Commutative doubly-idempotent semirings determined by chains and by preorder forests

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Abstract. A commutative doubly-idempotent semiring (cdi-semiring) $(S, \vee, \cdot, 0, 1)$ is a semilattice $(S, \vee, 0)$ with $x \vee 0 = 0$ and a semilattices $(S, \cdot, 1)$ with identity 1 such that $x0 = 0$, and $x(y \vee z) = xy \vee xz$ holds for all $x, y, z \in S$. Bounded distributive lattices are cdi-semirings that satisfy $xy = x \wedge y$, and the variety of cdi-semirings covers the variety of distributive lattices. Chajda and Länger showed in 2017 that the variety of all cdi-semirings is generated by the 3-element cdi-semiring.
We show that there are seven cdi-semirings with a $\vee$-semilattice of height less than or equal to 2. We construct all cdi-semirings for which their multiplicative semilattice is a chain with $n + 1$ elements, and we show that up to isomorphism the number of such algebras is the n^{th} Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. We also show that cdi-semirings with a complete atomic Boolean $\vee$-semilattice on the set of atoms A are determined by rooted preorder forests on the set A. From these results we obtain efficient algorithms to construct all multiplicatively linear cdi-semirings of size n and all Boolean cdi-semirings of size 2^n.

Keywords: idempotent semirings, distributive lattices, preorder forests

1 Introduction

The structure of distributive lattices is well understood since every distributive lattice is a subalgebra of a product of the 2 element lattice, i.e. a subalgebra of a Boolean lattice. The situation is more complicated for idempotent semirings $(A, \vee, \cdot, 0, 1)$, defined by the identities

$$(x \vee y) \vee z = x \vee (y \vee z) \quad x \vee y = y \vee x \quad x \vee 0 = x \quad x \vee x = x \quad x0 = 0 = 0x$$
$$(xy)z = x(yz) \quad x1 = x = 1x \quad (x \vee y)z = xz \vee yz \quad x(y \vee z) = xy \vee xz.$$

The subclass of *commutative doubly idempotent semirings*, or *cdi-semirings* for short, is obtained by adding the identities $xy = yx$ and $xx = x$. Even for this much smaller class of cdi-semirings there is no general structure theory. The classes of idempotent semirings and cdi-semirings are defined by a list of identities, hence they are varieties, i.e., closed under products, subalgebras and homomorphic images.

Since we are also assuming $\cdot$ is commutative and idempotent, there are two underlying semilattice orders $x \leq y \iff x + y = y$ and $x \sqsubseteq y \iff xy = x$.

A cdi-semiring is a bounded distributive lattice if and only if the two orders coincide, or equivalently if the absorption laws $x \vee xy = x$ and $x(x \vee y) = x$ hold. While the variety of cdi-semirings is quite special, it includes all distributive lattices and is small enough that there is hope for a general description of its finite members.

Recall that *Kleene algebras* are idempotent semirings with a unary operation x^* such that $1 \vee x \vee x^*x^* = x^*$, $xy \le y \implies x^*y = y$ and $yx \le y \implies yx^* = y$. It is well known that the class KA of all Kleene algebras is not closed under homomorphic images, hence the last two axioms cannot be replaced by identities and the class KA of Kleene algebras is only a quasivariety. Our first observation is that the results in this paper also apply to a special class of Kleene algebras.

Lemma 1. *The class of Kleene algebras that satisfy $x^* = 1 \vee x$ is a sub*variety *of KA, and the commutative doubly idempotent members are the variety of cdi-semirings.*

Proof. Assume $\mathbf{A}$ is a Kleene algebra that satisfies $x^* = 1 \vee x$ and let $x, y \in A$. If $xy \le y$ then $x^*y = (1 \vee x)y = y \vee xy = y$, and similarly $yx \le y \Rightarrow yx^* = y$. If $\mathbf{A}$ also satisfies $xx = x$ then $1 \vee x \vee x^*x^* = x^* \vee x^* = x^*$. □

There are two 3-element cdi-semirings, and in [1] it is proved that the variety CDI of cdi-semirings is generated by one of them, denoted by S_3, (the other one is the 3-element distributive lattice). In the literature of semirings there are several definitions depending on whether the algebra contains an identity and/or a zero element. S. V. Polin [11] studied minimal varieties of semirings without $0, 1$ as constant operations. A variety is minimal if it has no subvarieties other than the variety of one-element algebras. Polin showed there are 8 minimal varieties of semirings (without 0,1) generated by 2-element semirings and 2 countable sequences of minimal varieties of rings generated finite prime fields and by finite prime additive cyclic groups with constantly zero multiplication. If the constants are included, then there are still the two countable sequences and only one more minimal variety: the variety of distributive lattices.

