Relation algebras as expanded FL-algebras

Nikolaos Galatos and Peter Jipsen

Abstract. This paper studies generalizations of relation algebras to residuated lattices with a unary De Morgan operation. Several new examples of such algebras are presented, and it is shown that many basic results on relation algebras hold in this wider setting. The variety qRA of quasi relation algebras is defined, and is shown to be a conservative expansion of involutive FL-algebras. Our main result is that equations in qRA and several of its subvarieties can be decided by a Gentzen system, and that these varieties are generated by their finite members.

1. Introduction

Relation algebras and residuated Boolean monoids are part of classical algebraic logic, and they have found applications within computer science as algebraic semantics for programs and state-based systems. However both these classes of algebras have undecidable equational theories ([17][1] p.268 and [13] respectively), so we would like to identify a natural larger variety "close to" relation algebras that has a decidable equational theory. Previous generalizations to decidable varieties, such as [15] have weakened the associativity of multiplication (composition) to obtain nonassociative or weakly associative relation algebras. But for applications in computer science, the multiplication operation usually denotes sequential composition of programs, and associativity is an essential aspect of this operation that should be preserved in abstract models. Unfortunately equational undecidability already holds for the variety of all Boolean algebras with an associative operator, as well as for any subvarieties of an expansion that contains the full relation algebra on an infinite set ([13]). As we would like to keep associativity of multiplication, it is necessary to weaken the Boolean lattice structure.

Since the multiplication operation in relation algebras is residuated, it is natural to study relation algebras in the context of residuated lattices and Full Lambek (FL)-algebras (i.e., residuated lattices with a constant 0). These algebras originated in the 1930s from the study of ideal lattices in ring theory, they include diverse examples such as lattice-ordered groups and Boolean algebras, and they serve as algebraic models of substructural logics (see [6] for further details). This makes it possible to use methods and results from substructural logics (e.g., the decidability of involutive FL-algebras [5]) in this otherwise classical area of algebraic logic.

2010 *Mathematics Subject Classification*: Primary: 06F05, Secondary: 08B15, 03B47, 03G10,

[1] As mentioned in this reference, Tarski originally proved this result in the 1940s.

We define FL$'$-algebras as expansions of FL-algebras with a unary operation $'$ that is self-involutive, i.e., satisfies the identity $x'' = x$. This class of algebras includes relation algebras as a subvariety, as well as several natural generalizations of relation algebras. In particular, we define the (non-Boolean) variety of quasi relation algebras and the (Boolean) variety of skew relation algebras. Placing relation algebras within the uniform context of residuated lattices clarifies the connections between converse, involution and conjugation that was previously only studied in the context of Boolean algebras with operators. E.g., a result of Jónsson and Tsinakis [11] which shows that relation algebras are term-equivalent to a subvariety of residuated Boolean monoids (see Theorem 1 below) is generalized to the non-Boolean setting (see Theorem 17 below).

We prove the decidability of the equational theory of quasi relation algebras (qRAs) and define a functor from the category of involutive FL-algebras to the category of qRAs, with the property that the image of the finite involutive FL-algebras generates the variety qRA. We also give some natural examples of FL$'$-algebras "close to" representable relation algebras and lattice-ordered (pre)groups.

In Section 2 we give a brief review of the relevant definitions for this paper. The next section contains basic results about the arithmetic of FL$'$-algebras. Readers familiar with relation algebras and residuated Boolean monoids will recognize many of these properties, but here they hold in a much more general setting. Section 4 narrows the focus down to quasi relation algebras, defined as FL$'$-algebras in which the $'$-operation "dually commutes" with all the other fundamental operations, i.e., the equations $(x \wedge y)' = x' \vee y'$ (Dm), $({\sim}x)' = -(x')$ (Di), and $(x \cdot y)' = x' + y'$ (Dp) hold. Relation algebras are obtained if the lattice reducts are assumed to be Boolean algebras with $'$ as complementation, but other examples are given by expansions of lattice-ordered groups as well as a functorial way of mapping any involutive FL-algebra to a qRAs. We also prove in Theorem 17 the generalization of the Jónsson-Tsinakis result mentioned above and in Theorem 1, which gives an equational basis for qRA that is very similar to Tarski's equational basis for RA. In Section 5 we prove the decidability of the equational theory of quasi relation algebras, by reducing it to that of involutive FL-algebras. The same result holds for any self-dual subvariety of quasi relation algebras that is given by equations without $'$ that define a decidable subvariety of involutive FL-algebras. Finally Section 6 introduces skew relation algebras, defined as Boolean involutive FL$'$-algebras. They also have an equational basis that is close to the one for RAs (Corollary 28), and examples of skew RAs can be constructed in a natural way from algebras of binary relations. However the equational theory of skew RAs is undecidable.

2. Preliminaries

Chin and Tarski [4] defined *relation algebras* as algebras $\mathbf{A} = (A, \wedge, \vee, ', \bot, \top, \cdot, 1, ^\smile)$, such that $(A, \wedge, \vee, ', \bot, \top)$ is a Boolean algebra, $(A, \cdot, 1)$ is a monoid and for all $x, y, z \in A$

$$\text{(i) } x^{\smile\smile} = x \qquad \text{(ii) } (xy)^\smile = y^\smile x^\smile \qquad \text{(iii) } x(y \vee z) = xy \vee xz$$
$$\text{(iv) } (x \vee y)^\smile = x^\smile \vee y^\smile \qquad \text{(v) } x^\smile(xy)' \leq y'.$$

These five identities are equivalent to

$$xy \leq z' \iff x^\smile z \leq y' \iff zy^\smile \leq x'$$

so defining *conjugates* $x \rhd z = x^\smile z$ and $z \lhd y = zy^\smile$ we have

$$xy \leq z' \iff x \rhd z \leq y' \iff z \lhd y \leq x'.$$

Birkhoff [1] (cf. also Jónsson [9]) defined *residuated Boolean monoids* as algebras $(A, \wedge, \vee,', \bot, \top, \cdot, 1, \rhd, \lhd)$ such that $(A, \wedge, \vee,', \bot, \top)$ is a Boolean algebra, $(A, \cdot, 1)$ is a monoid and the *conjugation property* holds: for all $x, y, z \in A$,

$$xy \leq z' \iff x \rhd z \leq y' \iff z \lhd y \leq x'.$$

For example, given a monoid $\mathbf{M} = (M, *, e)$, the powerset monoid $\mathcal{P}(\mathbf{M}) = (\mathcal{P}(M), \cap, \cup,', \emptyset, M, \cdot, \{e\}, \rhd, \lhd)$ is a residuated Boolean monoid, where $XY = \{x * y : x \in X, y \in Y\}$, $X \rhd Y = \{z : x * z = y \text{ for some } x \in X, y \in Y\}$ and $X \lhd Y = \{z : z * y = x \text{ for some } x \in X, y \in Y\}$. If $\mathbf{G} = (G, *,^{-1})$ is a group, $\mathcal{P}(\mathbf{G})$ is a relation algebra with $X^\smile = \{x^{-1} : x {\in} X\}$.

Let RM denote the variety of residuated Boolean monoids and RA the variety of relation algebras.

Theorem 1. ([11] Thm 5.2) RA *is term-equivalent to the subvariety of* RM *defined by* $(x{\rhd}1)y = x{\rhd}y$. *The term-equivalence is given by* $x \rhd y = x^\smile y$, $x \lhd y = xy^\smile$ *and* $x^\smile = x \rhd 1$.

One of the aims of this paper is to lift this result to residuated lattices and FL-algebras (see Theorem 17). The conjugation condition

$$xy \leq z' \iff x \rhd z \leq y' \iff z \lhd y \leq x'$$

can be rewritten (replacing z by z') as

$$xy \leq z \iff y \leq (x \rhd z')' \iff x \leq (z' \lhd y)'$$

so by defining *residuals* $x\backslash z = (x{\rhd}z')'$ and $z/y = (z'{\lhd}y)'$ we get the equivalent *residuation property*

$$xy \leq z \iff y \leq x\backslash z \iff x \leq z/y$$

(hence the name *residuated* Boolean monoids).

