Frames and spaces for distributive quasi relation algebras and distributive involutive FL-algebras

Andrew Craig[1][0000−0002−4787−3760], Peter Jipsen[2,][0000−0001−8608−808X], and Claudette Robinson[1][0000−0001−7789−4880]

[1] University of Johannesburg, South Africa
{acraig,claudetter}@uj.ac.za
[2] Chapman University, Orange CA 92866, USA
jipsen@chapman.edu

Abstract. Analogous to atom structures for relation algebras, we define partially ordered frames and prove they are duals for complete perfect distributive quasi relation algebras and distributive involutive FL-algebras. We then extend this dual representation to all algebras and their corresponding frames with a Priestley topology.
For relation algebras up to size 16 it has been determined which algebras are representable by binary relations. We compute all finite distributive quasi relation algebras up to 8 elements and provide representations for some of them.

Keywords: Quasi relation algebras · involutive FL-algebras · Priestley spaces · representations.

1 Introduction

We investigate two classes of algebras which are connected to relation algebras: distributive involutive FL-algebras (DInFL-algebras) and distributive quasi relation algebras (DqRAs). This second class of algebras is a generalization of relation algebras, which includes all relation algebras and all commutative distributive involutive residuated lattices.

Quasi relation algebras (qRAs) were first studied by Galatos and Jipsen [9]. One of the features of the variety is that they have a decidable equational theory [9, Corollary 5.6]. Concrete distributive qRAs have been used to develop a notion of representability [4] for DqRAs which mimics the intensely studied concept of representability for relation algebras.

Our first goal is to describe dual frames for both DInFL-algebras and DqRAs. We do this in Section 3. Studying these algebras via (partially ordered) frames is motivated by the fact that a powerful tool for the study of finite relation algebras has been the so-called *atom structures*. Any finite relation algebra can be studied simply via an operation table of its atoms. Frames are useful for implementing a decision procedure for DqRA since, e.g., tableaux methods build frame-based counterexamples or prove that none exist.

In applications to computing, relations are often treated in the setting of heterogeneous relation algebras or within category theory as allegories. In the current paper we consider only unisorted DqRAs and their frames, but it is possible to adapt these concepts to the heterogeneous and categorical settings.

Another reason for wanting to develop frames is for the development of a game-based approach to determine whether certain algebras are representable. This has been done very successfully in the relation algebra case [21], [15] and also more recently for weakening relation algebras [17].

Our setting for the dual frames are complete perfect algebras (i.e. the completely join-irreducible elements are join-dense and the completely meet irreducible elements are meet-dense). All results obviously apply to the case of finite algebras. We then extend our frame-based approach to a Priestley-style representation in Section 4.

In the first part of Section 5 we present tables with the number of both DInFL-algebras and DqRAs up to size 8. These were calculated using Prover9/Mace4 [24]. We verified these numbers by calculating the numbers of corresponding frames. In the remainder of Section 5, we consider cyclic DqRAs which are $\{\vee, \cdot, 1, \sim\}$-subreducts of 16-element relation algebras. For the relation algebras which are representable, this leads to representability of the term-subreduct. We also identify the dual frames of particular DqRAs and relate them to the atom structures of the relation algebras in which they can be embedded.

2 Background

An *involutive full Lambek algebra* (briefly InFL-algebra) $\mathbf{A} = (A, \wedge, \vee, \cdot, 1, \sim, -)$ is a lattice $(A, \wedge, \vee)$ and a monoid $(A, \cdot, 1)$ such that for all $a, b, c \in A$

$$a \cdot b \leq c \iff a \leq -(b \cdot \sim c) \iff b \leq \sim(-c \cdot a). \tag{1}$$

Hence $-(b \cdot \sim c)$ and $\sim(-c \cdot a)$ are terms for the right residual c/b and the left residual $a\backslash c$ respectively. It follows that $-1 = \sim 1$ and this element is usually denoted 0. The operations $-, \sim$ are called *linear negations* and are order-reversing as well as *involutive*: $-\sim a = a = \sim - a$. If the residuals and 0 are included in the signature, then $\sim, -$ can be defined by $\sim a = a\backslash 0$ and $-a = 0/a$, and we obtain InFL-algebras (in a term-eqivalent form) if the identities $-\sim a = a = \sim - a$ hold.

The same variety can also be axiomatized by the following idempotent semiring axioms [18]: an InFL-algebra $\mathbf{A} = (A, \vee, \cdot, 1, \sim, -)$ is a semilattice $(A, \vee)$ and a monoid $(A, \cdot, 1)$ such that $\cdot$ distributes over $\vee$ and $a \leq b \iff a \cdot \sim b \leq -1 \iff -b \cdot a \leq -1$. Moreover $-\sim a = a = \sim - a$, and if we define $a \wedge b = -(\sim a \vee \sim b)$ then $(A, \wedge, \vee)$ is a lattice. The operation $+$ is defined by $a + b = -(\sim b \cdot \sim a)$, or equivalently by $a + b = \sim(-b \cdot -a)$ [12].

An InFL-algebra is *cyclic* if $\sim a = -a$, *commutative* if $a \cdot b = b \cdot a$ and distributive if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

A *De Morgan lattice* $\mathbf{A} = (A, \wedge, \vee, \neg)$ is a lattice $(A, \wedge, \vee)$ with a unary operation that satisfies $\neg\neg a = a$ and

$$\text{(Dm)} \qquad \neg(a \wedge b) = \neg a \vee \neg b.$$

A *De Morgan InFL-algebra* $\mathbf{A} = (A, \wedge, \vee, \cdot, 1, \sim, -, \neg)$ is a De Morgan lattice $(A, \wedge, \vee, \neg)$ and an InFL-algebra $(A, \vee, \cdot, 1, \sim, -)$.

A *quasi relation algebra* (qRA for short) is a De Morgan InFL-algebra such that the following identity holds:

$$\text{(Dp)} \qquad \neg(a \cdot b) = \neg a + \neg b.$$

Interesting qRAs that are not relation algebras include the finite Sugihara chains (where ${\sim}a = -a = a'$). See Table 1 and [4, Figure 1] for further examples of (non)cyclic qRAs.

In [9] the definition of qRAs included a third identity, (Di): $\neg{\sim}a = -\neg a$, but we note here that it is implied by the above definition. The proof follows from applications of (Dp) and (1) from above.

Lemma 1. *The identities* ${\sim}1 = \neg 1 = -1$ *and* (Di) *hold in any qRA.*

A relation algebra is a cyclic DqRA if the linear negations $\sim, -$ are defined as *complement-converse* $(\neg a)^{\smile} = \neg(a^{\smile})$ and a cyclic DqRA is a relation algebra if $\neg$ is complementation, i.e., $a \vee \neg a = \top$ and $a \wedge \neg a = \bot$.

The next result shows that qRAs can be obtained from commutative InFL-algebras, and if the lattice reduct is distributive this provides examples of DqRAs.

Lemma 2. *Commutative InFL-algebras are quasi relation algebras.*

Proof. Commutative InFL-algebras satisfy $a\backslash b = b/a$, hence they are cyclic. Defining $\neg a = {\sim}a$, it is immediate that (Di) and (Dm) hold, and (Dp) follows from the definition of $a + b = -({\sim}b \cdot {\sim}a)$ and commutativity. □

DqRAs can also be obtained from term-subreducts of relation algebras with respect to the signature $\{\vee, \cdot, 1, \sim\}$ where, as before, ${\sim}x = -x = (\neg x)^{\smile}$. In the case of representable relation algebras, these subreducts are known as *(representable) weakening relation algebras* [10], [11], [17]. If the subreducts are commutative, they are DqRAs by the preceding lemma. However, the next result implies that one should start with nonsymmetric relation algebras if the aim is to construct DqRAs that are not relation algebras.

An element a in a DqRA is said to be *symmetric* if $\neg a = {\sim}a = -a$. Since $\neg$ commutes with all basic operations, we get the following result.

Lemma 3. *If all generators are symmetric in a cyclic DqRA, then the generated term-subreduct with respect to the signature* $\{\vee, \cdot, 1, \sim\}$ *is a relation algebra.*

We now recall the definition of perfect distributive lattices. For a lattice $\mathbf{A}$ we denote by $J^{\infty}(\mathbf{A})$ and $M^{\infty}(\mathbf{A})$ the completely join-irreducible and completely meet-irreducible elements, respectively. A lattice $\mathbf{A}$ is *perfect* if for every $a \in A$, we have $a = \bigvee\{\, j \in J^{\infty}(\mathbf{A}) \mid j \leqslant a \,\} = \bigwedge\{\, m \in M^{\infty}(\mathbf{A}) \mid a \leqslant m \,\}$, i.e., $J^{\infty}(\mathbf{A})$ is join-dense and $M^{\infty}(\mathbf{A})$ is meet-dense. Some authors include completeness as part of the definition of perfect, but we state it separately. In Section 3 we give a representation for DInFL-algebras and DqRAs whose underlying lattices are complete perfect distributive lattices. Some equivalences in the theorem below are in the paper by Gehrke and Jónsson [13, Theorem 2.2].

Theorem 4. *For* $\mathbf{A}$ *a complete distributive lattice, the following are equivalent:*

(1) $\mathbf{A}$ *is completely distributive and* $J^\infty(\mathbf{A})$ *is join-dense in* $\mathbf{A}$*;*
(2) $J^\infty(\mathbf{A})$ *is join-dense in* $\mathbf{A}$ *and* $M^\infty(\mathbf{A})$ *is completely meet-dense in* $\mathbf{A}$*;*
(3) $\mathbf{A}$ *is isomorphic to the lattice of up-sets of some poset* P*.*

We remark that in a completely distributive lattice, all completely join-irreducible elements are completely join-prime and all completely meet-irreducible elements are completely meet-prime.