McKenzie and Romanovska [6] proved that the variety of doubly idempotent semirings without $0, 1$ has exactly 4 proper subvarieties: the trivial variety, the variety of distributive lattices (without constants for top, bottom), the variety of semilattices (defined by $xy = x \vee y$), and the join of the previous two varieties, called *distributive bisemilattices* and defined as commutative doubly idempotent semirings (without constants) where $x \vee yz = (x \vee y)(x \vee z)$. When 0 is in the signature of semirings with $0 \vee x = x$ and $x0 = 0$, then the distributivity of $\vee$ over $\cdot$ implies the absorption laws, hence the variety of distributive bisemilattices with 0 coincides with the variety of distributive lattices with 0. Likewise the identity $xy = x \vee y$ implies $0 = x0 = x \vee 0 = x$ hence the variety of semilattices coincides with the trivial variety. So with constants, the variety CDI has only two subvarieties, namely the variety of bounded distributive lattices, generated by the 2-element lattice $\mathbf{2}$ and the variety of one-element algebras.

2 Cdi-semirings of height two

In an idempotent semiring $\mathbf{S}$, the join-semilattice order is denoted by $x \le y$. If $(S, \le)$ is a linear order (or *chain* for short) then the *height* of $\mathbf{S}$ is $|S| - 1$. In general the height of an idempotent semirings is the maximal height over all subchains of $(S, \le)$.

It follows from a result of D. Stanovsky [12] about idempotent residuated lattices that there are only a small number of cdi-semirings of height 2. The proof below is self-contained and constructs all nonisomorphic cdi-semirings of height ≤ 2.

Recall that an *atom* of a poset with bottom element 0 is an element $a \ne 0$ such that $x < a$ implies $x = 0$.

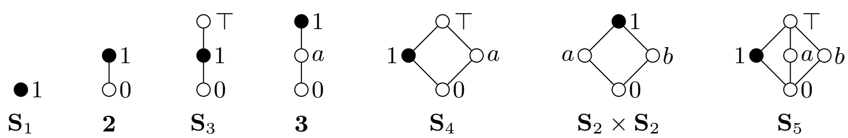

Fig. 1. All cdi-semirings of height 2 or less, ordered by $\le$. In $\mathbf{S}_4$ $a\top = a$, and in $\mathbf{S}_5$ $a \cdot b = 0$, $a\top = a$, $b\top = b$

Theorem 2. *There are, up to isomorphism, seven cdi-semirings of height two or less (Fig. 1).*

Proof. Let $\mathbf{S}$ be a cdi-semirings of height ≤ 2. For any elements $x, y \in S$ such that $x \in \{0, 1\}$ or $y \in \{0, 1\}$ the multiplication xy is fixed by the semiring axioms, and $xx = x$ hence the structure of $\mathbf{S}$ is determined by the join-semilattice order and the values of distinct elements $x, y \in S \setminus \{0, 1\}$. If $\mathbf{S}$ has height 0, it is the one-element semiring $(0 = 1)$, and if $\mathbf{S}$ has height 1, it is the 2-element lattice with $0 \ne 1$. In the remaining cases, $\mathbf{S}$ has height 2, so let A be the set of atoms of $\mathbf{S}$.

If $|A| = 1$, then $\mathbf{S}$ has three elements and either $A = \{1\}$ or $A = \{a\}$ for some $a \ne 1$. Therefore $\mathbf{S}$ is $\mathbf{S}_3$ or $\mathbf{3}$.

If $|A| = 2$, then $A = \{1, a\}$ for some $a \ne 1$ or $A = \{a, b\}$ for $a \ne b \ne 1$. In the first case $a\top = a(1 \vee a) = a \vee a = a$, and in the second case $a, b \le \top = 1$, hence $ab \le a, b$ and it follows that $ab = 0$. Therefore $\mathbf{S}$ is S_4 or $\mathbf{2} \times \mathbf{2}$.

If $|A| \ge 3$, then we have distinct elements $a, b, c \in A$. If $\top = 1$ then as in the previous case $ab = 0$ and similarly $ac, bc = 0$. We also have $b \vee c = 1$ since $\mathbf{S}$ has height 2. But now $0 = ab \vee ac = a(b + c) = a1 = a$ contradicts the assumption that a is an atom, hence we conclude that $\top \ne 1$ and therefore 1 is an atom. Since $\mathbf{S}$ has height 2, we have $a \vee 1 = \top = b \vee 1$ and

$$ab \vee b = ab \vee 1b = (a \vee 1)b = (b \vee 1)b = b \vee b = b.$$

It follows that $ab \leq b$, and similarly $ab \leq a$, hence $ab = 0$. In the case when $A = \{1, a, b\}$ we again have $a\top = a$ as well as $b\top = b$, therefore $\mathbf{S}$ is $\mathbf{S}_5$.