Ward and Dilworth [19] defined *residuated lattices*[2] as algebras of the form $(A, \wedge, \vee, \cdot, 1, \backslash, /)$ where $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a monoid, and the *residuation property* holds, i. e., for all $x, y, z \in A$

$$x \cdot y \leq z \quad \iff \quad x \leq z/y \quad \iff \quad y \leq x\backslash z.$$

A *Full Lambek* (or *FL-*)*algebra* $(A, \wedge, \vee, \cdot, 1, \backslash, /, 0)$ (cf. [16]) is a residuated lattice expanded with a constant 0 (no properties are assumed about this constant). The two unary term operations ${\sim}x = x\backslash 0$ and $-x = 0/x$ are called *linear negations*, and it follows from the residuation property that ${\sim}(x \vee y) = {\sim}x \wedge {\sim}y$ and $-(x \vee y) = -x \wedge -y$. An *involutive FL-algebra* (or InFL-algebra for short) is an FL-algebra in which ${\sim}, -$ satisfy the identities

(In) ${\sim}{-}x = x = -{\sim}x$.

Since ${\sim}, -$ are always order-reversing, they are both dual lattice isomorphisms, hence ${\sim}(x \wedge y) = {\sim}x \vee {\sim}y$ and $-(x \wedge y) = -x \vee -y$. A pair of operations $({\sim}, -)$ that satisfies (In) and these two De Morgan laws is said to form a *De Morgan involutive pair*. Examples of involutive FL-algebras include lattice ordered groups and a subvariety of relation algebras, namely *symmetric relation algebras*, defined by $x^{\smile} = x$ relative to RA. In the latter case $0 = 1'$, $x\backslash y = (xy')'$, $x/y = (x'y)'$, and complementation is defined by the term $x' = x\backslash 0 = 0/x$. However for (nonsymmetric) relation algebras $x\backslash 0 = (x^{\smile}1'')' = x^{\smile\prime}$ so in general complementation cannot be interpreted by one of the linear negations. Before we expand FL-algebras to remedy this issue, we recall a well-known alternative presentation of InFL-algebras that uses the linear negations and $\cdot$ to express $\backslash, /$, and gives a succinct equational basis for the variety InFL of all InFL-algebras.

Lemma 2. *InFL-algebras are term-equivalent to algebras $(A, \wedge, \vee, \cdot, 1, {\sim}, -)$ such that $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a monoid, and for all $x, y, z \in A$,*

$$xy \leq z \quad \Leftrightarrow \quad y \leq {\sim}(-z \cdot x) \quad \Leftrightarrow \quad x \leq -(y \cdot {\sim}z). \qquad (*)$$

Also, $(*)$ is equivalent to the following identities: $({\sim}, -)$ is a De Morgan involutive pair, multiplication distributes over joins and $-(xy) \cdot x \leq -y$, $y \cdot {\sim}(xy) \leq {\sim}x$.

Proof. In an InFL-algebra $y \leq {\sim}(-z \cdot x)$ iff $-z \cdot x \cdot y \leq 0$ iff $xy \leq z$ iff $y \leq x\backslash z$, hence ${\sim}(-z \cdot x) = x\backslash z$, and similarly $-(y \cdot {\sim}z) = z/y$, so $(*)$ holds. Conversely, if $(A, \wedge, \vee, \cdot, 1, {\sim}, -)$ satisfies the given conditions and one defines $x\backslash y = {\sim}(-y \cdot x)$, $x/y = -(y \cdot {\sim}x)$ and $0 = {\sim}1$ then (In) follows from $(*)$ with $x = 1$ (resp. $y = 1$). Similarly $(*)$ implies $(u \vee v)y = uy \vee vy$ (use $x = u \vee v$), ${\sim}(u \vee v) = {\sim}u \wedge {\sim}v$ (use $x = u \vee v$ and $z = {\sim}1$), ${\sim}1 = -1$ (use $x = 1, z = {\sim}1$), $x\backslash 0 = {\sim}x$ and $0/x = -x$. Hence $(A, \wedge, \vee, \cdot, 1, \backslash, /, 0)$ is an InFL-algebra.

[2] To be precise Ward and Dilworth's defintion assumed commutativity of multiplication and that 1 is the top element of the lattice. The more general definition given here is due to Blount and Tsinakis [2].

The identity $-(xy)\cdot x \leq -y$ follows from $(*)$, (In) with $z = xy$. On the other hand if the given identities hold, then $xy \leq z$ implies $-z\cdot x \leq -(xy)\cdot x \leq -y$, so $y \leq {\sim}(-z\cdot x)$. This in turn implies $xy \leq x\cdot{\sim}(-z\cdot x) \leq {\sim}{-z} = z$, and proving the second equivalence of $(*)$ is similar. $\square$

The binary operation $x + y$, called the dual of $\cdot$, is defined by $x + y = {\sim}(-y \cdot -x)$. We note that in any InFL-algebra $x + y = -({\sim}y \cdot {\sim}x)$ holds. Furthermore $+$ is associative, has 0 as unit, and is dually residuated (for detailed proofs see for example [6]).

3. FL′-algebras and RL′-algebras

As mentioned in the introduction, an *FL′-algebra* is defined as an expansion of an FL-algebra with a unary operation $'$ (called a self-involution) that satisfies the identity $x'' = x$. The operations ${\sim}x$, $-x$, $x + y$ are defined in the same way as above, and the following operations use $'$ in their definition:

- *converses* $x^{\cup} = ({\sim}x)' = (x\backslash 0)'$ and $x^{\sqcup} = (-x)' = (0/x)'$,
- *conjugates* $x \rhd y = (x\backslash y')'$ and $y \lhd x = (y'/x)'$, and
- *potential bounds* $\bot = 1 \wedge 1'$ and $\top = 1 \vee 1'$.

An *RL′-algebra* is defined as an FL′-algebra that satisfies $1' = 0$. Note that FL-algebras and residuated lattices are obtained from FL′-algebras and RL′-algebras by adding the identity $x' = x$. The corresponding varieties of algebras are denoted by FL', RL', FL and RL. We also use the convention that $'$,$^{\cup}$,$^{\sqcup}$ have higher priority than the other operations.

We now list some identities that are of primary interest in this paper, preceded by the abbreviation that is used to refer to each identity.

(Dm)	$(x \wedge y)' = x' \vee y'$	$(\Leftrightarrow (x \vee y)' = x' \wedge y')$	(De Morgan law)
(Di)	$({\sim}x)' = -x'$	$(\Leftrightarrow (-x)' = {\sim}x')$	(De Morgan involution)
(Dp)	$(x \cdot y)' = x' + y'$		(De Morgan product)
(Cy)	${\sim}x = -x$		(cyclic law)
(Cp)	$x' \wedge x = \bot$ and $x' \vee x = \top$		(complementation)
(D)	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$		(distributivity)
(B)	$=$ (Cp) and (D)		(Boolean, $\Rightarrow$ (Dm))

The names of these identities are also used as prefixes to refer to algebras that satisfy the respective identity. E.g., a DmFL′-algebra is an FL′-algebra that satisfies the (Dm) identity. A *De Morgan lattice* is an algebra $(A, \wedge, \vee, ')$ such that $(A, \wedge, \vee)$ is a lattice and $'$ is a unary operation that satisfies $x'' = x$ and (Dm). We emphasize that no assumption of distributivity or boundedness is made in our definition of a De Morgan lattice (in the literature De Morgan algebras are assumed to be distributive and bounded).