3 Frames for distributive quasi relation algebras and distributive involutive FL-algebras

In this section, we present complete perfect DqRAs and complete perfect DInFL-algebras by frames, similar to the frames of finite representable weakening relation algebras [17] and atom structures of atomic relation algebras [20].

The frames for DInFL-algebras are essentially Routley–Meyer style frames from relevance logic [1], but with the relevant negation replaced by two linear negations that are inverses of each other.

Definition 5. *A* DInFL-frame *is a tuple* $\mathbb{W} = (W, I, \preceq, R, {}^\sim, {}^-)$ *with* $W \neq \varnothing$*, a unary predicate* I*, a partial order* $\preceq$ *on* W*, a ternary relation* R *on* W *and functions* ${}^\sim : W \to W$ *and* ${}^- : W \to W$ *such that* I *is upward closed with respect to* $\preceq$ *and the following conditions hold for all* $u, v, x, y, z \in W$*:*

(1) $x \preceq y \iff \exists i\,(i \in I \wedge Rixy)$
(2) $x \preceq y \iff \exists i\,(i \in I \wedge Rxiy)$
(3) $x \preceq y \wedge Ruvx \implies Ruvy$
(4) $\exists s\,(Rxys \wedge Rsuv) \iff \exists t\,(Ryut \wedge Rxtv)$
(5) $Rxyz^\sim \iff Rzxy^-$
(6) $x^{\sim-} \preceq x$ *and* $x^{-\sim} \preceq x$

The following lemma shows that ${}^-$ and ${}^\sim$ are inverses of each other.

Lemma 6. *Let* $\mathbb{W} = (W, I, \preceq, R, {}^\sim, {}^-)$ *be a DInFL-frame. Then* $x^{\sim-} = x = x^{-\sim}$ *for all* $x \in W$*.*

Proof. We only have to prove that $x \preceq x^{\sim-}$ and $x \preceq x^{-\sim}$. We have $x^\sim \preceq x^\sim$, so by (1) in Definition 5 there is some $i \in I$ such that $i \in I$ and $Rix^\sim x^\sim$. An application of (5) gives $Rxix^{\sim-}$, and so, by (2), $x \preceq x^{\sim-}$.

We also have $x^- \preceq x^-$, so by (2) there is some $i \in W$ such that $i \in I$ and Rx^-ix^-. Applying (5) to the latter gives $Rixx^{-\sim}$. Hence, by (1), $x \preceq x^{-\sim}$. □

Using the above lemma we can show that ${}^-$ and ${}^\sim$ are order-reversing maps.

Proposition 7. *Let* $\mathbb{W} = (W, I, \preceq, R, {}^\sim, {}^-)$ *be a DInFL-frame. If* $x \preceq y$ *then* $y^- \preceq x^-$ *and* $y^\sim \preceq x^\sim$*.*

Proof. Assume $x \preceq y$. Then by item (1) in Definition 5 there is some $i \in W$ such that $i \in I$ and $Rixy$. The latter is equivalent to $Rixy^{-\sim}$ by Lemma 6. Applying (5) to this gives $Ry^{-}ix^{-}$. Therefore, by (2) $y^{-} \preceq x^{-}$.

For the second inequality, by (2) there is some $i \in W$ such that $i \in I$ and $Rxiy$. The latter is equivalent to $Rxiy^{\sim -}$ by Lemma 6. Hence, by (5), $Riy^{\sim}x^{\sim}$, and therefore, by item (1), $y^{\sim} \preceq x^{\sim}$. □

The next proposition says that the ternary relation R is order-reversing in the first and second coordinates. Galatos [8, Definition 3.4] includes both conditions of Proposition 8 in his dual structures for distributive residuated lattices. For us they are provable from conditions (3) and (5) of Definition 5.

Proposition 8. *Let $\mathbb{W} = (W, I, \preceq, R, ^{\sim}, ^{-})$ be a DInFL-frame.*

(a) *If $x \preceq y$ and $Ryuv$, then $Rxuv$.*
(b) *If $x \preceq y$ and $Ruyv$, then $Ruxv$.*

Proof. For (a), assume $x \preceq y$ and $Ryuv$. Applying Proposition 7 to the first part gives $y^{\sim} \preceq x^{\sim}$. The second part is equivalent to $Ryuv^{\sim -}$ by Lemma 6, which means $Ruv^{\sim}y^{\sim}$ (by (5) of Definition 5). Hence, since $y^{\sim} \preceq x^{\sim}$ and $Ruv^{\sim}y^{\sim}$, by (3) of Definition 5, $Ruv^{\sim}x^{\sim}$. This gives $Rxuv^{\sim -}$, which is equivalent to $Rxuv$.

For (b), assume $x \preceq y$ and $Ruyv$. Then $y^{-} \preceq x^{-}$ (by Proposition 7) and $Ruyv^{-\sim}$ (by Lemma 6). Hence, $y^{-} \preceq x^{-}$ and $Rv^{-}uy^{-}$, and so, by (3), $Rv^{-}ux^{-}$. Applying (5) to this gives $Ruxv^{-\sim}$, which means $Ruxv$. □

Proposition 9. *Let $\mathbb{W} = (W, I, \preceq, R, ^{\sim}, ^{-})$ be a DInFL-frame. Let $\mathsf{Up}(W, \preceq)$ be the set of all upsets of $(W, \preceq)$. For all $U, V \in \mathsf{Up}(W, \preceq)$ define $U \circ V = \{w \in W \mid (\exists u \in U)(\exists v \in V)(Ruvw)\}$, $\sim U = \{w \in W \mid w^{-} \notin U\}$ and $-U = \{w \in W \mid w^{\sim} \notin U\}$. Then $\mathbb{W}^{+} = (\mathsf{Up}(W, \preceq), \cap, \cup, \circ, I, \sim, -)$ is a DInFL-algebra.*

Proof. We first check if $\mathsf{Up}(W, \preceq)$ is closed under the above operations. It is well-known (and easy to check) that $\mathsf{Up}(W, \preceq)$ is closed under taking unions and intersections. It follows that $I \in \mathsf{Up}(W, \preceq)$ by assumption. Now let $x \in U \circ V$ and $y \in W$. Assume $x \preceq y$. Since $x \in U \circ V$, there are $u \in U$ and $v \in V$ such that $Ruvx$. Hence, by (3) of Definition 5, we get $Ruvy$, and so $y \in U \circ V$. This shows that $U \circ V \in \mathsf{Up}(W, \preceq)$.

Next let $x \in \sim U$ and $y \in W$. Assume $x \preceq y$. Since $x \in \sim U$, we have $x^{-} \notin U$. Applying Proposition 7 to $x \preceq y$, we obtain $y^{-} \preceq x^{-}$. Hence, since $U \in \mathsf{Up}(W, \preceq)$, we have $y^{-} \notin U$. This gives $y \in \sim U$. Thus, $\sim U \in \mathsf{Up}(W, \preceq)$ for all $U \in \mathsf{Up}(W, \preceq)$. Using Proposition 7 we can show in a similar way that $-U \in \mathsf{Up}(W, \preceq)$ for all $U \in \mathsf{Up}(W, \preceq)$.

Next we show that $I \circ U = U \circ I = U$ for all $U \in \mathsf{Up}(W, \preceq)$. Let $x \in I \circ U$. Then there is some $i \in I$ and $u \in U$ such that $Riux$. Hence, by (1) of Definition 5, $u \preceq x$. But $U \in \mathsf{Up}(W, \preceq)$, so $x \in U$.

Now let $x \in U$. We have $x \preceq x$, so by (1) there is some $i \in I$ such that $Rixx$. Hence, $x \in I \circ U$. Using (2) of Definition 5 we can show in a similar way that $U \circ I = U$ for all $U \in \mathsf{Up}(W, \preceq)$.

The associativity of $\circ$ follows from (4) in Definition 5.

Finally, we prove that $T \circ U \subseteq V$ iff $U \subseteq {\sim}(-V \circ T)$ iff $T \subseteq -(U \circ {\sim}V)$ for all $T, U, V \in \mathsf{Up}(W, \preceq)$. First, assume that $T \circ U \subseteq V$. Let $u \in U$ and suppose $u \notin {\sim}(-V \circ T)$. The latter gives $u^- \in (-V \circ T)$. Hence, there exist $v \in -V$ and $t \in T$ such that $Rvtu^-$, and so by (5), $Rtuv^\sim$. Since $t \in T$ and $u \in U$, we have $v^\sim \in T \circ U$, which means $v^\sim \in V$. This gives $v \notin -V$, a contradiction.

Conversely, assume $U \subseteq {\sim}(-V \circ T)$. Let $w \in T \circ U$. Then there exists $t \in T$ and $u \in U$ such that $Rtuw$. Since $u \in U$, we have $u \in {\sim}(-V \circ T)$. Hence, $u^- \notin -V \circ T$. This implies that for all $x, y \in W$, if $y \in T$ and $Rxyu^-$ then $x \notin -V$. Now $Rtuw$ is equivalent to $Rtuw^{-\sim}$ by Lemma 6, which is equivalent to Rw^-tu^- by (5). Since $t \in T$, we have $w^- \notin -V$. Hence, $w^{-\sim} \in V$, and so $w \in V$. This shows that $T \circ U \subseteq V$.

Now assume $T \circ U \subseteq V$ and let $t \in T$. Suppose $t \notin -(U \circ {\sim}V)$. Then $t^\sim \in U \circ {\sim}V$. Hence, there exists $u \in U$ and $v \in {\sim}V$ such that $Ruvt^\sim$. We thus have $t \in T$, $u \in U$ and $Rtuv^-$. This gives $v^- \in T \circ U$, and so $v^- \in V$. Thus, $v \notin {\sim}V$, a contradiction.