In all other cases $|A| > 3$, hence we have distinct $1, a, b, c \in A$ and $a + 1 = \top$. The same argument as above shows that $ac = 0$ and $bc = 0$, so

$$0 = ab \vee ac = a(b \vee c) = a\top = a(a \vee 1) = a \vee a = a$$

which again contradicts the assumption that a is an atom, so no further cdi-semirings of height 2 exist. □

3 Multiplicatively linear cdi-semirings

As mentioned in the introduction, cdi-semirings have a multiplicative semilattice order defined by $x \sqsubseteq y$ if and only if $x \cdot y = x$. A cdi-semiring is called *multiplicatively linear* if this multiplicative order is a chain. A search with Prover9/Mace4 [5] shows there are $1, 1, 2, 5, 14, 42$ such cdi-semirings of size up to 6. This sequence coincides with the sequence of Catalan numbers $C_n = \frac{1}{n}\binom{2n}{n}$ [9] and our next result shows that this coincidence continues for all n. Accordingly we refer to multiplicatively linear cdi-semirings as *Catalan* semirings. Using a result of [2] we construct all finite Catalan semirings by defining a *Catalan sum* © on this class.

Let $\mathbf{A}$ and $\mathbf{B}$ be two Catalan semirings and define $\mathbf{C} = \mathbf{A} © \mathbf{B}$ to be the structure over the disjoint union of A and B given in the following way. The $0, 1$ of $\mathbf{C}$ are $0^{\mathbf{A}}, 1^{\mathbf{A}}$ and the operations are given by

$$x \vee y = \begin{cases} x \vee^{\mathbf{A}} y & \text{if } x, y \in A \backslash \{0\} \\ x \vee^{\mathbf{B}} y & \text{if } x, y \in B \\ 1^{\mathbf{B}} \vee^{\mathbf{B}} y & \text{if } x \in A \backslash \{0\}, y \in B \\ 1^{\mathbf{B}} \vee^{\mathbf{B}} x & \text{if } x \in B, y \in A \backslash \{0\} \\ y & \text{if } x = 0^{\mathbf{A}} \\ x & \text{if } y = 0^{\mathbf{A}} \end{cases} \qquad xy = \begin{cases} x \cdot^{\mathbf{A}} y & \text{if } x, y \in A \backslash \{0\} \\ x \cdot^{\mathbf{B}} y & \text{if } x, y \in B \\ y & \text{if } x \in A \backslash \{0\}, y \in B \\ x & \text{if } x \in B, y \in A \backslash \{0\} \\ 0^{\mathbf{A}} & \text{if } x = 0^{\mathbf{A}} \text{ or } y = 0^{\mathbf{A}} \end{cases}$$

Recall that for two partially ordered sets P_1, P_2 the *ordinal sum* $P_1 \oplus P_2$ is given by the disjoint union of P_1, P_2 with every element of P_1 below every element of P_2. Using this construction, the multiplicative semilattice of $\mathbf{C}$ is simply the ordinal sum $\{0^{\mathbf{A}}\} \oplus (B, \sqsubseteq) \oplus (A \setminus \{0^{\mathbf{A}}\}, \sqsubseteq)$, and the join-semilattice of C is described by Figure 2. Note that if $\mathbf{A}$ or $\mathbf{B}$ is a one-element algebra, the underlying lattice of $\mathbf{A} © \mathbf{B}$ is the ordinal sum of the lattices of $\mathbf{A}$ and $\mathbf{B}$.

The next lemma is proved in [2] for finite commutative conservative idempotent residuated lattices. Every finite idempotent semiring uniquely expands to a finite residuated lattice, and it is conservative precisely when the semiring is multiplicatively linear. Hence we can state the result in the following way.

Lemma 3. *1. If $\mathbf{A}, \mathbf{B}$ are finite Catalan semirings then $\mathbf{A} © \mathbf{B}$ is a Catalan semiring of size $|A| + |B|$.*

2. *Suppose* $\mathbf{C}$ *is a finite Catalan semiring of cardinality* $n \geq 2$. *Then* $\mathbf{C} = \mathbf{A} © \mathbf{B}$ *for a unique pair* $\mathbf{A}, \mathbf{B}$ *of smaller Catalan semirings.*

Now start with the one-element Catalan semiring $\mathbf{C}_1^1$. The next one is $\mathbf{C}_1^2 = \mathbf{C}_1^1 © \mathbf{C}_1^1$, the two-element distributive lattice. The two 3-element cdi-semirings are $\mathbf{C}_1^3 = \mathbf{C}_1^1 © \mathbf{C}_1^2$ and $\mathbf{C}_2^3 = \mathbf{C}_1^2 © \mathbf{C}_1^1$. In general, the algebras of size n are built by constructing all Catalan sums of algebras $\mathbf{A}$ and $\mathbf{B}$ of size $n-k$ and k respectively, as k ranges from 1 to $n-1$ (see Figure 2). This yields the following result.