Lemma 3. *The following properties hold in every FL′-algebra:*

(1) $(xy) \rhd z = y \rhd (x \rhd z)$ *and* $z \lhd (yx) = (z \lhd x) \lhd y$
(2) $(xy)^{\cup} = y \rhd x^{\cup}$ *and* $(xy)^{\sqcup} = y^{\sqcup} \lhd x$

(3) $1 \triangleright x = x$ *and* $x \triangleleft 1 = x$
(4) ${\sim}x = x^{\cup\prime}$, $-x = x^{\sqcup\prime}$, $x \leq x^{\cup\prime\sqcup\prime}$ *and* $x \leq x^{\sqcup\prime\cup\prime}$
(5) ${\sim}x = -x$ *iff* $x^{\cup} = x^{\sqcup}$ *(cyclic/balanced).*

If (Dm) *is assumed then we also have*

(6) $x^{\prime\cup\prime\sqcup} \leq x$ *and* $x^{\prime\sqcup\prime\cup} \leq x$
(7) $xy \leq z'$ $\Leftrightarrow$ $x \triangleright z \leq y'$ $\Leftrightarrow$ $z \triangleleft y \leq x'$ *(conjugation)*
(8) $(x \vee y)^{\cup} = x^{\cup} \vee y^{\cup}$ *and* $(x \vee y)^{\sqcup} = x^{\sqcup} \vee y^{\sqcup}$
(9) $(x \vee y) \triangleright z = (x \triangleright z) \vee (y \triangleright z)$ *and* $z \triangleleft (x \vee y) = (z \triangleleft x) \vee (z \triangleleft y)$
(10) $(x \vee y) \triangleleft z = (x \triangleleft z) \vee (y \triangleleft z)$ *and* $z \triangleright (x \vee y) = (z \triangleright x) \vee (z \triangleright y)$.

Proof. (1) follows from the definition of conjugation and the corresponding properties for divisions: $(xy) \triangleright z = ((xy)\backslash z')' = (y\backslash(x\backslash z'))' = (y\backslash(x\backslash z')'')' = y \triangleright (x \triangleright z)$ and the second identity is derived similarly. (2) follows from (1) if we let $z = 0'$. For (3), we have $1 \triangleright x = (1\backslash x')' = x'' = x$. Note that $x \leq -{\sim}x$ and $x \leq {\sim}{-x}$ hold in any FL-algebra, so properties (4) and (5) follow from the definition of the converses. (Dm) implies that $'$ is an order-reversing involution, hence (4) implies (6). Finally, (7) follows from residuation and (Dm), while (8), (9) and (10) follow from the De Morgan properties of $', \sim, -$. □

Recall that RL' is defined as FL' with the additional equation $1' = 0$. Since ${\sim}1 = 0 = -1$ holds in any FL-algebra, the next lemma shows that $1' = 0$ is equivalent to $1^{\cup} = 1$ as well as to $1^{\sqcup} = 1$.

Lemma 4. *In an RL′-algebra the following properties hold:*

(1) $x \triangleright 1 = x^{\cup}$ *and* $1 \triangleleft x = x^{\sqcup}$.
(2) $1^{\cup} = 1^{\sqcup} = 1$.

If (Dm) *holds then*

(3) $1 \leq x'$ *iff* $1 \leq {\sim}x$ *iff* $1 \leq -x$,
(4) $x \leq 1$ *implies* $x^{\cup} \leq 1$ *and* $x^{\sqcup} \leq 1$.

Proof. For (1) we have $x \triangleright 1 = (x\backslash 0)' = x^{\cup}$. Likewise, $1 \triangleleft x = x^{\sqcup}$. (2) follows from (1) and Lemma 3(3). For (3), we have $1 \leq x'$ iff $x \leq 1'$ iff $x1 \leq 0$ iff $1 \leq x\backslash 0 = {\sim}x$, and similarly $1 \leq x'$ iff $1 \leq -x$. Finally, for (4) $x \leq 1$ implies $x0 \leq 0$, so $0 \leq x\backslash 0$ gives $1 = 0' \geq x^{\cup}$. A symmetrical argument shows $x^{\sqcup} \leq 1$. □

An FL′-algebra is called *complemented*, if (Cp) holds, in which case $\bot$ is the smallest element and $\top$ is the largest element. A *Boolean* FL′-algebra (or *BFL′-algebra*) is a complemented and distributive FL′-algebra.

For Boolean FL′-algebras conjugation takes the more familiar form

$$xy \wedge z = \bot \quad \Leftrightarrow \quad x \triangleright z \wedge y = \bot \quad \Leftrightarrow \quad z \triangleleft y \wedge x = \bot$$

Note that residuated Boolean monoids are (term-equivalent to) Boolean RL′-algebras.

Some properties of FL$'$-algebras. For an ordered monoid $\mathbf{A}$, the set $A^- = \{a \in A : a \le 1\}$ is called the *negative cone*. It is well known that for relation algebras elements below the identity element are symmetric ($x^{\smile} = x$) and satisfy $xy = x \wedge y$. The next lemma shows these properties hold in a more general setting.

Lemma 5. *If the negative cone of an FL$'$-algebra $\mathbf{A}$ is a complemented lattice, then for $x, y \in A^-$, $xy = x \wedge y$. Furthermore, if $\mathbf{A}$ is a BRL$'$-algebra then $x^{\cup} = x^{\sqcup} = x$, for all $x \in A^-$.*

Proof. If $x, y \le 1$, then $xy \le x \wedge y$. Also, for every $u \le 1$ with complement u^* in A^-, $u = u1 = u(u \vee u^*) = u^2 \vee uu^* \le u^2 \vee (u \wedge u^*) = u^2$. Hence $x \wedge y \le (x \wedge y)^2 \le xy$.

Now suppose $\mathbf{A}$ is a BRL$'$-algebra, hence (Dm) holds. By Lemma 4(4), from $x \le 1$ we obtain $x^{\cup} \le 1$. For $u \in A^-$ with complement u^* in A^-, we have $x^{\cup} \le u^*$ iff $x^{\cup} \wedge u = \bot$ iff $(x \triangleright 1) \wedge u = \bot$ iff $xu \wedge 1 = \bot$ iff $x \wedge u \wedge 1 = \bot$ iff $x \le u^*$, where (B) was used in the last steps. Therefore $x^{\cup} = x$, and $x^{\sqcup} = x$ is similar. $\square$

4. Quasi relation algebras

We now explore some implications between several identities and then define a subvariety of RL$'$-algebras that generalizes relation algebras. Recall that (Di) refers to the De Morgan involution identity $({\sim}x)' = -x'$ or its equivalent $(-x)' = {\sim}x'$.

Lemma 6. *In a DmFL$'$-algebra we have the following equivalences:*

(i) $x \le x^{\cup\cup} \quad\Leftrightarrow\quad x^{\sqcup\prime} \le x^{\prime\cup} \quad\Leftrightarrow\quad x^{\cup\prime} \le x^{\prime\sqcup} \quad\Leftrightarrow\quad x \le x^{\sqcup\sqcup}$

(ii) $x^{\cup\cup} \le x \quad\Leftrightarrow\quad x^{\prime\cup} \le x^{\sqcup\prime} \quad\Leftrightarrow\quad x^{\prime\sqcup} \le x^{\cup\prime} \quad\Leftrightarrow\quad x^{\sqcup\sqcup} \le x$

(iii) $x^{\cup\cup} = x \quad\Leftrightarrow\quad x^{\prime\cup} = x^{\sqcup\prime} \quad\Leftrightarrow\quad x^{\prime\sqcup} = x^{\cup\prime} \quad\Leftrightarrow\quad x^{\sqcup\sqcup} = x \quad\Leftrightarrow\quad$ (Di)

Moreover, (Di) *and hence each of the identities in* (iii) *implies*

(In) ${\sim}{-x} = x = -{\sim}x$ (*linear*) *involutive.*

Proof. (i) Assuming $x \le x^{\cup\cup}$, Lemma 3(6) implies $x^{\sqcup\prime} \le x^{\prime\prime\sqcup\prime\cup\cup} \le x^{\prime\cup}$. Conversely, assuming $x^{\sqcup\prime} \le x^{\prime\cup}$, Lemma 3(4) implies $x \le x^{\cup\prime\sqcup\prime} \le x^{\cup\prime\prime\cup} = x^{\cup\cup}$. The third equivalence is similar, while the middle one is proved by applying $'$ on both sides and replacing x by x'.