Conversely, assume $T \subseteq -(U \circ {\sim}V)$ and let $w \in T \circ U$. This implies there are $t \in T$ and $u \in U$ such that $Rtuw$. Since $t \in T$, we get $t \in -(U \circ {\sim}V)$, and so $t^\sim \notin U \circ {\sim}V$. Hence, for all $x, y \in W$, if $x \in U$ and $Rxyt^\sim$, then $y \notin {\sim}V$. Now $Rtuw$ is equivalent to $Rtuw^{\sim -}$, which in turn is equivalent to $Ruw^\sim t^\sim$. Therefore, since $u \in U$, we get $w^\sim \notin {\sim}V$. This gives $w^{\sim -} \in V$, thus $w \in V$. $\square$

The algebra $\mathbb{W}^+$ in Proposition 9 is called the *complex algebra* of $\mathbb{W}$. It is easy to check that $\mathbb{W}^+$ is complete and perfect.

Recall that for any complete perfect distributive lattice $\mathbf{A} = (A, \wedge, \vee)$, the map $\kappa : J^\infty(\mathbf{A}) \to M^\infty(\mathbf{A})$ defined by $\kappa(j) = \bigvee\{a \in A \mid j \nleqslant a\}$ is an order isomorphism. (cf. [6, Section II.5]). We now define two useful unary operations on the completely join-irreducibles of a complete perfect distributive InFL-algebra.

Definition 10. *For every completely join-irreducible a of a complete perfect DInFL-algebra $\mathbf{A}$, define $a^\sim = {\sim}\kappa(a)$ and $a^- = -\kappa(a)$.*

Since κ maps completely join-irreducibles to completely meet-irreducibles and $-$ and $\sim$ are order-reversing, we obtain the following lemma.

Lemma 11. *Let $\mathbf{A} = (A, \wedge, \vee, \cdot, 1, \sim, -)$ be a complete perfect DInFL-algebra. If a is a completely join-irreducible, then so are $a^\sim$ and a^-.*

The following propositions adapt a well-known method (see [5]) for obtaining dual frames from complete perfect algebras:

Proposition 12. *Let $\mathbf{A} = (A, \wedge, \vee, \cdot, 1, \sim, -)$ be a complete perfect DInFL-algebra. Let $J^\infty(\mathbf{A})$ be the set of completely join-irreducibles of $\mathbf{A}$. Set $I_1 = \{i \in J^\infty(\mathbf{A}) \mid i \leqslant 1\}$ and, for all $a, b, c \in J^\infty(\mathbf{A})$, define $\preceq\, = \,\geqslant$, $R.abc$ iff $c \leqslant a \cdot b$, $a^\sim = {\sim}\kappa(a)$ and $a^- = -\kappa(a)$. Then the structure $\mathbf{A}_+ = (J^\infty(\mathbf{A}), I_1, \preceq, R., {}^\sim, {}^-)$ is a DInFL-frame.*

Proof. First, to see that I_1 is upward closed w.r.t. $\preceq$, let $i \in I_1$ and $j \in J^\infty(\mathbf{A})$, and assume $i \preceq j$. Then $i \leqslant 1$ and $j \leqslant i$. Hence, $j \leqslant 1$, and so $j \in I_1$.

Next we show that $\mathbf{A}_+$ satisfies (1) of Definition 5. Let $a, b \in J^\infty(\mathbf{A})$ and assume $a \preceq b$. Then $b \leqslant a$. Since $\mathbf{A}$ is completely join-generated by $J^\infty(\mathbf{A})$, we have $b \leqslant 1 \cdot a = \bigvee\{j \in J^\infty(\mathbf{A}) \mid j \leqslant 1\} \cdot a = \bigvee\{j \cdot a \mid j \in J^\infty(\mathbf{A}) \text{ and } j \leqslant 1\}$. Now b is completely join-prime, so we have $b \leqslant i \cdot a$ for some $i \in J^\infty(\mathbf{A})$ such that $i \leqslant 1$. Hence, there is some $i \in J^\infty(\mathbf{A})$ such that $i \in I_1$ and $R.iab$.

Conversely, let $i \in J^\infty(\mathbf{A})$ such that $i \in I_1$ and $R.iab$. Then $i \leqslant 1$ and $b \leqslant i \cdot a$. From $i \leqslant 1$, we obtain $i \cdot a \leqslant 1 \cdot a = a$, and so $b \leqslant i \cdot a = a$. This shows that $a \preceq b$. The proof of (2) is similar.

For (3), assume $a \preceq b$ and $R.cda$. Then $b \leqslant a$ and $a \leqslant c \cdot d$, and so $b \leqslant c \cdot d$, which shows that $R.cdb$.

Condition (4) follows from the associativity of the monoid operation.

We now show that $R.abc^\sim$ iff $R.cab^-$ for all $a, b, c \in J^\infty(\mathbf{A})$. First assume $R.abc^\sim$. Then $c^\sim \leqslant a \cdot b$, and so $\sim\kappa(c) \leqslant a \cdot b$. Since $-$ is order-reversing and $-$ and $\sim$ are inverses of each other, we have $-(a \cdot b) \leqslant \bigvee \kappa(c)$ To prove that $R.cab^-$, we have to show that $b^- \leqslant c \cdot a$, i.e., $-\kappa(b) \leqslant c \cdot a$. This is equivalent to showing that $\sim(c \cdot a) \leqslant \kappa(b)$ Hence, if we can show that $b \nleqslant \sim(c \cdot a)$, we would be done. So suppose $b \leqslant \sim(c \cdot a)$. Then $b \leqslant \sim(-\sim c \cdot a)$, and so $a \cdot b \leqslant \sim c$ by (1) in Section 2, which means $c \leqslant -(a \cdot b)$. We thus obtain $c \leqslant \kappa(c)$. Therefore, since c is completely join-prime, there is some $s \in A$ such that $c \leqslant s$ and $c \nleqslant s$, a contradiction. The converse implication can be proved in a similar way.

Finally, we have to show that $a^{\sim -} \preceq a$ and $a^{-\sim} \preceq a$ for all $a \in J^\infty(\mathbf{A})$. We will show $a^{\sim -} \preceq a$ for all $a \in J^\infty(\mathbf{A})$ and leave the other case for the reader. Proving that $a^{\sim -} \preceq a$ is equivalent to showing that $a \leqslant a^{-\sim}$, which is equivalent to $a \leqslant \sim\kappa(a^-)$ Since $-$ is order-reversing and $\sim$ and $-$ are inverses of each other, this is equivalent to showing that $\kappa(a^-) \leqslant -a$. That is, we have to show that $b \leqslant -a$ for all $b \in A$ such that $a^- \nleqslant b$. Let b be an arbitrary element of A such that $a^- \nleqslant b$. For the sake of a contradiction, suppose $b \nleqslant -a$. Then $a = \sim - a \nleqslant \sim b$. Since $a^- \nleqslant b$, we have $\sim b \nleqslant \kappa(a)$. Hence, $\sim b \nleqslant c$ for all $c \in A$ such that $a \nleqslant c$. But $a \nleqslant \sim b$, so in particular, $\sim b \nleqslant \sim b$, a contradiction. □

The next theorem shows that every complete perfect DInFL-algebra is isomorphic to the complex algebra of its DInFL-frame of completely join-irreducibles.

Theorem 13. *If* $\mathbf{A} = (A, \wedge, \vee, \cdot, 1, \sim, -)$ *is a complete perfect DInFL-algebra, then* $\mathbf{A} \cong (\mathbf{A}_+)^+$.

Proof. The fact that the map $\psi : A \to \mathsf{Up}(J^\infty(\mathbf{A}), \preceq)$ defined by $\psi(a) = \{j \in J^\infty(\mathbf{A}) \mid j \leqslant a\}$ is a lattice isomorphism is well known (cf. [14]).

First, we note that $\psi(1) = \{j \in J^\infty(\mathbf{A}) \mid j \leqslant 1\} = I_1$. For the monoid operation, first let $i \in \psi(a \cdot b)$. Then $i \in J^\infty(\mathbf{A})$ and $i \leqslant a \cdot b$. Since $\mathbf{A}$ is completely join-generated by $J^\infty(\mathbf{A})$, we have

$$\begin{aligned} i \leqslant a \cdot b &= \bigvee\{j \in J^\infty(\mathbf{A}) \mid j \leqslant a\} \cdot \bigvee\{k \in J^\infty(\mathbf{A}) \mid k \leqslant b\} \\ &= \bigvee\left\{j \cdot \bigvee\{k \in J^\infty(\mathbf{A}) \mid k \leqslant b\} \mid j \in J^\infty(\mathbf{A}) \text{ and } j \leqslant a\right\} \\ &= \bigvee\{j \cdot k \mid j, k \in J^\infty(\mathbf{A}), j \leqslant a \text{ and } j \leqslant k\} \end{aligned}$$

Since i is completely join-prime, it follows that $i \leqslant j \cdot k$ for some $j, k \in J^{\infty}(\mathbf{A})$ such that $j \leqslant a$ and $k \leqslant b$. Hence, $j \in \psi(a)$, $k \in \psi(b)$ and $R.jki$, which shows that $i \in \psi(a) \circ \psi(b)$.

For the other inclusion, let $i \in \psi(a) \circ \psi(b)$. Then there are $j \in \psi(a)$ and $k \in \psi(b)$ such that $R.jki$. Hence, $j \leqslant a$, $k \leqslant b$ and $i \leqslant j \cdot k$. From the first part we get $j \cdot k \leqslant a \cdot k$ and from the second part we get $a \cdot k \leqslant a \cdot b$. It follows that $i \leqslant j \cdot k \leqslant a \cdot k \leqslant a \cdot b$, and so $i \in \psi(a \cdot b)$.

Next we show that $\psi(\sim a) = \sim\psi(a)$. First, let $j \in \psi(\sim a)$. Then $j \in J^{\infty}(\mathbf{A})$ and $j \leqslant \sim a$. For the sake of a contradiction, suppose $j \notin \sim\psi(a)$. Then we have $j^{-} \in \psi(a)$, and so $j^{-} \leqslant a$. Since $j \leqslant \sim a$ and $-$ is order-reversing, we obtain $a = -\sim a \leqslant -j$. Hence, $-\kappa(j) = j^{-} \leqslant -j$, which means $j \leqslant \kappa(j)$. Since j is completely join-prime, there is some $s \in A$ such that $j \leqslant s$ and $j \nleqslant s$, a contradiction. It must therefore be the case that $j \in \sim\psi(a)$.