Theorem 4. *The number of Catalan semirings with* $n+1$ *elements, up to isomorphism, is the* n^{th} *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Proof. Let $CS(n)$ denote the number of Catalan semirings of cardinality n. The result is proved by induction. The sequence $\langle C_i : i \geq 0\rangle$ of Catalan numbers is determined recursively by $C_0 = 1$ and $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$. Obviously, $CS(1) = 1 = C_0$. Suppose now that $n \geq 1$ and $CS(n) = C_{n-1}$. Using the preceding lemma and the induction hypothesis, we have that

$$CS(n+1) = \sum_{k=1}^{n} CS(k) \cdot CS(n+1-k) = \sum_{k=1}^{n} C_{k-1}C_{n-k} = \sum_{i=0}^{n-1} C_i C_{n-1-i} = C_n.$$

□

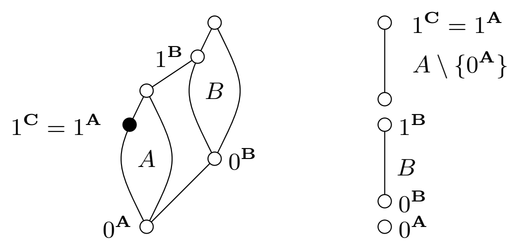

Fig. 2. The Catalan sum $\mathbf{C} = \mathbf{A} © \mathbf{B}$

The construction of finite Catalan semirings is very efficient and can be implemented, for example, with the following short Python program that computes all Catalan semirings of size $\leq n$. The output (after conversion to TikZ) is shown in Figure 3. The black dot marks the identity element and the elements are numbered in increasing order of the multiplicative semilattice. Note that these algebras are rigid (i.e., have trivial automorphism group) and are all pairwise nonisomorphic.