(ii) is similar to (i), and the first three equivalences of (iii) are obvious consequences of (i), (ii). From the definition of $x^{\cup} = ({\sim}x)'$, $x^{\sqcup} = (-x)'$ it follows that $x^{\prime\cup} = x^{\sqcup\prime}$ is equivalent with (Di).

(Di)$\Rightarrow$(In): Using the identities of (iii), we have $x^{\cup\prime\sqcup\prime} = x^{\cup\prime\prime\cup} = x^{\cup\cup} = x = x^{\sqcup\sqcup} = x^{\sqcup\prime\cup\prime}$ which is a translated version of (In). $\square$

The following algebra shows that (In) is not equivalent to (Di), even for commutative RL$'$-algebras.

Example 7. Let $A = \{0, a, b, c, d, e, 1\}$ with $0 < a < b, c, d < e < 1$ and b, c, d pairwise incomparable. The operation $\cdot$ is commutative and satisfies $0 \cdot x = 0$, $1 \cdot x = x$ for $x \in A$, $a \cdot y = 0$ for $y \in \{a, b, c, d, e\}$, $b \cdot b = c \cdot d = 0$, $b \cdot c = b \cdot d = c \cdot c = d \cdot d = a$ (the remaining products follow from distribution over joins, and one can check that makes $\cdot$ a well-defined associative operation). The residuals $\backslash, /$ are implicitly defined since A is finite. The unary operations are given by $0' = {\sim}0 = 1$, $a' = {\sim}a = e$, $b' = d$, $c' = c$, ${\sim}b = b$, ${\sim}c = d$ and $-x = {\sim}x$. Then $({\sim}c)' = d' = b$ whereas ${\sim}(c') = {\sim}c = d$.

The next result shows that involutive DmFL$'$-algebras can be defined using operations and equations similar to Tarski's original axiomatisation of relation algebras.

Theorem 8. *DmInFL$'$-algebras are term equivalent to algebras of the form $\mathbf{A} = (A, \wedge, \vee, ', \cdot, 1, ^{\cup}, ^{\sqcup})$, such that $(A, \wedge, \vee, ')$ is a De Morgan lattice, $(A, \cdot, 1)$ is a monoid and for all $x, y, z \in A$,*

$$xy \leq z' \iff (z'^{\sqcup\prime}x)^{\cup} \leq y' \iff (yz'^{\cup\prime})^{\sqcup} \leq x'.$$

The above conjugation property can also be replaced by the following identities:

(i) $x^{\cup\prime\sqcup\prime} = x = x^{\sqcup\prime\cup\prime}$,
(ii) $x(y \vee z) = xy \vee xz, \quad (x \vee y)z = xz \vee yz$,
(iii) $(x \vee y)^{\cup} = x^{\cup} \vee y^{\cup}, \quad (x \vee y)^{\sqcup} = x^{\sqcup} \vee y^{\sqcup}$,
(iv) $((xy)^{\sqcup\prime}x)^{\cup} \leq y', \quad (x(yx)^{\cup\prime})^{\sqcup} \leq y'$.

Proof. A DmInFL$'$-algebra satisfies the given conditions with $x^{\cup} = ({\sim}x)'$ and $x^{\sqcup} = (-x)'$, and the conjugation property is equivalent to

$$xy \leq z \iff y \leq (z^{\sqcup\prime}x)^{\cup\prime} \iff x \leq (yz^{\cup\prime})^{\sqcup\prime}$$

which is a direct translation of the residuation property for InFL if one defines ${\sim}x = x^{\cup\prime}$ and $-x = x^{\sqcup\prime}$. Hence the term equivalence follows from Lemma 2. The identities (i)-(iv) are also (equivalent to) direct translations of the identities in Lemma 2, so they are equivalent to the conjugation property. $\square$

Adding (Di) to the previous theorem and invoking Lemma 6 gives the following result.

Corollary 9. *DiDmFL$'$-algebras are term equivalent to algebras of the form $\mathbf{A} = (A, \wedge, \vee, ', \cdot, 1, ^{\cup}, ^{\sqcup})$, such that $(A, \wedge, \vee, ')$ is a De Morgan lattice, $(A, \cdot, 1)$ is a monoid and for all $x, y, z \in A$, $x^{\cup\cup} = x$, and*

$$xy \leq z' \iff (z'^{\sqcup\prime}x)^{\cup} \leq y' \iff (yz'^{\cup\prime})^{\sqcup} \leq x'.$$

The above conjugation property can also be replaced by the following identities:

(i) $x(y \vee z) = xy \vee xz, \quad (x \vee y)z = xz \vee yz$,
(ii) $(x \vee y)^{\cup} = x^{\cup} \vee y^{\cup}, \quad (x \vee y)^{\sqcup} = x^{\sqcup} \vee y^{\sqcup}$
(iii) $((xy)^{\sqcup\prime}x)^{\cup} \leq y', \quad (x(yx)^{\cup\prime})^{\sqcup} \leq y'$.

However DiDmFL$'$-algebras do not satisfy the analogue of the relation algebra axiom $(xy)^{\smile} = y^{\smile}x^{\smile}$, which leads us to the next result. Recall that the dual of $\cdot$ is defined as $x + y = {\sim}(-y \cdot -x)$.

Lemma 10. *In every InFL$'$-algebra the following are equivalent and they imply* $0 = 1'$.

(1) $(xy)^{\cup} = y^{\cup}x^{\cup}$
(2) $(xy)^{\sqcup} = y^{\sqcup}x^{\sqcup}$
(3) $x \triangleright y = x^{\cup}y$
(4) $y \triangleleft x = yx^{\sqcup}$
(5) $(xy)' = x' + y'$

Proof. (1)$\Leftrightarrow$(5): We have $(xy)^{\cup} = y^{\cup}x^{\cup} \Leftrightarrow ({\sim}(xy))' = ({\sim}y)'({\sim}x)' \Leftrightarrow$ $[{\sim}(-x' \cdot -y')]' = y \cdot x \Leftrightarrow (y' + x')' = yx \Leftrightarrow y' + x' = (yx)'$.

(3)$\Leftrightarrow$(5): We have $x \triangleright y = (x \backslash y')' = ({\sim}x + y')'$, by (In). So, (3) is equivalent to $({\sim}x + y')' = ({\sim}x)'y$, which for x equal to $-(x')$ gives (5), in view of (In).