For the other inclusion, let $j \in \sim\psi(a)$. Then $j^{-} \notin \psi(a)$, and so $j^{-} = -\kappa(j) \nleqslant a$, which gives $\sim a \nleqslant \kappa(j)$. Hence, $\sim a \nleqslant s$ for all $s \in A$ such that $j \nleqslant s$. Therefore it must be the case that $j \leqslant \sim a$; for otherwise, $\sim a \nleqslant \sim a$, a contradiction. Thus $j \in \psi(\sim a)$. The proof that $\psi(-a) = -\psi(a)$ is similar. $\square$

As mentioned, the complex algebra of a DInFL-frame $\mathbb{W} = (W, I, \preceq, R, {}^{\sim}, {}^{-})$ is a complete perfect DInFL-algebra. The following proposition says that a DInFL-frame is isomorphic to the frame of completely join-irreducibles of its complex algebra. It is well known that $J^{\infty}(\mathbb{W}^{+}) = \{\uparrow x \mid x \in W\}$ the set of all principal upsets $\uparrow x = \{y \in W \mid x \preceq y\}$. The standard map $x \mapsto \uparrow x$ gives us the required isomorphism.

Theorem 14. *If $\mathbb{W} = (W, I, \preceq, R, {}^{\sim}, {}^{-})$ is a DInFL- frame, then $\mathbb{W} \cong (\mathbb{W}^{+})_{+}$.*

We now equip DInFL-frames with additional conditions to dually represent DqRAs.

Definition 15. *A* DqRA-frame *is a tuple $\mathbb{W} = (W, I, \preceq, R, {}^{\sim}, {}^{-}, {}^{\neg})$ such that $(W, I, \preceq, R, {}^{\sim}, {}^{-})$ is a DInFL-frame and the following additional conditions hold for all $x, y, z \in W$:*

(7) $x^{\neg\neg} = x$
(8) $x \preceq y \implies y^{\neg} \preceq x^{\neg}$.
(9) $Rxyz^{-} \iff Ry^{\sim\neg}x^{\sim\neg}z^{\neg}$.

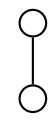
As in the case of qRAs, we can deduce that $x^{\sim\neg} = x^{\neg-}$ for all $x \in W$.

Lemma 16. *Let $\mathbb{W} = (W, I, \preceq, R, {}^{\sim}, {}^{-}, {}^{\neg})$ be a DqRA-frame. Then $x^{\sim\neg} = x^{\neg-}$ for all $x \in W$.*

Proof. We first show that $x^{\sim\neg} \preceq x^{\neg-}$. Since $x^{-\sim} \preceq x^{-\sim}$, there exists $i \in W$ such that $i \in I$ and $Rx^{-\sim}ix^{-\sim}$ by (2). Applying (5) to the second part gives $Rx^{-}x^{-\sim}i^{-}$, and so, by (9), $Rx^{-\sim\sim\neg}x^{-\sim\neg}i^{\neg}$. Hence, using Lemma 6, we obtain $Rx^{\sim\neg}x^{\neg}i^{\neg-\sim}$. Another application of (5) yields $Ri^{\neg-}x^{\sim\neg}x^{\neg-}$. Therefore, by (9), $Rx^{\sim\neg\sim\neg}i^{\neg-\sim\neg}x^{\neg\neg}$, and so, by (7) and Lemma 6, $Rx^{\sim\neg\sim\neg}ix$. Since $i \in I$, we can

1

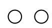
$\mathbb{A}_1$	1	$\sim$	$\neg$
1	1	1	1



2

$\mathbb{L}_3$	a	1	$\sim$	$\neg$
a		a	1	1
1	a	$\top$	a	a

$\mathbb{S}_3$	1	a	$\sim$	$\neg$
1	1	$\top$	a	a
a	$\top$	$\top$	1	1



1+1

$\mathbb{A}_2$	1	a	$\sim$	$\neg$
1	1	a	1	1
a	a	1	a	a

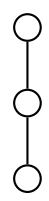
$\mathbb{A}_3$	1	a	$\sim$	$\neg$
1	1	a	1	1
a	a	$\top$	a	a

$\mathbb{A}_4$	1	a	$\sim$	$\neg$
1	1	a	a	a
a	a	1	1	1

$\mathbb{A}_5$	1	a	$\sim$	$\neg$
1	1	a	a	a
a	a		1	1

$\mathbb{A}_6$	e	f	$\sim$	$\neg$
e	e		e	e
f		f	f	f

$\mathbb{A}_7$	e	f	$\sim$	$\neg$
e	e		e	f
f		f	f	e



3

$\mathbb{L}_4$	b	a	1	$\sim$	$\neg$
b			b	1	1
a		b	ab	a	a
1	b	ab	$\top$	b	b

$\mathbb{A}_8$	b	a	1	$\sim$	$\neg$
b			b	1	1
a		ab	ab	a	a
1	b	ab	$\top$	b	b

$\mathbb{S}_4$	0	1	a	$\sim$	$\neg$
b	b	b	$\top$	a	a
1	b	$1b$	$\top$	1	1
a	$\top$	$\top$	$\top$	b	b

$\mathbb{A}_9$	b	1	a	$\sim$	$\neg$
b	1	b	$\top$	a	a
1	b	$1b$	$\top$	1	1
a	$\top$	$\top$	$\top$	b	b

Table 1. DqRA-frames for distributive quasi relation algebras of cardinality ≤ 4. A string of elements denotes the join of them and $\top$ is the join of all join-irreducibles.

apply (2) again to get $x^{\sim\neg\sim\neg} \preceq x$. Hence, by (7) and (8), $x^{\neg} \preceq x^{\sim\neg\sim\neg\neg} = x^{\sim\neg\sim}$. Using Proposition 7 and Lemma 6 we obtain $x^{\sim\neg} = x^{\sim\neg\sim-} \preceq x^{\neg-}$.

Now $x \preceq x^{\neg\neg}$ by (7), so there is some $i \in W$ such that $i \in I$ and $Rxix^{\neg\neg}$. Hence, by (7) and Lemma 6, we have $R^{\neg-\sim\neg}i^{\neg-\sim\neg}x^{\neg\neg}$. Applying (9) to this gives $Ri^{\neg-}x^{\neg-}x^{\neg-}$, and so $Rx^{\neg-}x^{\neg}i^{\neg-\sim}$ by (5). Thus, by (7) and Lemma 6, $Rx^{\neg-\neg-\sim\neg}x^{\neg\neg-\sim\neg}i^{\neg}$. Applying (9) again gives $Rx^{\neg\neg-}x^{\neg-\neg-}i^{-}$, and therefore, $Rx^{-}x^{\neg-\neg-}i^{-}$. By (5), $Rx^{\neg-\neg-}ix^{-\sim}$, which means $Rx^{\neg-\neg-}ix$. We thus get $x^{\neg-\neg-} \preceq x$. Hence, $x^{\sim} \preceq x^{\neg-\neg-\sim} = x^{\neg-\neg}$, and so $x^{\neg-} = x^{\neg-\neg\neg} \preceq x^{\sim\neg}$. $\square$

The following lemma shows that $x^{\sim\neg} = x^{\neg-}$ is equivalent to $x^{-\neg} = x^{\neg\sim}$, as expected. The proof is left for the reader.

Lemma 17. *Let $\mathbb{W} = (W, I, \preceq, R, ^{\sim}, ^{-}, ^{\neg})$ be a DqRA-frame. Then $x^{\sim\neg} = x^{\neg-}$ for all $x \in W$ iff $x^{-\neg} = x^{\neg\sim}$ for all $x \in W$.*

The above lemma allows us to prove the following result.

Proposition 18. *Let $\mathbb{W} = (W, I, \preceq, R, ^{\sim}, ^{-}, ^{\neg})$ be a DqRA-frame. Let $\mathsf{Up}(W, \preceq)$ is the set of all upsets of $(W, \preceq)$. For all $U, V \in \mathsf{Up}(W, \preceq)$, define $\circ$, $\sim$ and $-$ as in Proposition 9 and $\neg$ by $\neg U = \{x \in W \mid x^{\neg} \notin U\}$. Then the structure $\mathbb{W}^{+} = (\mathsf{Up}(W, \preceq), \cap, \cup, \circ, I, \sim, -, \neg)$ is a DqRA.*

Proof. Since $(\mathsf{Up}(W, \preceq), \cap, \cup, \circ, I, \sim, -)$ is a DInFL-algebra (Proposition 9), it will follow that $\mathbb{W}^{+}$ is a DqRA if we can show that $\neg U \in \mathsf{Up}(W, \preceq)$ for all $U \in \mathsf{Up}(W, \preceq)$, $\neg$ is involutive, and (Dm) and (Dp) hold. Using the fact that $\neg$ is order-reversing, we can show that if $U \in \mathsf{Up}(W, \preceq)$, then $\neg U \in \mathsf{Up}(W, \preceq)$.

Next we show that $\neg$ is involutive. Let $U \in \mathsf{Up}(W, \preceq)$. Then $x \in \neg\neg U$ iff $x^{\neg} \notin \neg U$ iff $x^{\neg\neg} \in U$ iff $x \in U$. Here the last equivalence follows from (7).

For (Dm), let $U, V \in \mathsf{Up}(W, \preceq)$. Then $x \in \neg(U \cap V)$ iff $x^{\neg} \notin U \cap V$ iff $x^{\neg} \notin U$ or $x^{\neg} \notin V$ iff $x \in \neg U$ or $x \in \neg V$ iff $x \in \neg U \cup \neg V$.