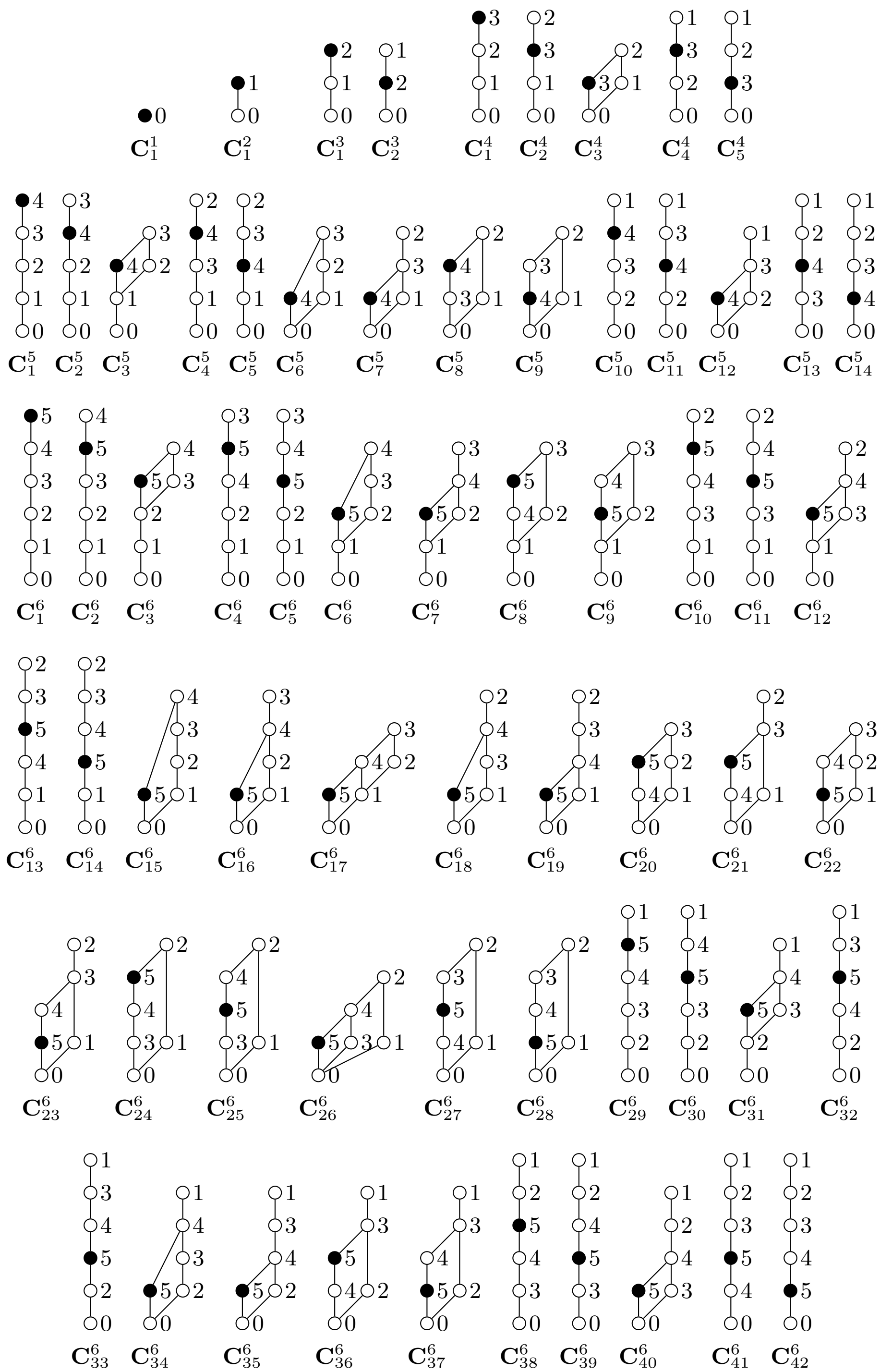

Fig. 3. Catalan semirings, i.e., multiplicatively linear cdi-semirings of size ≤ 6

```
def catalan_sum(A,B):
# A,B are tuples with A[0] a list of upper covers, topologically sorted
# A[1]=[s[0],...,s[n-1]] a permutation of range(n) s.t. x*y=x iff s[x]<=s[y]
# A[2]=[p[0],...,p[n-1]] a list of coordinates p[i]=(x,y) for display
    m = len(A[0])
    n = len(B[0])
    id_B = B[1].index(n-1)
    uc = [A[0][0]+([m] if n!=1 or m==1 else [])] + A[0][1:-1]\
        + ([A[0][-1]+[id_B+m]] if m!=1 else [])\
        + [[x+m for x in u] for u in B[0]]
    s = [0] + [x+n for x in A[1][1:]] + [x+1 for x in B[1]]
    x = A[2][-1][0] if m==1 or n==1 else max([p[0] for p in A[2]]) + 1
    y = (A[2][-1][1] + 1) if m==1 or n==1 else \
        max(1, A[2][-1][1] - B[2][id_B][1] + 1)
    pos = A[2] + [(B[2][0][0]+x, 1 if m!=1 and n!=1 else B[2][0][1]+y)] + \
        [(p[0]+x,p[1]+y) for p in B[2][1:]]
    return (uc,s,pos)

def catalan_semirings(n):
# calculate all Catalan semirings of size 1 to n
    if n==0: return [[([[]],[0],[(0,0)])]]
    CL = catalanRL(n-1)
    return CL + [[catalan_sum(A,B) for i in range(len(CL))
        for A in CL[i] for B in CL[n-1-i]]]
```

4 Boolean cdi-semirings and directed graphs

An idempotent semiring is *Boolean* if its join-semilattice is the reduct of a Boolean algebra. In this section we analyze the structure of finite Boolean cdi-semirings. We use ideas from the theory of Boolean algebras with operators and relation algebras [3,4] to recover the semiring operations from a ternary relation on the atoms of the Boolean algebra. The lemma below is a standard result that states this works in general for nonassociative nonunital complete atomic Boolean idempotent semirings. These algebras are also known as nonassociative atomic Boolean quantales. A *nonassociative quantale* $\mathbf{B} = (B, \bigvee, \cdot)$ is a complete join-semilattice $(B, \bigvee)$ with a binary operation $\cdot$ such that $x(\bigvee Y) = \bigvee_{y\in Y} xy$ and $(\bigvee Y)x = \bigvee_{y\in Y} yx$ for all $x \in B$ and $Y \subseteq B$. A *quantale* in addition satisfies the identity $(xy)z = x(yz)$. By completeness, every quantale has a least and a greatest element, denoted by 0 and $\top$ respectively. The complete distributivity of $\cdot$ over $\bigvee$ implies $x0 = 0 = 0x$. If it also has an *identity*, denoted by 1, then it is a *unital* quantale. Hence a join-complete idempotent semiring is the same as a unital quantale. As for semirings, a quantale is *Boolean* if its order is that of a complete Boolean algebra, and *atomic* if every nonzero element has an atom below it. The set of atoms of $\mathbf{B}$ is denoted by $At(\mathbf{B})$.

Lemma 5. *1. Let $\mathbf{B}$ be a nonassociative atomic Boolean quantale with $A = At(\mathbf{B})$ and define a ternary relation $R \subseteq A^3$ by $R(x, y, z) \iff x \leq yz$.*

Then for all $b, c \in B$,

$$ab = \bigvee\{x : \exists y \leq b\, \exists z \leq c\ R(x, y, z)\}.$$

2. *Suppose $R \subseteq A^3$ is a ternary relation on a set A, and define $\mathbf{B} = (\mathcal{P}(A), \bigcup, \cdot)$ where for $Y, Z \in P(A)$*

$$Y \cdot Z = \{x : \exists y \in Y\, \exists z \in Z\ R(x, y, z)\}.$$

Then $\mathbf{B}$ is a nonassociative atomic Boolean quantale.

As in the theory of Boolean algebras with operators or modal logic, the relational structure $\mathbf{A} = (A, R)$ from the preceding lemma is called the *atom structure* or *Kripke frame* of the Boolean quantale $\mathbf{B}$. Correspondence theory from modal logic also applies to Boolean quantales. For example, $\mathbf{B}$ is commutative if and only if $R(x, y, z) \Leftrightarrow R(x, z, y)$. It is convenient to split associativity into two inequalities $(xy)z \leq x(yz)$, called *subassociativity*, and $(xy)z \geq x(yz)$, called *supassociativity*.

