\tau$. In the following lemma, we summarize what we can say about the images of $P_3$'s of various types.

**Lemma [Image].** Suppose every nonterminal vertex is bound. Let $abc \in \Pi_3(G)$ with $\deg(a) \geq \deg(c)$.

(i) If $abc$ is a 2-thorn, $(abc)'$ is a 2-thorn.

(ii) If $abc$ is a terminal 1-thorn, with neighbor $dab$, then $d$ is bound and $(abc)'$ is either $a'b'x$ or $a'd'x$ for some unbound $x$.

(iii) If $abc$ is one of a diamond pair, the other being $ade$, then $(abc)'$ is either $a'b'c'$ or $a'd'c'$.

(iv) If $abc$ is binding with all of $a$, $b$, $c$ bound, then $(abc)' = a'b'c'$.

(v) If $abc$ is binding with not all of $a$, $b$, $c$ bound, then only $c$ is unbound, and $(abc)' = a'b'x$ for some unbound $x$.

(vi) If $abc$ is a diamond path that is not a 2-thorn or terminal 1-thorn, then it is binding, and $a$, $b$, $c$ are all bound.

**Proof.** Part (i) is obvious. Part (iv) follows from (ii) and (iii) of the Binding Theorem. For (v), only $c$ is unbound since $a$ and $b$ are nonterminal, and $(abc)' = a'b'x$ by (ii) and (iii) of the Binding Theorem, where $x$ is unbound since otherwise we could apply (iv) to $\tau^{-1}$.

For (vi), a diamond path cannot also be one of a diamond pair, or else $G_0 \cong C_4$ which has unbound nonterminal vertices, so $abc$ is binding and $b$ is strongly bound. Also, $\deg(a) \geq 2$ so $a$ is bound, and hence $c$ is bound by (i) of the Weak Binding Lemma.

For (iii), parts (vi) and (iv) apply to the diamond paths $bad$ and $bcd$, so $N((abc)') = \{b'a'd', b'c'd'\}$, and (iii) follows.

For (ii), $dab$ cannot be a terminal 1-thorn, otherwise $G_0 \cong P_3$ which has unbound nonterminal vertices, so (vi) and (iv) apply to $dab$. Now $(abc)' \sim (dab)' = d'a'b'$, so $(abc)' = a'b'x$ or $a'd'x$ for some $x$. If $x$ is bound, then $c$ is bound by (iii) of the Binding Theorem, but $c$ is not strongly bound since it is terminal, and not weakly bound since there is no binding $P_3$ ending at $c$. Thus, $x$ is unbound.

**Theorem [Nontrivial Bipartite Type].** Suppose that Situations 1 and 2 do not occur and every nonterminal vertex is bound. Then there is a bound vertex $a$ with $\deg(a) \neq \deg(a')$ if and only if $\tau$ is $D$-related to a $P_3$-isomorphism of bipartite type starting from $F \not\cong K_2$.

**Proof.** The ‘if’ part is not difficult, so consider the ‘only if’ part. Without loss of generality, suppose that $a$ is bound and $\deg(a') = \deg(a) + k$ for some $k > 0$. If $a$ has a neighbour $c$ of degree 3 or more, then $a$ cannot be weakly bound, so $a$ is strongly bound. By (ii) and (iii) of
the Binding Theorem, applied to both \( \tau \) and \( \tau^{-1} \), there is a bijection between \( a \rightarrow c \) and \( a' \rightarrow c' \), contradicting \( \deg(a) \neq \deg(a') \).

Thus, all edges incident with \( a \) are terminal or edges of diamonds with an end at \( a \). The same applies to \( a' \). Let \( b \) be a vertex at distance 2 from \( a \), which must be an end of a diamond path \( \alpha = acb \). By (iii) of the Binding Theorem, \( a' = a'wx \) for some \( w, x \). Now \( \deg(a) + \deg(b) - 2 = \deg(a') = \deg(a') + \deg(x) - 2 \), so \( \deg(b) = \deg(x) + k \geq 2 \), and \( b \) is bound. Thus, \( x = b' \) and \( \deg(b') = \deg(b) - k \). By (ii) of the Diamond Lemma, \( D_{a,b} \cong D_{a',b'} \).