Finally, we show that $\mathbb{W}^{+}$ satisfies (Dp). Let $U, V \in \mathsf{Up}(W, \preceq)$. To see that $\neg(U \circ V) \subseteq {\sim}(-\neg V \circ -\neg U)$, suppose $x \notin {\sim}(-\neg V \circ -\neg U)$. Then we have $x^{-} \in -\neg V \circ -\neg U$. This means there exist $v \in -\neg V$ and $u \in -\neg U$ such that $Rvux^{-}$. Applying condition (9) to $Rvux^{-}$ gives $Ru^{\sim\neg}v^{\sim\neg}x^{\neg}$. From $v \in -\neg V$

and $u \in -\neg U$ we get $v^{\sim\neg} \in V$ and $u^{\sim\neg} \in U$. Hence, $x^{\neg} \in U \circ V$, and so $x \notin \neg(U \circ V)$.

For the other inclusion, suppose $x \notin \neg(U \circ V)$. Then $x^{\neg} \in U \circ V$. This means there exist $u \in U$ and $v \in V$ such that $Ruvx^{\neg}$. Now $u = u^{\sim-\neg\neg}$ by (7) and Lemma 6. By Lemma 17, $u^{\sim-\neg\neg} = u^{\sim\neg\sim\neg}$, and hence $u = u^{\sim\neg\sim\neg}$. Likewise, $v = v^{\sim-\neg\neg} = v^{\sim\neg\sim\neg}$. We thus have $u^{\sim\neg\sim\neg} \in U$, $v^{\sim\neg\sim\neg} \in V$ and $Ru^{\sim\neg\sim\neg}v^{\sim\neg\sim\neg}x^{\neg}$. Applying (9) to this gives $Rv^{\sim\neg}u^{\sim\neg}x^{-}$. From $u^{\sim\neg\sim\neg} \in U$ and $v^{\sim\neg\sim\neg} \in V$ we get $u^{\sim\neg} \in -\neg U$ and $v^{\sim\neg} \in -\neg V$. It thus follows that $x^{-} \in -\neg V \circ -\neg U$, and so $x \notin \sim(-\neg V \circ -\neg U)$. □

Definition 19. *For every completely join-irreducible a of a complete perfect DqRA $\mathbf{A}$, define $a^{\neg} = \neg\kappa(a)$.*

Lemma 20. *Let $\mathbf{A} = (A, \wedge, \vee, \cdot, 1, \sim, -, \neg)$ be a complete perfect DqRA. If a is a completely join-irreducible, then so is $a^{\neg}$.*

Proposition 21. *Let $\mathbf{A} = (A, \wedge, \vee, \cdot, 1, \sim, -, \neg)$ be a complete perfect DqRA. Let $J^{\infty}(\mathbf{A})$ be the set of completely join-irreducibles of $\mathbf{A}$. Define I_1, $\preceq$, $R.$, ${}^{\sim}$, ${}^{-}$ as in Proposition 12 and for all $a \in J^{\infty}(\mathbf{A})$, define $a^{\neg} = \neg\kappa(a)$. Then the structure $\mathbf{A}_{+} = (J^{\infty}(\mathbf{A}), I_1, \preceq, R., {}^{\sim}, {}^{-}, {}^{\neg})$ is a DqRA-frame.*

Proof. We only have to prove that $\mathbf{A}_{+}$ satisfies conditions (7) to (9) of Definition 15. We first show that $a^{\neg\neg} = a$. To prove that $a \preceq a^{\neg\neg}$, we have to show that $\neg\kappa(a^{\neg}) \leqslant a$, i.e., $\neg a \leqslant \kappa(a^{\neg})$. If we can show that $a^{\neg} \nleqslant \neg a$, we would be done, so suppose $a^{\neg} \leqslant \neg a$. Then $\neg\kappa(a) \leqslant \neg a$, and so $a \leqslant \kappa(a)$ Since a is completely join-prime, there is some $s \in A$ such that $a \leqslant s$ and $a \nleqslant s$, a contradiction.

To prove that $a^{\neg\neg} \preceq a$, we have to show that $a \leqslant \neg\kappa(a^{\neg})$, which is equivalent to showing that $\kappa(a^{\neg}) \leqslant \neg a$. Let b be an arbitrary element of A such that $a^{\neg} \nleqslant b$. Then $\neg\kappa(a) \nleqslant b$, and therefore $\neg b \nleqslant \kappa(a)$ This means $\neg b \nleqslant s$ for all $s \in A$ such that $a \nleqslant s$. It follows that it must be the case that $b \leqslant \neg a$; for otherwise, $a \nleqslant \neg b$, which means $\neg b \nleqslant \neg b$, a contradiction. Since b was an arbitrary element of A we have $b \leqslant \neg a$ for all $b \in A$ such that $a^{\neg} \nleqslant b$, and hence $\kappa(a^{\neg}) \leqslant \neg a$, as required.

To see that ${}^{\neg}$ is order-reversing, assume $a \preceq b$. Then we have $b \leqslant a$. We have to show that $a^{\neg} \leqslant b^{\neg}$; that is, we have to show that $\neg\kappa(a) \leqslant \neg\kappa(b)$. This is equivalent to showing that $\kappa(b) \leqslant \kappa(a)$. Let s be an arbitrary element of A such that $b \nleqslant s$. Then it must be the case that $a \nleqslant s$; for otherwise, $b \leqslant a \leqslant s$, a contradiction. It follows that $s \leqslant \kappa(a)$. Hence, since s was arbitrary, $s \leqslant \kappa(a)$ for all $s \in A$ such that $b \nleqslant s$. This proves that $\kappa(b) \leqslant \kappa(a)$, as required.

Finally, we show that $R.abc^{-}$ iff $R.b^{\sim\neg}a^{\sim\neg}c^{\neg}$ for all $a, b, c \in J^{\infty}(\mathbf{A})$. Let $a, b, c \in J^{\infty}(\mathbf{A})$. Then $R.b^{\sim\neg}a^{\sim\neg}c^{\neg}$ iff $c^{\neg} \leqslant b^{\sim\neg} \cdot a^{\sim\neg}$ iff $\neg\kappa(c) \leqslant b^{\sim\neg} \cdot a^{\sim\neg}$ iff $\neg(b^{\sim\neg} \cdot a^{\sim\neg}) \leqslant \kappa(c)$ iff $\neg(b^{\sim\neg}) + \neg(a^{\sim\neg}) \leqslant \kappa(c)$ iff $\neg\neg\kappa(b^{\sim}) + \neg\neg\kappa(a^{\sim}) \leqslant \kappa(c)$ iff $\kappa(b^{\sim}) + \kappa(a^{\sim}) \leqslant \kappa(c)$ iff $\sim(-\kappa(a^{\sim}) \cdot -\kappa(b^{\sim})) \leqslant \kappa(c)$ iff $-\kappa(c) \leqslant -\kappa(a^{\sim}) \cdot -\kappa(b^{\sim})$ iff $c^{-} \leqslant a^{\sim-} \cdot b^{\sim-}$ iff $R.abc^{-}$. The 4th equivalence follows from (Dp). □

Theorem 22. *If $\mathbf{A}$ is a complete perfect DqRA, then $\mathbf{A} \cong (\mathbf{A}_{+})^{+}$.*

Proof. All that is left here to do is to show that the map $\psi : A \to \mathsf{Up}(J^{\infty}(\mathbf{A}), \preceq)$ defined by $\psi(a) = \{j \in J^{\infty}(\mathbf{A}) \mid j \leqslant a\}$ preserves $\neg$, and the proof of this is analogous to the proof that $\psi(\sim a) = \sim\psi(a)$. □

Theorem 23. *If* $\mathbb{W} = (W, I, \preceq, R, ^{\sim}, ^{-}, ^{\neg})$ *is a DqRA-frame, then* $\mathbb{W} \cong (\mathbb{W}^{+})_{+}$.

4 Priestley-style representation for DInFL-algebras and DqRAs

Our goal in this section is to use the results of Section 3 to define Priestley spaces with additional structure that will be dual to DInFL-algebras and DqRAs. Since the signatures of both DInFL-algebras and DqRAs do not include lattice bounds, our dual spaces will in fact be *doubly-pointed* Priestley spaces. That is, the partially ordered set must have both a least and greatest element. Such spaces have also been called *bounded Priestley spaces* (cf. Section 1.2 and Theorem 4.3.2 of Clark and Davey [3]). More recent papers by Cabrer and Priestley (cf. [2]) refer to the spaces as doubly-pointed.

We recall that a partially ordered topological space $(X, \leqslant, \tau)$ is *totally order-disconnected* if whenever $x \nleqslant y$ there exists a clopen up-set U of X such that $x \in U$ and $y \notin U$. A *doubly-pointed Priestley space* is a compact totally order-disconnected space with bounds $0 \neq 1$. When recovering an unbounded distributive lattice from a doubly-pointed Priestley space, the proper, non-empty, clopen upsets form a lattice [3, Theorem 1.2.4].

We note two examples that show the importance of this version of Priestley duality for our setting. The integers (with $+$ as the monoid operation) are a commutative distributive residuated lattice and hence a DInFL-algebra. As the underlying lattice is unbounded, we cannot represent it as the clopen upsets of a Priestley space $(X, \leqslant, \tau)$ as this would introduce bounds $\varnothing$ and X. Secondly, consider the two-element Sugihara monoid $\mathbf{S}_2$ and the homomorphism $h(1) = 1$ and $h(0) = 0$ into the four-element Sugihara monoid $\mathbf{S}_4$ (whose elements are $a < 0 < 1 < b$). The usual dual of a homomorphism h would be h^{-1}, but this might not be defined if the dual spaces of the algebras consisted of the usual prime filters. In particular, ${\uparrow}b$ is a prime filter of $\mathbf{S}_4$, but $h^{-1}({\uparrow}b) = \varnothing$ is not a prime filter of $\mathbf{S}_2$.