Theorem 6. *Let $\mathbf{B}$ be a nonassociative atomic Boolean quantale with R defined on $A = At(\mathbf{B})$ as in the preceding lemma. Then $\mathbf{B}$ is*

mult. idempotent	$\Leftrightarrow$	$R(x, x, x)\,\&\,(R(x, y, z) \Rightarrow x = y \text{ or } x = z)$
subassociative	$\Leftrightarrow$	$(R(u, x, y)\,\&\,R(w, u, z) \Rightarrow \exists v(R(v, y, z)\,\&\,R(w, x, v)))$
right unital	$\Leftrightarrow$	$\exists I \subseteq A(x = y \Leftrightarrow \exists z \in I\, R(x, y, z))$

Proof. Assume $\mathbf{B}$ in multiplicatively idempotent, let x, y, z be atoms of $\mathbf{B}$ and assume $x \leq yz$. Then $y \vee z = (y \vee z)^2 = y^2 \vee yz \vee z^2 = y \vee z \vee yz$. Therefore $yz \leq y \vee z$. Since $x \leq yz$ we have $x \leq y \vee z$, and we assumed x, y, z are atoms, hence it follows that $x = y$ or $x = z$.

Now suppose R(x,x,x) and $(R(x, y, z) \Rightarrow x = y$ or $x = z)$ holds for all atoms $x, y, z \in A$. Let $c \in B$ and $C = \{x \in A : x \leq c\}$. Since $R(x, x, x)$ holds $c \leq c \cdot c$. Let x be an atom such that $x \leq c \cdot c$. Then $x \leq y \cdot z$ for some atoms $y, z \leq c$, therefore $R(x, y, z)$ holds and by assumption $x = y$ or $x = z$. Hence $x \leq c$ and it follows that $xx = x$.

Since all variables in subassociativity are distinct, this property holds for all elements of $\mathbf{B}$ if and only if it holds for all atoms. Now let $x, y, z \in A$. Then $(xy)z \leq x(yz)$ is equivalent to $w \leq (xy)z \Rightarrow w \leq x(yz)$ for all $w \in A$. This in turn is equivalent to

$$\exists u \in A\,(u \leq xy\ \&\ w \leq uz) \Rightarrow \exists v \in A\,(v \leq yz\ \&\ w \leq xv).$$

The first existential quantifier can move out of the premise to the front of the formula and switches to a universal quantifier, hence the formula translates to the given condition for R.

If $\mathbf{B}$ is unital then it has a 1 and we can define $I = \{z \in A : z \leq 1\}$. If $x = y$ then $x = y1$, hence there exists $z \leq 1$ such that $x \leq yz$, which shows $R(x, y, z)$. Likewise $R(x, w, y)$ for some $w \leq 1$ is derived from $x = 1y$. Conversely, assume $x \leq yz$ where $z \in I$. Then $yz \leq y1 = y$ implies $x \leq y$ and since both are atoms, $x = y$.

Now assume a set I of atoms with the given property exists in A and define $1 = \bigvee I$. It suffices to show that $y = y1$ for all atoms $y \in A$ since this identity lifts to all of $\mathbf{B}$. Let $x \leq y1$, then $x \leq yz$ for some $z \in I$. Hence $x = y$, which shows that y is the only atom below $y1$. It follows that $y = y1$. □

A ternary relation R is called commutative, (multiplicatively) idempotent, subassociative or unital if its corresponding Boolean quantale has the same property.

We now observe that if multiplication is idempotent then the ternary relation can be replaced by two reflexive relations P and Q. In the commutative case they coincide, so the structure of nonassociative Boolean cdi-semirings is determined by a single reflexive relation Q. The proof follows directly from the formula $R(x, y, z) \Rightarrow x = y$ or $x = z$.

Lemma 7. *An idempotent ternary relation $R \subseteq A^3$ is definitionally equivalent to a pair of reflexive binary relations $P, Q \subset A^2$ via the definitions*

$$(\mathsf{Pdef}) \quad P(x,y) \Leftrightarrow R(x,y,x) \qquad (\mathsf{Qdef}) \quad Q(x,y) \Leftrightarrow R(x,x,y)$$
$$(\mathsf{Rdef}) \quad R(x,y,z) \Leftrightarrow (x = y \ \&\ Q(y,z)) \text{ or } (x = z \ \&\ P(z,y)).$$

Moreover, the relation R is commutative if and only if $P = Q$.

The existentially quantified subassociative property for ternary relation is not easy to work with, hence it is noteworthy that, in the presence of idempotence, subassociativity can be replaced by the following three universal formulas for P and Q.