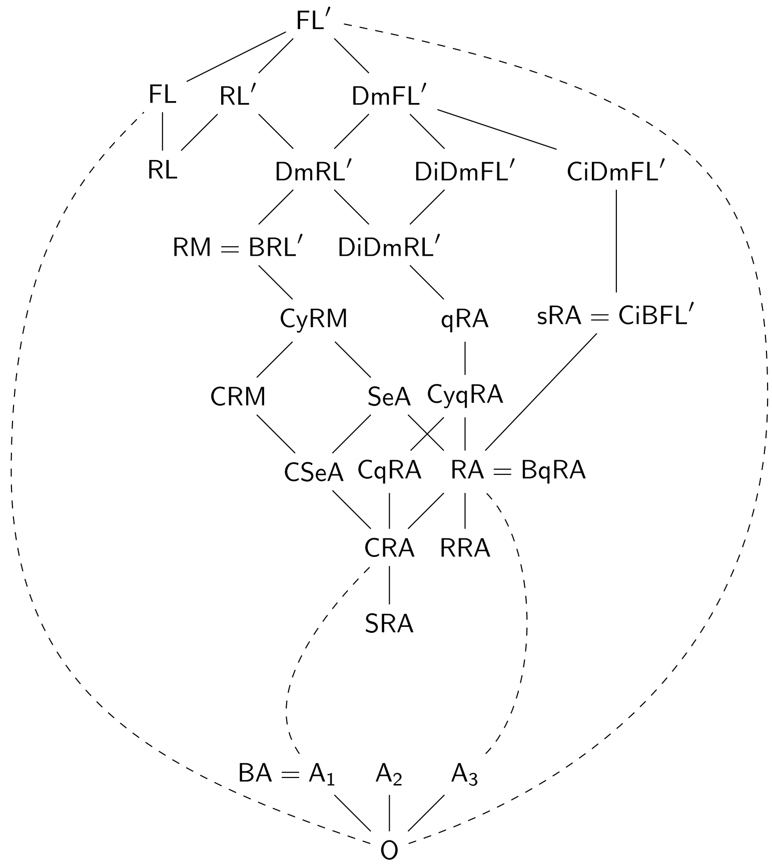
The equivalences (2)$\Leftrightarrow$(5)$\Leftrightarrow$(4) follow by symmetry. By setting $x = y = 1$ in (3), we get $1 = 1 \triangleright 1 = 1^{\cup}$, which is equivalent to $0 = 1'$. □

The following example shows that there are commutative Boolean InRL$'$-algebras that do not satisfy (Dp).

Example 11. Let $\mathbf{A}$ be an 8-element Boolean algebra with atoms $1, a, b$ and define $a \cdot a = a$, $a \cdot b = b \cdot a = \top$, $b \cdot b = a \vee b = 0 = 1'$. Since $\cdot$ is join-preserving the remaining products and residuals are determined. It suffices to check associativity for the atoms, and $a^{\cup} = b$, $b^{\cup} = a$, hence $(aa)^{\cup} = b$ while $a^{\cup}a^{\cup} = a \vee b$.

A *quasi relation algebra* (qRA) is defined to be a DiDpDmFL$'$-algebra. Hence $'$ is a dual automorphism in these algebras. The variety of quasi relation algebras is denoted by qRA.

Figure 1 shows a (necessarily incomplete) picture of the lattice of subvarieties of FL$'$-algebras, highlighting a few of the important members (the dotted lines around the outside are meant to indicate that there are many more varieties within their boundaries). The variety qRA is below RL', above the variety of relation algebras RA and incomparable to varieties such as residuated Boolean monoids RM and skew relation algebras (defined later as Boolean InFL$'$-algebras). In fact relation algebras are Boolean qRAs, hence RA is the intersection of qRA with RM as well as with sRA. The ideal below RA (outlined by the dotted lines inside the figure) is a brief indication of the lattice of subvarieties of RA, which includes the varieties CRA, SRA and RRA of commutative, symmetric and representable relation algebras respectively. By Jónsson and Tarski [10], there are three minimal subvarieties of RA, generated by the 2-element relation algebra and two 4-element relation algebras that satisfy $1' \cdot 1' = 1$ and $1' \cdot 1' = 1 \vee 1'$ respectively. The variety O of one-element algebras is the smallest variety of qRAs. The variety SeA of sequential algebras [8]



Dm: $(x \wedge y)' = x' \vee y'$, Di: $({\sim}x)' = -x'$, Ci: $({\sim}x)' = {\sim}x'$ and $(-x)' = -x'$
B: Boolean, Cy: ${\sim}x = -x$, C: $xy = yx$, S: ${\sim}x = x' = -x$
qRA: DiDmRL′ + $(xy)^\cup = y^\cup x^\cup$, SeA: CyRM+$(x \rhd y)z \leq x \rhd (yz)$

FIGURE 1. Some subvarieties of FL′ ordered by inclusion

is fairly close to RA, but since its members are Boolean, it is not a subvariety of qRA. Non-Boolean examples of quasi relation algebras can be constructed as follows.

Example 12. Let $G = \mathrm{Aut}(C)$ be the lattice-ordered group of all order-automorphisms of a chain C, and assume that C has a dual automorphism ${}^\partial : C \to C$. Note that G is a cyclic involutive FL-algebra with ${\sim}x = -x = x^{-1}$, $x + y = xy$, and $0 = 1$. This algebra can be expanded to a FL′-algebra in the following way. For $g \in G$, define $g'(x) = g(x^\partial)^\partial$. Note that g' is the point symmetry at the origin (namely the fixed element of ${}^\partial$, if it exists), or 'rotation by 180°' of g. Moreover, the converse of g is the reflection along the 'line' with graph $x \mapsto x^\partial$. Then $g'' = g$, $(g \vee h)'(x) = (g(x^\partial) \vee h(x^\partial))^\partial =$

$g(x^\partial)^\partial \wedge h(x^\partial)^\partial = (g' \wedge h')(x)$ and $(gh)'(x) = (g(h(x^\partial)))^\partial = g((h(x^\partial)^\partial)^\partial)^\partial = (g'h')(x) = (g' + h')(x)$. Hence G expanded with $'$ is a cyclic quasi relation algebra.

Noncyclic examples can be obtained from proper ℓ-pregroups [14]. In this case the algebras above are enlarged by considering all endomorphisms of C that have iterated unary residuals and dual residuals. A specific example would be the orderpreserving maps f on $\mathbb{Z}$ for which each element has a finite preimage, so $({\sim}f)(n) = \max\{m : f(m) \leq n\}$ and $(-f)(n) = \min\{m : f(m) \geq n\}$.

For an InFL-algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, 1, 0, \sim, -)$ define $\mathbf{A}^\partial = (A, \vee, \wedge, +, 0, 1, -, \sim)$. Then $\mathbf{A}^\partial$ is an InFL-algebra called the dual of $\mathbf{A}$.

Consider the categories of InFL-algebras and qRAs, and define $F : \mathsf{InFL} \to \mathsf{qRA}$ by $F(\mathbf{A}) = \mathbf{A} \times \mathbf{A}^\partial$ expanded with the operation $(a, b)' = (b, a)$. For a homomorphism $h : \mathbf{A} \to \mathbf{B}$ define $F(h) : F(\mathbf{A}) \to F(\mathbf{B})$ by $F(h)(a, b) = (h(a), h(b))$. The following theorem is a generalization of [3], where it is proved for distributive De Morgan lattices.

Theorem 13. *F is a functor from InFL to qRA, and if G is the reduct functor from qRA to InFL then for any quasi relation algebra $\mathbf{C}$, the map $\sigma_{\mathbf{C}} : \mathbf{C} \to FG(\mathbf{C})$ given by $\sigma_{\mathbf{C}}(a) = (a, a')$ is an embedding.*

Proof. Let $\mathbf{A}$ be an InFL-algebra. Since $\mathbf{A}^\partial$ is also an InFL-algebra, it will follow that $F(\mathbf{A})$ is a qRA as soon as we observe that (Dm), (Di) and (Dp) hold.

(Dm): $((a, b) \wedge (c, d))' = (a \wedge c, b \vee d)' = (b \vee d, a \wedge c) = (b, a) \vee (d, c) = (a, b)' \vee (c, d)'$.

(Di): ${\sim}(a, b)' = {\sim}(b, a) = ({\sim}b, -a) = (-a, {\sim}b)' = (-(a, b))'$ and similarly $-(a, b)' = ({\sim}(a, b))'$.