Given a DInFL-algebra or DqRA $\mathbf{A}$, we will call $F \subseteq A$ a *generalised prime filter* if F is a prime filter, $F = A$ or $F = \varnothing$. This terminology follows, for instance, Fussner and Galatos [7], although we note that there they only allow $F = X$ as their algebras have a top element, but not a bottom element. (Their dual spaces are hence *pointed* Priestley spaces rather than doubly-pointed Priestley spaces.)

We first state some necessary results about generalised prime filters. The proof of the lemma below follows from a standard argument, using the involutiveness of $\mathbf{A}$ and considering the special cases of $F = A$ and $F = \varnothing$.

Lemma 24. *Let* $\mathbf{A} = (A, \wedge, \vee, \cdot, 1, \sim, -)$ *be a DInFL-algebra. If* F *is a generalised prime filter of the lattice reduct of* $\mathbf{A}$*, then so are* $F^{\sim} = \{{\sim}a \mid a \notin F\}$ *and* $F^{-} = \{-a \mid a \notin F\}$.

The following lemma is a restatement of [8, Lemma 3.5] but for generalised prime filters (see also [25, Lemma 2.2]). The proof follows [8, Lemma 3.5] with straightforward adaptations to account for the empty set and the whole algebra.

Lemma 25. *Let F, G, H be filters (possibly empty or total) of a distributive residuated lattice such that H is a generalised prime filter and $F \circ G \subseteq H$. Then there exist generalised prime filters F' and G' such that $F' \circ G \subseteq H$ and $F \circ G' \subseteq H$.*

A *doubly-pointed DInFL-frame* will be a DInFL-frame where the poset is bounded, and the set I is a proper, non-empty upset. We use Lemmas 24 and 25 to show that that the set of generalised prime filters can be equipped with the necessary structure to be a doubly-pointed DInFL-frame.

Proposition 26. *Let $\mathbf{A} = (A, \wedge, \vee, \cdot, 1, \sim, -)$ be a DInFL-algebra. Let $W_{\mathbf{A}}$ be the set of generalised prime filters of the lattice reduct of $\mathbf{A}$. For all F, G, H in $W_{\mathbf{A}}$, define $F \in \mathcal{I}$ iff $1 \in F$, $F \preceq G$ iff $F \subseteq G$, $R(F, G, H)$ iff for all $a \in F$ and all $b \in G$ we have $a \cdot b \in H$, $F^{\sim} = \{\sim a \mid a \notin F\}$ and $F^{-} = \{-a \mid a \notin F\}$. Then the structure $\mathfrak{F}(\mathbf{A}) = (W_{\mathbf{A}}, \mathcal{I}, \preceq, R, ^{\sim}, ^{-}, \varnothing, A)$ is a doubly-pointed DInFL-frame.*

Proof. The relation $\preceq$ is clearly a partial order. To see that $\mathcal{I}$ is a upset of $(W, \preceq)$, let $F \in \mathcal{I}$ and assume $F \preceq G$. Then $1 \in F \subseteq G$, so $G \in \mathcal{I}$. Since $\varnothing \notin \mathcal{I}$ and $A \in \mathcal{I}$, it is a proper, non-empty subset of $W_{\mathbf{A}}$.

Conditions (1) and (2) of Definition 5 can be proven using Lemma 25 and the filter $\uparrow 1$. Condition (3) follows easily from the fact that the order on $W_{\mathbf{A}}$ is set containment. For (4), see [8, Theorem 3.7(2)], but apply Lemma 25 above.

Next we show that (5) holds, i.e. that $R(F, G, H^{\sim})$ iff $R(H, F, G^{-})$. Assume $R(F, G, H^{\sim})$. If at least one of F, G or H is empty, then $R(H, F, G^{-})$ is trivially satisfied. Now assume they are all non-empty and let $a \in H$ and $b \in F$. We must show that $a \cdot b \in G^{-}$. Suppose $a \cdot b \notin G^{-}$. Then $\sim(a \cdot b) \in G$. Hence, since $b \in F$ and $R(F, G, H^{\sim})$, we get $b \cdot \sim(a \cdot b) \in H^{\sim}$. This means $-(b \cdot \sim(a \cdot b)) \notin H$. Now $a \cdot b \leqslant a \cdot b$, so we have $a \leqslant -(b \cdot \sim(a \cdot b))$. Thus, since $a \in H$ and H is upward closed, we get $-(b \cdot \sim(a \cdot b)) \in H$, a contradiction.

Conversely, assume $R(H, F, G^{-})$. Again, if any of F, G or H are empty, $R(F, G, H^{\sim})$ is trivially satisfied. Let $a \in F$ and $b \in G$. We have to show that $a \cdot b \in H^{\sim}$. Suppose $a \cdot b \notin H^{\sim}$. Then $-(a \cdot b) \in H$, and so, since $a \in F$ and $R(H, F, G^{-})$, we have $-(a \cdot b) \cdot a \in G^{-}$. This means $\sim(-(a \cdot b) \cdot a) \notin G$. Now $a \cdot b \leqslant a \cdot b$, so we have $b \leqslant \sim(-(a \cdot b) \cdot a)$, and therefore, since G is upward closed and $b \in G$, we have $\sim(-(a \cdot b) \cdot a) \in G$, a contradiction.

For (6), to see that $F^{\sim -} \subseteq F$, let $a \in F^{\sim -}$. Then $a = -b$ for some $b \notin F^{\sim}$. Hence, $\sim a = \sim -b = b$, and so $\sim a \notin F^{\sim}$. This means $-\sim a = a \in F$. A similar proof gives $F^{-\sim} \subseteq F$. The cases $F = \varnothing$ and $F = A$ are trivial. $\square$

Recall from Proposition 9 the definitions of $U \circ V$, $\sim U$ and $-U$. We remark that (3) below is equivalent to both maps $x \mapsto x^{\sim}$ and $x \mapsto x^{-}$ being continuous.

Definition 27. *A* DInFL-space $(W, I, \preceq, R, ^{\sim}, ^{-}, \tau)$ *is a doubly-pointed DInFL-frame with a compact totally order-disconnected topology τ satisfying:*

(1) *I is clopen.*
(2) *If U and V are clopen proper non-empty up-sets, then $U \circ V$ is clopen.*
(3) *If U is a clopen proper non-empty up-set, then $\sim U$ and $-U$ are clopen.*

For a DInFL-algebra $\mathbf{A}$, we consider the structure $\mathfrak{W}(\mathbf{A}) = (\mathfrak{F}(\mathbf{A}), \tau)$ where τ is the topology on the set of generalised prime filters with subbasic open sets of the form $X_a = \{F \in W_{\mathbf{A}} \mid a \in F\}$ and $X_a^c = \{F \in W_{\mathbf{A}} \mid a \notin F\}$. For a DInFL-frame $\mathbb{W}$, we denote by $K_{\mathbb{W}}$ the set of clopen proper non-empty upsets of $\mathbb{W}$ and define $\mathfrak{A}(\mathbb{W})$ to be the algebra $(K_{\mathbb{W}}, \cap, \cup, \circ, I, \sim, -)$.

Proposition 28. *If* $\mathbf{A}$ *is a DInFL-algebra, then* $\mathfrak{W}(\mathbf{A})$ *is a DInFL-space and if* $\mathbb{W}$ *is a DInFL-space then* $\mathfrak{A}(\mathbb{W})$ *is a DInFL-algebra.*

Proof. The fact that $\mathfrak{W}(\mathbf{A})$ has an underlying DInFL-frame structure is the result of Proposition 26 and the compact totally order-disconnectedness follows from the Priestley duality. By definition, $\mathcal{I} = X_1$ so it is clopen. For (2), the fact that $U \circ V$ is clopen follows from, for instance, [16, Theorem 6.3], noting that for any $a \in A$, we have $\varnothing \notin X_a$ and $A \in X_a$. Let U be a clopen proper non-empty upset. From the duality we have that $U = X_a = \{F \in W_{\mathbf{A}} \mid a \in F\}$ for some $a \in A$. Now $-U = \{F \mid F^{\sim} \notin U\} = \{F \mid a \notin F^{\sim}\} = \{F \mid {\sim}{-a} \notin F^{\sim}\}$. But ${\sim}{-a} \notin F^{\sim}$ iff $-a \in F$. Hence $-U = X_{-a}$, which is clopen. A similar proof shows that ${\sim}U$ is clopen.

For a DInFL-space $\mathbb{W}$, the lattice structure of $\mathfrak{A}(\mathbb{W})$ follows from Priestley duality. The algebra structure follows from the definition of a DInFL-space, Proposition 9 and the fact that the elements of $K_{\mathbb{W}}$ are special upsets. $\square$

Theorem 29. *Let* $\mathbf{A}$ *be a DInFL-algebra and* $\mathbb{W}$ *a DInFL-space. Then we have* $\mathbf{A} \cong \mathfrak{A}(\mathfrak{W}(\mathbf{A}))$ *and* $\mathbb{W} \cong \mathfrak{W}(\mathfrak{A}(\mathbb{W}))$.

Proof. The standard maps $a \mapsto X_a$ and $x \mapsto \{U \in K_{\mathbb{W}} \mid x \in U\}$ give us the required isomorphisms. $\square$

Here we will define the dual spaces of DqRAs. They will be topologised versions of the DqRA frames from Section 3.

Lemma 30. *Let* $\mathbf{A} = (A, \wedge, \vee, \cdot, 1, \sim, -, \neg)$ *be a DqRA. If* F *is a generalised prime filter of the lattice reduct of* $\mathbf{A}$*, then so is* $F^{\neg} = \{\neg a \mid a \notin F\}$.

As for DInFL-frames, we will consider doubly-pointed DqRA-frames, which have the additional constraint that I must be proper and non-empty.