Theorem 8. *An idempotent ternary relation $R \subseteq A^3$ is subassociative if and only if the corresponding reflexive relations P, Q satisfy*

$$(\mathsf{P}_1) \quad P(x,y) \ \&\ P(y,z) \Rightarrow P(x,z) \quad \textit{i.e. } P\textit{-transitivity}$$
$$(\mathsf{P}_2) \quad Q(x,y) \ \&\ Q(x,z) \Rightarrow Q(y,z) \text{ or } P(z,y)$$
$$(\mathsf{P}_3) \quad P(x,y) \ \&\ Q(y,z) \ \&\ x \neq y \Rightarrow P(x,z)$$

To characterize supassociativity of R, it suffices to interchange P, Q in these conditions to obtain (P'_1), (P'_2), (P'_3). Hence R is associative if and only if P, Q satisfy all six conditions.

Proof. Suppose (P_1)–(P_3) hold and recall that subassociativity of R is given by

$$R(u,x,y) \,\&\, R(w,u,z) \Rightarrow \exists v (R(v,y,z) \,\&\, R(w,x,v)).$$

Assume $R(u,x,y)$ and $R(w,u,z)$ holds. From (Rdef) we get

$$[u = x \mathbin{\&} Q(x,y) \ \text{ or } \ u = y \mathbin{\&} P(y,x)] \ \text{ and }$$
$$[w = u \mathbin{\&} Q(u,z) \ \text{ or } \ w = z \mathbin{\&} P(z,u)].$$

We consider 4 cases, with the aim of showing that in each case there exists a v that satisfies the conclusion of subassociativity.

Case 1: Suppose $u = x$, $Q(x,y)$, $w = u$ and $Q(u,z)$. Then we have $u = x = w$, $Q(x,y)$ and $Q(x,z)$. From (P_2) we deduce $Q(y,z)$ or $P(z,y)$, and we want to find v such that

$$[(\mathsf{A}) \ \ v = y \mathbin{\&} Q(y,z) \text{ or } (\mathsf{B}) \ v = z \mathbin{\&} P(z,y)] \ \text{ and }$$
$$[(\mathsf{C}) \ w = x \mathbin{\&} Q(x,v) \text{ or } (\mathsf{D}) \ w = v \mathbin{\&} P(v,x)].$$

If $Q(y,z)$ holds, we choose $v = y$, then (A) and (C) hold, and if $P(z,y)$, we choose $v = z$, then (B) and (C) hold.

Case 2: Suppose $u = y$, $P(y,x)$, $w = u$ and $Q(u,z)$. Then $u = w = y$ and $P(y,x)$ and $Q(y,z)$ holds. Taking $v = y$ we get $v = y$ and $Q(y,z)$ and $w = v$ and $P(v,x)$. Hence (A) and (D) are true.

Case 3: Suppose $u = x$, $Q(x,y)$, $w = z$ and $P(z,u)$, hence $P(z,x)$. We want to find v such that $((\mathsf{A})$ or $(\mathsf{B}))$ and $((\mathsf{C})$ or $(\mathsf{D}))$ hold (see Case 1). First assume $z = y$, then choosing $v = y$ makes (A) and (D) true. Next assume $z \neq y$. Then by (P_3) we have $P(z,y) \mathbin{\&} Q(y,y) \mathbin{\&} z \neq y \Rightarrow P(z,y)$. Now choosing $v = z$ shows (B) and (D) hold.

Case 4: Suppose $u = y$, $P(y,x)$, $w = z$ and $P(z,u)$, hence $P(z,y)$. From (P_1) (transitivity) we deduce $P(z,x)$. Now taking $v = z$ we see that (B) and (D) are true.

Hence in all four cases we have proved subassociativity.

Conversely, assume that subassociativity holds for R:

$$R(u,x,y) \mathbin{\&} R(w,u,z) \Rightarrow \exists v(R(v,y,z) \mathbin{\&} R(w,x,v)).$$

We show that (P_1)–(P_3) hold.

For (P_1) assume $P(x,y)$ and $P(y,z)$. Then we have $R(x,y,x)$ and $R(y,z,y)$ by definition of P. Matching $R(y,z,y) \mathbin{\&} R(x,y,x)$ to the premise of subassociativity with $u = y$, $x = z$ and $w = x$, there exists v such that $R(v,y,x)$ and $R(x,z,v)$ holds. By idempotence of R it follows that $v = x$ and hence we get $P(x,z)$ from $R(x,z,x)$.

For (P_2) assume $Q(x,y)$ and $Q(x,z)$. By definition of Q we get $R(x,x,y)$ and $R(x,x,z)$. Let $u = x$ and $w = x$, then by subassociativity there exists v such that $R(v,y,z)$ and $R(x,x,v)$ holds. By idempotence there are two options for v: if $v = y$ we have $Q(y,z)$ and if $v = z$ we have $P(z,y)$. Hence (P_2) holds.