(Dp): $((a, b) \cdot (c, d))' = (ac, b + d)' = (b + d, ac) = ({\sim}(-d \cdot -b), -({\sim}c + {\sim}a)) = {\sim}((-d, {\sim}c) \cdot (-b, {\sim}a)) = {\sim}(-(d, c) \cdot -(b, a)) = (b, a) + (d, c) = (a, b)' + (c, d)'$. ☐

Corollary 14. *The equational theory of* qRA *is a conservative extension of that of* InFL*; i.e., every equation over the language of* InFL *that holds in* qRA*, already holds in* InFL*.*

Euclidean algebras. An FL$'$-algebra is called *Euclidean* if it satisfies the inequality

$$(x \triangleright y)z \leq x \triangleright (yz).$$

It is called *strongly Euclidean* if it satisfies $(x \triangleright y)z = x \triangleright (yz)$. The Euclidean inequality holds in any powerset-algebra of a (partial) monoid, and cyclic Euclidean residuated Boolean monoids have been studied under the terminology of *sequential algebras* in [18], [8], [7].

Lemma 15. *For a DmRL$'$-algebra $\mathbf{A}$, the following results hold:*

(1) $\mathbf{A}$ *is Euclidean if and only if it satisfies* $x(y \triangleleft z) \leq (xy) \triangleleft z$,
(2) $\mathbf{A}$ *is strongly Euclidean if and only if any one of the following equivalent identities hold in* $\mathbf{A}$*:* $x(y \triangleleft z) = (xy) \triangleleft z$, $x^{\cup}z = x \triangleright z$, $xz^{\sqcup} = x \triangleleft z$.

Proof. (1) From the conjugation property we see that the Euclidean inequality is equivalent to the following four equivalent statements (each universally quantified over x, y, z, w):

$$\begin{array}{rcl} x \triangleright (yz) \leq w & \Rightarrow & (x \triangleright y)z \leq w \\ xw' \leq (yz)' & \Rightarrow & w' \triangleleft z \leq (x \triangleright y)' \\ yz \leq (xw')' & \Rightarrow & x \triangleright y \leq (w' \triangleleft z)' \\ (xw') \triangleleft z \leq y' & \Rightarrow & x(w' \triangleleft z) \leq y'. \end{array}$$

and the last line is equivalent to the inequality $x(y \triangleleft z) \leq (xy) \triangleleft z$.

(2) Replacing the four implications in the proof of (1) by bi-implications proves the corresponding result for the strongly Euclidean identity. To see that the three identities are equivalent, let $y = 1$ in the first identity to derive the last one, using Lemma 4(1). Conversely if $xz^{\sqcup} = x \triangleleft z$ holds, then $x(y \triangleleft z) = xyz^{\sqcup} = (xy) \triangleleft z$. Likewise the second identity is equivalent to the strongly Euclidean identity. □

Theorem 16. *The variety* qRA *coincides with the variety of strongly Euclidean DmRL′-algebras.*

Proof. By Lemma 10 and the previous lemma, every quasi relation algebra is a strongly Euclidean DmRL′-algebra. Conversely, in a strongly Euclidean DmRL′-algebra we have $x^{\cup}y = x \triangleright y$, hence $x^{\cup}y \leq z'$ is equivalent to $xz \leq y'$ by the conjugation property. So from $x \leq x$ we successively deduce $x1 \leq x''$, $x^{\cup}x' \leq 1'$, $x^{\cup\cup}1 \leq x''$, hence $x^{\cup\cup} \leq x$.

For the reverse inequality, note that $x^{\cup\cup\prime} = (x^{\cup}\backslash 0)'' \leq x^{\cup}\backslash 0$ from which we deduce $x^{\cup}x^{\cup\cup\prime} \leq 0''$, and by the equivalence in the previous paragraph we obtain $x0' \leq x^{\cup\cup}$. Therefore $x^{\cup\cup} = x$, and similarly $x^{\sqcup\sqcup} = x$ (using the previous lemma). By Lemma 6 we conclude that (Di) and (In) hold. Finally, since $x^{\cup}y = x \triangleright y$ is a consequence of the strongly Euclidean identity, Lemma 10 shows that (Dp) holds. □

Combining the preceding result with Theorem 9 gives the following generalization of the Jónsson-Tsinakis result [11] stated in Theorem 1. Recall that residuated Boolean monoids are Boolean RL′-algebras, hence they are contained in the variety of DmRL′-algebras.

Theorem 17. *Strongly Euclidean DmRL′-algebras, namely qRAs, are term-equivalent to algebras* $\mathbf{A} = (A, \wedge, \vee, ', \cdot, 1, ^{\cup}, ^{\sqcup})$*, such that* $(A, \wedge, \vee, ')$ *is a De Morgan lattice,* $(A, \cdot, 1)$ *is a monoid and for all* $x, y, z \in A$

$$xy \leq z' \iff x^{\cup}z \leq y' \iff zy^{\sqcup} \leq x' \qquad (**)$$

Alternatively, the above conjugation property $(**)$ *can be replaced by the following identities:*

(i) $x \le x^{\cup\cup}$, $x \le x^{\sqcup\sqcup}$,
(ii) $x(y \vee z) = xy \vee xz$, $(x \vee y)z = xz \vee yz$,
(iii) $x^{\cup}(xy)' \le y'$, $(yx)'x^{\sqcup} \le y'$.

For cyclic qRAs only one of the identities in (iii) *is needed.*

Proof. Quasi relation algebras satisfy the given conditions and the conjugation property $(**)$. Conversely, suppose $\mathbf{A}$ is an algebra with the stated properties, and define ${\sim}x = x^{\cup\prime}$ and $-x = x^{\sqcup\prime}$. From $(**)$ $x \le z'$ iff $x^{\cup}z \le 1'$ iff $x^{\cup\cup} \le z'$, hence $x^{\cup\cup} = x$ and similarly $x^{\sqcup\sqcup} = x$. Also $y^{\cup}x^{\cup} \le z'$ iff $yz \le x^{\cup\prime}$ iff $x^{\cup} \le (yz)'$ iff $xyz \le 1'$ iff $(xy)^{\cup} \le z'$, where we made use of associativity. Thus $(xy)^{\cup} = y^{\cup}x^{\cup}$. A standard argument shows that $\cdot$ is join-preserving. Likewise $^{\cup}$ is join preserving: $(x \vee y)^{\cup} \le z'$ iff $(x \vee y)z \le 1'$ iff $xz \le 1'$ and $yz \le 1'$ iff $x^{\cup} \vee y^{\cup} \le z'$. Now (Di) holds since $x \le y'^{\cup}$ iff $x^{\cup} \le y'^{\cup\cup} = y'$ iff $xy \le 1'$ iff $y^{\sqcup} \le x'$ iff $x \le y^{\sqcup\prime}$. Thus by Corollary 9 $\mathbf{A}$ is term equivalent to a DiDmFL'-algebra, and by Lemma 10 we have (Dp), hence $\mathbf{A}$ is a qRA.

To see that (i)-(iii) are equivalent to $(**)$, suppose $\mathbf{A}$ is an algebra with the stated properties, and let $xy \le z'$. Then $z \le (xy)'$, so by (i), (iii) $x^{\cup}z \le x^{\cup}(xy)' \le y'$. On the other hand let $x^{\cup}z \le y'$. Then $y \le (x^{\cup}z)'$, so using (i), (ii), (iii) $xy \le x^{\cup\cup}(x^{\cup}z)' \le z'$. Together with a symmetric argument for $^{\sqcup}$ we obtain $(**)$. Conversely, from $(**)$ we already derived (i), and proving (ii), (iii) is routine.