Proposition 31. *Let* $\mathbf{A} = (A, \wedge, \vee, \cdot, 1, \sim, -, \neg)$ *be a DqRA and let* $W_{\mathbf{A}}$ *be the set of all generalised prime filters of the lattice reduct of* $\mathbf{A}$. *Define* $\mathcal{I}$, $\preceq$, R, ${}^{-}$ *and* ${}^{\sim}$ *as in Proposition 26 and for all* $F \in W_{\mathbf{A}}$*, define* $F^{\neg} = \{\neg a \mid a \notin F\}$. *Then the structure* $(W_{\mathbf{A}}, \mathcal{I}, \preceq, R, {}^{\sim}, {}^{-}, {}^{\neg}, \varnothing, A)$ *is a doubly-pointed DqRA-frame.*

Proof. We must show that (7), (8) and (9) from Definition 15 hold. For (7), since $\neg\neg a = a$, we have $a \in F$ iff $\neg a \notin F^{\neg}$ iff $\neg\neg a \in F^{\neg\neg}$ iff $a \in F^{\neg\neg}$.

To see that $F \preceq G$ implies $G^{\neg} \preceq F^{\neg}$, assume $F \subseteq G$ and let $a \in G^{\neg}$. The latter implies that $a = \neg b$ for some $b \notin G$. Hence, $\neg a = \neg\neg b = b$, and therefore $\neg a \notin G$. Since $F \subseteq G$, we have $\neg a \notin F$, and so $\neg\neg a \in F^{\neg}$, which means $a \in F^{\neg}$.

Finally, we show (9), i.e. that $R\,(F, G, H^{-})$ iff $R\,(G^{\sim\neg}, F^{\sim\neg}, H^{\neg})$. Assume that $R\,(F, G, H^{-})$. Notice that if $G^{\sim\neg} = \varnothing$ or $F^{\sim\neg} = \varnothing$, or $H^{\neg} = A$, then

$R(G^{\sim\neg}, F^{\sim\neg}, H^{\neg})$ is trivially satisfied. Hence let $a \in G^{\sim\neg}$ and $b \in F^{\sim\neg}$. We have to show that $a \cdot b \in H^{\neg}$. Since $a \in G^{\sim\neg}$, there is some $c \in A$ such that $c \notin G^{\sim}$ and $a = \neg c$. Hence, $\neg a = \neg\neg c = c$, and so $\neg a \notin G^{\sim}$. This implies that $-\neg a \in G$. Likewise, since $b \in F^{\sim\neg}$, we have $-\neg b \in F$. Therefore, by our assumption that $R(F, G, H^{-})$, we get $-\neg b \cdot -\neg a \in H^{-}$. It must therefore be the case that $\sim(-\neg b \cdot -\neg a) \notin H$. Applying (Dp), we get $\neg(a \cdot b) \notin H$, which means $a \cdot b \in H^{\neg}$, as required.

Conversely, assume $R(G^{\sim\neg}, F^{\sim\neg}, H^{\neg})$. If $F = \varnothing$, $G = \varnothing$ or $H = \varnothing$ then $R(F, G, H^{-})$ is trivially true. So, let $a \in F$ and $b \in G$. We must show that $a \cdot b \in H^{-}$. Since $a \in F$, we have $\sim a \notin F^{\sim}$, and so $\neg{\sim}a \in F^{\sim\neg}$. Hence, by (Di), $-\neg a \in F^{\sim\neg}$. Likewise, since $b \in G$, we can show that $-\neg b \in G^{\sim\neg}$. It thus follows from our assumption that $-\neg b \cdot -\neg a \in H^{\neg}$, which means $\neg(-\neg b \cdot -\neg a) \notin H$. This gives $-\neg(-\neg b \cdot -\neg a) \in H^{-}$, and therefore, by (Di), $\neg{\sim}(-\neg b \cdot -\neg a) \in H^{-}$. Applying (Dp) to this, we get $\neg\neg(a \cdot b)$, i.e., $a \cdot b \in H^{-}$. □

We now define the Priestley-style dual objects of DqRAs. As before, for U an upset of a DqRA frame, we have $\neg U = \{x \in W \mid x^{\neg} \notin W\}$.

Definition 32. *A* DqRA-space $(W, I, \preceq, R, ^{\sim}, ^{-}, ^{\neg}, \tau)$ *is a doubly-pointed DqRA-frame with a compact totally order-disconnected topology* τ *which satisfies:*

(1) I *is clopen.*
(2) *If* U *and* V *are clopen proper non-empty up-sets, then* $U \circ V$ *is clopen.*
(3) *If* U *is a clopen proper non-empty upset, then* ${\sim}U$, $-U$ *and* $\neg U$ *are clopen.*

We extend the maps $\mathfrak{A}$ and $\mathfrak{W}$ to DqRAs and DqRA-spaces, and denote these extensions by $\mathfrak{A}_q$ and $\mathfrak{W}_q$.

Proposition 33. *For a DqRA* $\mathbf{A}$*, let* $\mathfrak{W}_q(\mathbf{A}) = (W_{\mathbf{A}}, \mathcal{I}, R, ^{\sim}, ^{-}, ^{\neg}, \varnothing, A)$ *and for a DqRA-space* $\mathbb{W}$*, let* $\mathfrak{A}_q(\mathbb{W}) = (K_{\mathbb{W}}, \cap, \cup, \circ, I, \sim, -, \neg)$*. Then* $\mathfrak{W}_q(\mathbf{A})$ *is a DqRA-space and* $\mathfrak{A}_q(\mathbb{W})$ *is a DqRA.*

Proof. Most of the work has been done by Proposition 28. If U is a clopen proper non-empty upset, then $U = X_a$ for some $a \in A$. Then $\neg U = \{F \mid F^{\neg} \notin U\} = \{F \mid a \notin F\} = \{F \mid \neg\neg a \notin F^{\neg}\} = \{F \mid \neg a \in F\} = X_{\neg a}$. To show $\mathfrak{A}_q(\mathbb{W})$ is a DqRA, we use Proposition 18 to show that $\neg U$ has the required properties. □

The same maps from Theorem 29 are used to prove the theorem below.

Theorem 34. *For any DqRA* $\mathbf{A}$ *and DqRA-space* $\mathbb{W}$*, we have* $\mathbf{A} \cong \mathfrak{A}_q(\mathfrak{W}_q(\mathbf{A}))$ *and* $\mathbb{W} \cong \mathfrak{W}_q(\mathfrak{A}_q(\mathbb{W}))$.

Based on the results of Urquhart [25] and Jipsen and Litak [16], the morphisms for DqRA-spaces will need to be continuous, order-preserving, bound-preserving maps that satisfy properties 6,7 and 10 from [16, p.17]. Additionally, such a morphism should preserve $^{\sim}$, $^{-}$ and $^{\neg}$ (hence they would not need to satisfy 8, 9 from [16]). We delay a detailed study of morphisms for future work.

5 Applications to finite models

5.1 Counting DInFL-frames and DqRA-frames

Using our descriptions of DInFL-frames and DqRA-frames from Section 3 we are able to calculate the number of algebras with eight elements or less for both DInFL-algebras and DqRAs. Further, we are able to classify them in terms of their lattice structure.

We note that amongst the DInFL-algebras, only one seven-element algebra is non-cyclic, and only one eight-element algebra is non-cyclic. Their dual frames are $\bowtie$ and $\mathbf{X}$, respectively. For DqRAs, there are two seven-element non-cyclic algebras (both have $\bowtie$ as their dual frame), and two eight-element non-cyclic algebras (both with $\mathbf{X}$ as their dual frame).

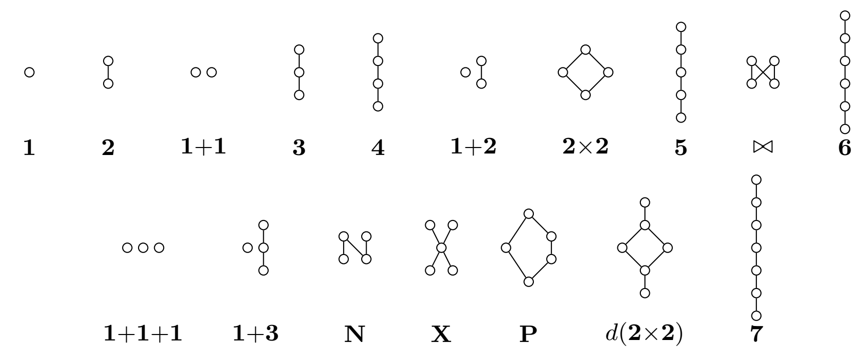


Fig. 1. Posets of join-irreducibles for self-dual distributive lattices of cardinality ≤ 8

Poset	**1**	**2**	**1+1**	**3**	**4**	**1+2**	**2×2**	**5**	$\bowtie$	**6**	**1+1+1**	**1+3**	**N**	**X**	**P**	$d(\mathbf{2}\times\mathbf{2})$	**7**
DInFL-frames	1	2	5	4	8	10	16	17	11	38	25	25	22	21	28	70	91
DqRA-frames	1	2	6	4	8	10	23	17	12	36	31	25	22	23	26	106	81

Size of algebra	1	2	3	4	5	6	7	8
Number of DInFL-algebras	1	1	2	9	8	43	49	282
Number of DqRAs	1	1	2	10	8	50	48	314

Table 2. Number of nonisomorphic quasi relation algebras of cardinality up to 8

5.2 Representions for some small quasi relation algebras

In this subsection we use DqRA-frames to describe the DqRAs that are $\{\vee, \cdot, 1, {\sim}\}$-subreducts of small relation algebras. We refer to these algebras as DqRA-subreducts. Moreover, we only present information on DqRAs that are not relation algebras and, for now, we do not consider the more general concept of representation that is defined in [4].