For (P_3) assume $Q(y,z)$ and $P(x,y)$ and $x \neq y$ hold. From the definition of Q and P we get $R(y,y,z)$ and $R(x,y,x)$. Let $u := y$, $x := y$, $y := z$, $w := x$ and $z := x$, then by subassociativity there exists v such that $R(v,z,x)$ and $R(x,y,v)$ hold. By idempotence the second conjunct implies $v = x$, so $P(x,z)$ follows from the first conjunct. Hence (P_3) is true. $\square$

Corollary 9. *An atomic Boolean idempotent quantale is determined by two reflexive binary relations P, Q on its set of atoms such that the condition* (P_1)*,* (P_2)*,* (P_3)*,* (P_1')*,* (P_2')*,* (P_3') *from the previous theorem hold.*

However the conditions (P_1), (P_2), (P_3) are nonintuitive, and it is fortunate that in the commutative case they reduce to a much simpler pair of axioms. Recall that a *preorder* is a reflexive transitive binary relation and a *partial order* is a preorder that is antisymmetric: $P(x, y)$ & $P(y, x) \Rightarrow x = y$. A *forest* is a partial order such that

$$(*) \qquad P(x, y) \ \&\ P(x, z) \Rightarrow P(y, z) \text{ or } P(z, y)$$

i.e., all the elements above a given element are linearly ordered. A forest can have many connected components, each of which is a *tree*. If each tree has a top element (called the root) then forest is said to be *rooted*. Finite forests are always rooted and they are easy to enumerate up to isomorphism. In fact they are in one-one correspondence with finite tree since one can add a new root to convert any forest into a tree with one more element. The number of finite trees with n unlabeled elements (i.e., up to isomorphism) is the sequence [8].

A preorder $P \subset A^2$ is determined by the equivalence relation $\equiv\, = P \cap P^{-1}$ and the induced partial order on the set of equivalence classes $P/\equiv$.

A *preorder forest* is a preorder that satisfies property $(*)$ and it is *rooted* if each component has a largest equivalence class. Finite preorder forests are always rooted, and the number of finite preorder forests with n unlabeled elements (i.e., up to isomorphism) is also easy to count and given in [7]. Finally, a preorder forest is said to have *singleton roots* if it is rooted and all largest equivalence classes contain only one element.

Theorem 10. *Atomic Boolean commutative idempotent unital quantales are definitionally equivalent to preorder forests with singleton roots.*

In the finite case these algebras are Boolean cdi-semirings, hence all finite Boolean cdi-semirings can be constructed by enumerating preorder forests with singleton roots.

Proof. Let P, Q be the reflexive binary relations on the atoms that exist by idempotence. From commutativity it follows that $P = Q$ hence (P_2) reduces to $(*)$ and (P_1) implies (P_3). This means the relation P is a preorder forest. For any atom z below 1, $P(z, x)$ implies $R(z, x, z)$, and it follows from unitality that $x = z$. Hence z is a unique maximal element of the preorder.

Conversely, from a preorder forest with singleton roots we define I to be the set of all roots of the forest to get a unit element for the quantale. □

Fig. 4 shows the preorder forests with singleton roots up to cardinality 4. They correspond to Boolean semirings of size $2, 4, 8$ and 16. It is interesting to note that there are $1, 2, 5, 14, \ldots$ such semirings of each size, but this is *not* related to the Catalan numbers since the sequence continues with 41 followed by 127 (while the Catalan numbers are $42, 132$).

Note that every finite forest is a preorder forest with singleton roots, and it is interesting to investigate the multiplicative semilattices a obtained from specific finite forests. As a simple example, the forests where each component is a singleton poset correspond to cdi-semirings that are Boolean lattices.

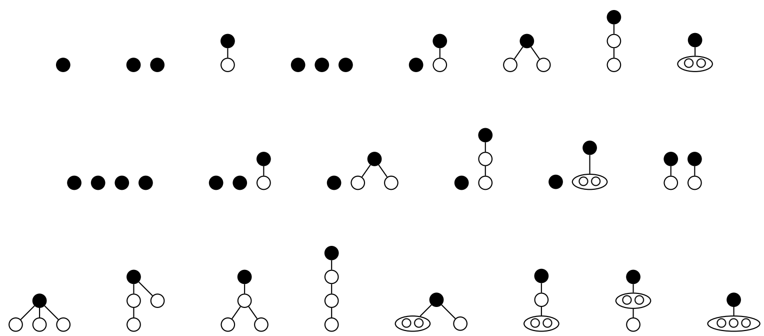

Fig. 4. Preorder forests with singleton roots

5 Conclusion

In the theory of rings and other algebras, multiplicatively idempotent elements often play a central role in controlling some structural aspects of the algebra. The structure of idempotent semirings in general is quite challenging, but with suitable restrictions some nice characterizations can be found. Here we considered commutative doubly idempotent semirings of height ≤ 2, or with a multiplicative linear order or with a Boolean join-semilattice. In each case it was possible to give detailed descriptions of the finite members that allow them to be enumerated easily up to isomorphism. It is likely that some of the techniques explored here can be applied to larger classes of idempotent semirings by, for example, weakening the assumption of commutativity or allowing distributive join-semilattices.

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