If $\mathbf{A}$ is cyclic, i.e., $x^{\cup} = x^{\sqcup}$, then $x^{\cup}(xy)' \le y'$ implies $(x^{\cup}(xy)')^{\cup} \le y'^{\cup}$, and we obtain the second inequality of (iii) by using (Dp), (Di) and replacing x, y by $x^{\cup}, y^{\cup}$. $\square$

If we assume that the algebras above have Boolean reducts, then $x^{\cup} = x^{\sqcup}$ holds (see [11] 4.3), and the result reduces to the term-equivalence of Jónsson and Tsinakis.

5. Decidability of quasi relation algebras

In this section we reduce checking an equation in qRA to checking a related equation in InFL; the same applies to subvarieties of qRA defined by self-dual sets of InFL-identities. We conclude that the equational theory of qRA is decidable. We actually show the stronger result that the variety qRA is generated by its finite members.

For an InFL-term t, we define the *dual* term t^{∂} inductively by

$$\begin{array}{rl} x^{\partial} = x & (s \wedge t)^{\partial} = s^{\partial} \vee t^{\partial} \\ 0^{\partial} = 1 & (s \vee t)^{\partial} = s^{\partial} \wedge t^{\partial} \\ 1^{\partial} = 0 & (s \cdot t)^{\partial} = s^{\partial} + t^{\partial} \\ ({\sim}s)^{\partial} = -s^{\partial} & (s + t)^{\partial} = s^{\partial} \cdot t^{\partial} \\ (-s)^{\partial} = {\sim}s^{\partial} & \end{array}$$

We also define $(s = t)^{\partial}$ to be $s^{\partial} = t^{\partial}$. For a set $\mathcal{E}$ of InFL identities, we define $\mathcal{E}^{\partial} = \{\varepsilon^{\partial} : \varepsilon \in \mathcal{E}\}$. The set $\mathcal{E}$ is called *self-dual* if $\mathcal{E}^{\partial} = \mathcal{E}$.

Now let $\mathcal{E}$ be a self-dual set of InFL identities, and consider the varieties $\mathsf{V} = \{\mathbf{A} \in \mathsf{InFL} : \mathbf{A} \models \mathcal{E}\}$ and $\mathsf{V}' = \{\mathbf{A} \in \mathsf{qRA} : \mathbf{A} \models \mathcal{E}\}$.

Lemma 18. *An equation ε is valid in V iff ε^∂ is also valid in V.*

Proof. The axioms of V are self-dual. Therefore, if we uniformly dualize the whole equational proof of ε, we obtain an equational proof of ε^∂. The converse follows from the fact that $\varepsilon^{\partial\partial} = \varepsilon$. $\square$

We fix a partition of the denumerable set of variables into two denumerable sets X and $X^\bullet$, and fix a bijection $x \mapsto x^\bullet$ from the first set to the second (hence $x^{\bullet\bullet}$ denotes x). We will assume that all qRA-terms are written over the set of variables X, but we will consider InFL-terms over $X \cup X^\bullet$. For a qRA-term t (over X), we define the InFL-term t° inductively by

$$\begin{array}{cc}
x^\circ = x & (s'')^\circ = s^\circ \\
0^\circ = 0, & ((s \wedge t)')^\circ = s'^\circ \vee t'^\circ, \\
1^\circ = 1, & ((s \vee t)')^\circ = s'^\circ \wedge t'^\circ, \\
(s \diamond t)^\circ = s^\circ \diamond t^\circ, \text{ for } \diamond \in \{\wedge, \vee, \cdot, +\}, & ((s \cdot t)')^\circ = s'^\circ + t'^\circ, \\
(\diamond s)^\circ = \diamond s^\circ, \text{ for } \diamond \in \{\sim, -\}, & ((s + t)')^\circ = s'^\circ \cdot t'^\circ, \\
 & (0')^\circ = 1,\ (1')^\circ = 0 \\
 & ((\sim s)')^\circ = -(s'^\circ),\ ((-s)')^\circ = \sim(s'^\circ) \\
 & (x')^\circ = x^\bullet
\end{array}$$

We also define $t^\downarrow$ by the same clauses except for the last one: $(x')^\downarrow = x'$. Both t° and $t^\downarrow$ represent a term obtained from t by 'pushing' all primes to the variables in a natural way consistent with qRA equations. The only difference is their behavior on the variables. The next result shows that we may assume in qRA that all negations have been pushed down to the variables.

Lemma 19. $\mathsf{qRA} \models t = t^\downarrow$.

Proof. The induction is clear for variables and InFL connectives. For $t = s'$, we proceed by induction on s. For $s = p \wedge q$, we have $(p \wedge q)'^\downarrow = p'^\downarrow \vee q'^\downarrow = p' \vee q' = (p \wedge q)'$ in qRA, where the last equality holds because of the DeMorgan properties of qRA. Similarly, we proceed for InFL connectives for s. For $s = p'$, we have $p''^\downarrow = p^\downarrow = p = p''$. $\square$

For a term t, we denote by $t^\bullet$ the result of applying the substitution that extends the bijection $x \mapsto x^\bullet$.

Lemma 20. *For every qRA-term t, $t^{\circ\partial} = (t'^\circ)^\bullet$ in InFL.*

Proof. We proceed by induction on t. If $t = x$, a variable, then clearly $x^{\circ\partial} = x = x^{\bullet\bullet} = (x'^\circ)^\bullet$. If $t = s \wedge r$, then $(s \wedge r)^{\circ\partial} = (s^\circ \wedge r^\circ)^\partial = s^{\circ\partial} \vee r^{\circ\partial} = (s'^\circ)^\bullet \vee (r'^\circ)^\bullet = ((s \wedge r)'^\circ)^\bullet$. The same argument holds for all other InFL connectives. For $t = s'$, we need to do further induction on s. If $s = p \wedge q$, then $(p \wedge q)'^{\circ\partial} = (p'^\circ \vee q'^\circ)^\partial = p'^{\circ\partial} \wedge q'^{\circ\partial} = (p''^\circ)^\bullet \wedge (q''^\circ)^\bullet = (p^\circ)^\bullet \wedge (q^\circ)^\bullet = ((p \wedge q)^\circ)^\bullet = ((p \wedge q)''^\circ)^\bullet$, and likewise for the other InFL connectives. For $s = p'$, we have $p''^{\circ\partial} = p^{\circ\partial} = (p'^\circ)^\bullet = (p'''^\circ)^\bullet$. $\square$

For a substitution σ, we define a substitution σ° by $\sigma^\circ(x) = (\sigma(x))^\circ$, if $x \in X$, and $\sigma^\circ(x) = (\sigma(x)')^\circ$, if $x \in X^\bullet$.

Lemma 21. *For every* qRA*-term* t *and* qRA*-substitution* σ*,* $(\sigma(t))^\circ = \sigma^\circ(t^\circ)$*.*

Proof. For $t = s \wedge r$, we have $(\sigma(s \wedge r))^\circ = (\sigma(s) \wedge \sigma(r))^\circ = (\sigma(s))^\circ \wedge (\sigma(r))^\circ = \sigma^\circ(s^\circ) \wedge \sigma^\circ(r^\circ) = \sigma^\circ(s^\circ \wedge r^\circ) = \sigma^\circ((s \wedge r)^\circ)$. The proof is similar for other InFL connectives. For $t = s'$, we proceed by induction on s. For $s = p \wedge q$, we have $(\sigma((p \wedge q)'))^\circ = ((\sigma(p) \wedge \sigma(q))')^\circ = (\sigma(p))'^\circ \vee (\sigma(q))'^\circ = (\sigma(p'))^\circ \vee (\sigma(q'))^\circ = \sigma^\circ(p'^\circ) \vee \sigma^\circ(q'^\circ) = \sigma^\circ(p'^\circ \vee q'^\circ) = \sigma^\circ((p \wedge q)'^\circ)$. For $s = p'$, $(\sigma(p''))^\circ = (\sigma(p)'')^\circ = (\sigma(p))^\circ = \sigma^\circ(p^\circ) = \sigma^\circ(p''^\circ)$. $\square$

Let V, V' be as above.