Now we can apply the same reasoning, replacing \( a \) with \( b' \). Continuing in this way, we can decompose \( G_0 \) and \( H_0 \) into terminal edges with unbound ends and corresponding diamonds, each of which has \( \deg(a') - \deg(a) = k \) for one end, \( a \), and \( \deg(b') - \deg(b) = -k \) for the other end, \( b \). Thus, \( G_0 \) and \( H_0 \) are isomorphic to diamond inflations of some bipartite graph \( F \not\cong K_2 \), and from (i)–(v) of the Image Lemma, \( \tau \) is D-related to \( \tau' \) of bipartite type.

6. CONCLUSION

Finally, we show that we have now identified all \( P_3 \)-isomorphisms.

**Main Theorem.** Let \( \tau \) be a \( P_3 \)-isomorphism from \( G \) to \( H \) with at least one of \( G \) or \( H \) connected. Then \( \tau \) is one of the following:

(i) \( T \)-related to a \( P_3 \)-isomorphism of generalized \( K_{3,3} \)-type;
(ii) of special Whitney type;
(iii) \( D \)-related to a \( P_3 \)-isomorphism of Whitney type 3, 4, 5 or 6;
(iv) \( D \)-related to a \( P_3 \)-isomorphism of bipartite type; or
(v) TBSD-related to an induced \( P_3 \)-isomorphism.

**Proof.** We may suppose that situations 1 and 2 do not occur, every nonterminal vertex is bound, and \( \deg(a') = \deg(a) \) for every bound \( a \), as we have seen that (i)–(v) cover all other situations. By (iv) of the Binding Theorem, the subgraphs \( G_1 \) and \( H_1 \) induced by the bound vertices of \( G \) and \( H \) respectively are isomorphic, with isomorphism \( \sigma : a \mapsto a' \). So \( G_0 \) and \( H_0 \) differ only in the placement of some terminal edges with unbound ends. But since \( \deg(a) = \deg(a') \) for all bound vertices \( a \), \( \sigma \) extends to an isomorphism from \( G_0 \) to \( H_0 \), which we still call \( \sigma \). Applying (i)–(v) of the Image Lemma shows that \( \tau \) is TBSD-related to \( \sigma^* \).

One special case is easily stated.

**Corollary.** If \( \tau \) is a \( P_3 \)-isomorphism from \( G \) to \( H \), where \( G \) has minimum degree at least 3, then \( G \cong H \). Moreover, \( \tau \) is induced unless \( \tau \) is equivalent to \( \tau_0 \) as in the Construction on \( K_{3,3} \).

It is also not difficult to extract a result for when \( G \) and \( H \) both have minimum degree at least 2; we omit the details.

We remark that although our results are stated for finite graphs, our reasoning appears to hold without change for infinite graphs that are locally finite (every vertex has finite degree). In this case, the situation for each infinite component must be as in (iv) or (v) of our Main Theorem. If we allow vertices of infinite degree, however, it becomes easier to construct \( P_3 \)-isomorphisms by diamond inflation, including some that are of neither Whitney nor bipartite type. As the simplest example, consider two diamond inflations \( I \) and \( I' \) of a \( K_3 \) with vertices \( a, b, c \), where \( s_e = s'_e \) for each edge \( e \), and each set \( \{ t_a, t_b, t_c \} \) and \( \{ t'_a, t'_b, t'_c \} \) has two countably infinite elements while the third element is finite or countably infinite. For each edge \( e = uv \) we have \( t_u + t_v = t'_u + t'_v = \aleph_0 \),
and so there is a $P_3$-isomorphism $\tau$ from $I$ to $I'$ that is a diamond inflation of the identity edge isomorphism of the $K_3$. In general $\tau$ is of a type not mentioned in our Main Theorem.

We conclude with some obvious questions about extensions of our work. Can we characterize $P_k$-isomorphisms for $k \geq 4$? What can we say about $P_3$-isomorphisms if we allow multiple edges? Or if we restrict ourselves to induced $P_3$'s? Or if we allow $P_3$'s to be adjacent whenever they share any edge? We know the answer to only the last of these questions. The graph with vertex set $\Pi_3(G)$, with two $P_3$'s adjacent when they share an edge, is just $L(L(G))$, and it is a relatively simple exercise to characterize isomorphisms between $L(L(G))$ and $L(L(H))$ using Whitney's characterization of edge isomorphisms.

References