$\cdot_1$	a	r	s
a	1	r	s
r	r	r	$\top$
s	s	$\top$	s

$\cdot_2$	a	r	s
a	$1a$	r	s
r	r	r	$\top$
s	s	$\top$	s

$\cdot_3$	a	r	s
a	1	r	s
r	r	s	$1a$
s	s	$1a$	r

$\cdot_4$	a	r	s
a	$1a$	r	s
r	r	s	$1a$
s	s	$1a$	r

$\cdot_5$	a	r	s
a	1	r	s
r	r	rs	$\top$
s	s	$\top$	rs

$\cdot_6$	a	r	s
a	$1a$	r	s
r	r	rs	$\top$
s	s	$\top$	rs

$\cdot_7$	a	r	s
a	$1rs$	a	a
r	a	r	$1rs$
s	a	$1rs$	s

$\cdot_8$	a	r	s
a	$\top$	a	a
r	a	r	$1rs$
s	a	$1rs$	s

$\cdot_9$	a	r	s
a	$1rs$	a	a
r	a	s	1
s	a	1	r

$\cdot_{10}$	a	r	s
a	$\top$	a	a
r	a	s	1
s	a	1	r

$\cdot_{11}$	a	r	s
a	$1rs$	a	a
r	a	rs	$1rs$
s	a	$1rs$	rs

$\cdot_{12}$	a	r	s
a	$\top$	a	a
r	a	rs	$1rs$
s	a	$1rs$	rs

$\cdot_{13}$	a	r	s
a	$\top$	ar	a
r	a	r	$\top$
s	as	$1rs$	s

$\cdot_{14}$	a	r	s
a	$1rs$	ar	as
r	ar	r	$\top$
s	as	$\top$	s

$\cdot_{15}$	a	r	s
a	$\top$	ar	as
r	ar	r	$\top$
s	as	$\top$	s

$\cdot_{16}$	a	r	s
a	$1rs$	ar	as
r	ar	rs	$\top$
s	as	$\top$	rs

$\cdot_{17}$	a	r	s
a	$\top$	ar	as
r	ar	rs	$\top$
s	as	$\top$	rs

$\cdot_{18}$	a	r	s
a	1	s	r
r	s	a	1
s	r	1	a

$\cdot_{19}$	a	r	s
a	1	s	r
r	s	ars	$1rs$
s	r	$1rs$	ars

$\cdot_{20}$	a	r	s
a	$1a$	rs	rs
r	rs	a	$1a$
s	rs	$1a$	a

$\cdot_{21}$	a	r	s
a	$1a$	rs	rs
r	rs	ar	$\top$
s	rs	$\top$	as

$\cdot_{22}$	a	r	s
a	$1a$	rs	rs
r	rs	ars	$\top$
s	rs	$\top$	ars

$\cdot_{23}$	a	r	s
a	$1rs$	as	ar
r	as	ar	$1rs$
s	ar	$1rs$	as

$\cdot_{24}$	a	r	s
a	$\top$	as	ar
r	as	ar	$1rs$
s	ar	$1rs$	as

$\cdot_{25}$	a	r	s
a	$1rs$	as	ar
r	as	ars	$1rs$
s	ar	$1rs$	ars

$\cdot_{26}$	a	r	s
a	$\top$	as	ar
r	as	ars	$1rs$
s	ar	$1rs$	ars

$\cdot_{27}$	a	r	s
a	$1rs$	ars	ar
r	as	ar	$\top$
s	ars	$1rs$	as

$\cdot_{28}$	a	r	s
a	$\top$	ars	ar
r	as	ar	$\top$
s	ars	$1rs$	as

$\cdot_{29}$	a	r	s
a	$1rs$	ars	ar
r	as	ars	$\top$
s	ars	$1rs$	ars

$\cdot_{30}$	a	r	s
a	$\top$	ars	ar
r	as	ars	$\top$
s	ars	$1rs$	ars

$\cdot_{31}$	a	r	s
a	$\top$	ars	ars
r	ars	a	$1a$
s	ars	$1a$	a

$\cdot_{32}$	a	r	s
a	$1rs$	ars	ars
r	ars	ar	$\top$
s	ars	$\top$	as

$\cdot_{33}$	a	r	s
a	$\top$	ars	ars
r	ars	ar	$\top$
s	ars	$\top$	as

$\cdot_{34}$	a	r	s
a	$1rs$	ars	ars
r	ars	as	$1a$
s	ars	$1a$	ar

$\cdot_{35}$	a	r	s
a	$\top$	ars	ars
r	ars	as	$1a$
s	ars	$1a$	ar

$\cdot_{36}$	a	r	s
a	$1rs$	ars	ars
r	ars	ars	$\top$
s	ars	$\top$	ars

$\cdot_{37}$	a	r	s
a	$\top$	ars	ars
r	ars	ars	$\top$
s	ars	$\top$	ars

Table 3. Atom structures (= frames) for the 37 nonsymmetric RAs of cardinality 16. The identity atom 1 is not shown, a string of elements denotes the join of them, and ${\sim}a = 1rs$, ${\sim}r = 1ar$, ${\sim}s = 1as$.

We consider small relation algebras with up to 16 elements, with the aim to compute all $\{\vee, \cdot, {\sim}\}$-subreducts of these algebras that are proper quasi relation algebras. Since symmetric relation algebras satisfy $x = x^{\smile}$, it follows that ${\sim}x = \neg x$, hence the subreducts of such algebras only produce relation algebras. This means we only need to consider nonsymmetric relation algebras. In Roger Maddux's book [22] there are lists of all finite integral relation algebras with up to 5 atoms. Recall that a relation algebra is *integral* if the identity element is an atom. For nonsymmetric ones with up to 4 atoms, these relation algebras are denoted $1_3, 2_3, 3_3, 1_{37} - 37_{37}$.

We begin by recalling some information on the smallest nonrepresentable relation algebras. Lyndon [19] showed that all relation algebras with 8 elements or less are representable, and McKenzie [23] found a 16-element relation algebra (now referred to as 14_{37} in Roger Maddux's list [22]) that is nonrepresentable. There are 10 further algebras in the list of 37 that are nonrepresentable: $16_{37}, 21_{37}, 24_{37} - 29_{37}, 32_{37}, 34_{37}$. The representations of the remaining 26 relation algebras were found by Steven Comer and Roger Maddux (see [22]).

We now describe the maximal subreducts of these 37 relation algebras that are proper qRAs When they occur as a subreduct of a representable relation algebra, they are themselves representable (indicated by bold names below).

The frames for the first type of subreducts are based on the poset 1+1+2, and there are 20 (nonisomorphic) frames of this kind. The corresponding DqRAs have 12 elements forming the lattice $\mathbf{2}\times\mathbf{2}\times\mathbf{3}$ and occur as subreducts of the relation algebras $\mathbf{1}_{37}$, $\mathbf{2}_{37}$, $\mathbf{5}_{37}$, $\mathbf{6}_{37}$, $\mathbf{7}_{37}$, $\mathbf{8}_{37}$, $\mathbf{11}_{37}$, $\mathbf{12}_{37}$, 14_{37}, $\mathbf{15}_{37}$, 16_{37}, $\mathbf{17}_{37}$, $\mathbf{20}_{37}$, 21_{37}, $\mathbf{22}_{37}$, $\mathbf{31}_{37}$, 32_{37}, $\mathbf{33}_{37}$, $\mathbf{36}_{36}$, $\mathbf{37}_{37}$.

In each case the 2-element chain in the frame is given by $s \prec r$ (or isomorphically by $r \prec s$). To see that this frame corresponds to a subreduct of the listed relation algebras, it suffices to check that $s \le x \cdot y \implies r \le x \cdot y$ for all $x, y \in \{a, r, r \vee s\}$, while this formula fails for the other 17 nonsymmetric integral relation algebras with 16 elements (see Table 3). Hence there are at least 16 representable DqRA with poset 1+1+2 as their frame. Using the representation game in [17] it has been checked that the DqRA-subreduct of McKenzie's algebra 14_{37} is not representable, but for $16_{37}, 21_{37}, 32_{37}$ it has not (yet) been determined if the DqRA-subreduct is representable.

Ten of the remaining 17 relation algebras in the list have a maximal DqRA-subreduct with 1+3 as poset: $\mathbf{13}_{37}$, $\mathbf{19}_{37}$, $\mathbf{23}_{37}$, 24_{37}, 25_{37}, 26_{37}, 27_{37}, 28_{37}, 29_{37}, $\mathbf{30}_{37}$.

In this case, the poset of the frame satisfies $s \prec a \prec r$ (or isomorphically $r \prec a \prec s$), and such a frame corresponds to an 8-element subreduct of a relation algebras if it satisfies $s \le x \cdot y \implies a \vee r \le x \cdot y$ and $a \le x \cdot y \implies r \le x \cdot y$ for all $x, y \in \{r, a \vee r, a \vee r \vee s\}$.

Note that the algebras $\mathbf{13}_{37}, 27_{37} - \mathbf{30}_{37}$ are noncommutative, but the DqRA-subreducts are commutative, hence they can be expanded to DqRAs. Four of the relation algebras in this list are representable, but the DqRA-subreducts of $\mathbf{19}_{37}$ and $\mathbf{30}_{37}$ are isomorphic, so this gives representations for three 8-element DqRAs. Other representable 8-element DqRAs can be found as subalgebras of the sixteen 12-element DqRAs described above.

Finally, 7 algebras do not have subreducts that produce proper quasi relation algebras: $\mathbf{3}_{37}, \mathbf{4}_{37}, \mathbf{9}_{37}, \mathbf{10}_{37}, \mathbf{18}_{37}, 34_{37}, \mathbf{35}_{37}$.

6 Conclusion

We have found first-order axiomatizations for DInFL-frames and DqRA-frames that are dual to complete perfect distributive involutive FL-algebras and distributive quasi relation algebras respectively. By adding Priestley-space topologies to these frames we obtain dual spaces for these algebras without requiring them to be complete and perfect. For small nonsymmetric relation algebras, DqRA-frames have been used to provide representations for 16 DqRAs with 12 elements and for 3 DqRAs with 8 elements.

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