Theorem 22. *An equation* ε *over* X *holds in* V' *iff the equation* ε° *holds in* V*.*

Proof. For the backward direction assume that ε° holds in V. Then V also satisfies the equation $\hat{\varepsilon}$ obtained by substituting in ε° the variables $x^\bullet$ of $X^\bullet$ with new (namely they do not appear in ε°) and distinct variables $\hat{x} \in X$. Since every InFL-equation that holds in V also holds in V', we have that $\hat{\varepsilon}$ holds in V'. If we substitute the term x' for each $\hat{x}$ in $\hat{\varepsilon}$ then the resulting equation, which is actually $\varepsilon^\downarrow$, holds in V', as well. By Lemma 19, we get that ε holds in V'.

For the forward direction we assume that there is a proof of ε in the equational logic over V'. Without loss of generality, we may assume that all variables in the proof are contained in X. We will show that ε° is provable in the equational logic over V, by induction over the rules of equational logic.

Assume first that ε is an axiom of V'. If it does not involve prime, then $\varepsilon = \varepsilon^\circ$ and it is also an axiom of V. If it involves prime, say, $(x \wedge y)' = x' \vee y'$, then ε° is $x^\bullet \vee y^\bullet = x^\bullet \vee y^\bullet$; the argument for the other axioms is very similar.

If the last step of the proof of $\varepsilon = (t = s)$ was symmetry, then $s = t$ is provable in V' and, by the induction hypothesis, $s^\circ = t^\circ$ is provable in V. Then by symmetry, $\varepsilon^\circ = (t^\circ = s^\circ)$ is provable in V. The same argument works if the last step in the proof is transitivity.

Suppose that the last rule was replacement (for unary basic terms), say deriving $s \wedge p = t \wedge p$ from $s = t$ in V'. By induction, $s^\circ = t^\circ$ is provable in V. Then by replacement we get $s^\circ \wedge p^\circ = t^\circ \wedge p^\circ$, namely $(s \wedge p)^\circ = (t \wedge p)^\circ$. Likewise we argue for the other InFL connectives. Now assume that the last rule is the derivation of $s' = t'$ from $s = t$ in V'. By induction, $s^\circ = t^\circ$ is provable in V. By Lemma 18, $s^{\circ\partial} = t^{\circ\partial}$ is provable in V, hence also $(s'^\circ)^\bullet = (t'^\circ)^\bullet$ is provable, by Lemma 20. So $s'^\circ = (s'^\circ)^{\bullet\bullet} = (t'^\circ)^{\bullet\bullet} = t'^\circ$ is provable in V.

Finally, assume that the last rule was substitution, deriving $\sigma(s) = \sigma(t)$ from $s = t$ in V'. By induction, $s^\circ = t^\circ$ is provable in V. By substitution, $\sigma^\circ(s^\circ) = \sigma^\circ(t^\circ)$ is provable in V. By Lemma 21, $(\sigma(s))^\circ = (\sigma(t))^\circ$ is provable in V. $\square$

In [5] it is shown that the equational theory of InFL is decidable by a Gentzen system. It is also known that cyclic InFL-algebras [21, 20], cyclic distributive InFL-algebras [12], commutative InFL-algebras and lattice-ordered groups have decidable equational theories. This, together with Theorem 22 and the above definition of V, V', yields the following result.

Corollary 23. *If* V *has a decidable equational theory then so does* V'*. Hence the equational theories of* qRA*, cyclic* qRA*, cyclic distributive* qRA*, commutative* qRA *and the variety of qRAs that have ℓ-group reducts* ($= \{\mathbf{A} \in \mathsf{qRA} : \mathbf{A} \models x^{\cup} \cdot x = 1\}$) *are decidable.*

Let F be the functor defined ahead of Theorem 13, and let V, V' be as above. The varieties InFL, cyclic InFL and commutative InFL are generated by their finite members [5].

Theorem 24. *If* V *is generated by its finite members, so is* V'*. In fact, the finite members of the form* $F(\mathbf{A})$*, for* $\mathbf{A} \in \mathsf{V}$*, generate* V'*. Hence the varieties* qRA*, cyclic* qRA *and commutative* qRA *are generated by their finite members.*

Proof. Assume V is generated by its finite members, and let $\varepsilon = (s = t)$ be an equation in the language of qRA, over the variables $x_1, \ldots, x_n$, that fails in the variety V'. Then, by Theorem 22, the equation $s^\circ = t^\circ$ (over the variables $x_1, \ldots, x_n, x_1^\bullet, \ldots, x_n^\bullet$) fails in V. Since the variety V is generated by its finite members, there is a finite $\mathbf{A} \in \mathsf{V}$ and $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$, such that $(s^\circ)^{\mathbf{A}}(\bar{a}, \bar{b}) \neq (t^\circ)^{\mathbf{A}}(\bar{a}, \bar{b})$. In view of Lemma 19, without loss of generality we can assume that $s = s^\downarrow$ and $t = t^\downarrow$. Note that $s^\downarrow$ and s° are almost identical, except for occurrences of variables x' and $x^\bullet$. Therefore, $s(x_1, \ldots, x_n) = s^\circ(x_1, \ldots, x_n, x_1', \ldots, x_n')$, and the same for t. We have

$$
\begin{aligned}
& s^{F(\mathbf{A})}((a_1, b_1), \ldots, (a_n, b_n)) \\
&= (s^\circ)^{F(\mathbf{A})}((a_1, b_1), \ldots, (a_n, b_n), (a_1, b_1)', \ldots, (a_n, b_n)') \\
&= (s^\circ)^{F(\mathbf{A})}((a_1, b_1), \ldots, (a_n, b_n), (b_1, a_1), \ldots, (b_n, a_n)) \\
&= ((s^\circ)^{\mathbf{A}}(\bar{a}, \bar{b}), (s^\circ)^{\mathbf{A}^\partial}(\bar{b}, \bar{a})) \\
&\neq ((t^\circ)^{\mathbf{A}}(\bar{a}, \bar{b}), (t^\circ)^{\mathbf{A}^\partial}(\bar{b}, \bar{a})) \\
&= t^{F(\mathbf{A})}((a_1, b_1), \ldots, (a_n, b_n)).
\end{aligned}
$$

In other words, the equation $s = t$ fails in $F(\mathbf{A})$, which is a finite algebra in V'. □

Currently cyclic distributive qRA is the smallest known variety that includes all relation algebras, has an associative operation $\cdot$, and has a decidable equational theory. It is an interesting question whether this variety is generated by its finite members.

6. Skew relation algebras

In this section we investigate adding the identities ${\sim}(x') = ({\sim}x)'$ and $-(x') = (-x)'$ to DmFL$'$-algebras. While such algebras are perhaps not as well-behaved as quasi relation algebras, they do still inherit many equational properties of relation algebras and have several natural examples.

Lemma 25. *In a DmFL$'$-algebra* $\mathbf{A}$ *the following are equivalent:*

(Ci) ${\sim}(x') = ({\sim}x)'$ *and* $-(x') = (-x)'$ *(commuting involution)*

(ii) $x^{\cup\prime} = x^{\prime\cup}$ *and* $x^{\sqcup\prime} = x^{\prime\sqcup}$ *(commuting converses involution)*

(iii) $x^{\cup\sqcup} = x = x^{\sqcup\cup}$ *(converse involutive)*

(iv) $-x^{\cup} = x' = {\sim}x^{\sqcup}$

Moreover, each of these properties implies that

(In) ${\sim}{-}x = x = -{\sim}x$ *(linear) involutive.*

If $\mathbf{A}$ *is Boolean, then all four properties are equivalent. Hence* $\mathsf{BInFL}' = \mathsf{BCiFL}'$.

Proof. The equivalence of (Ci) and (ii) follows from the definition of the converses.

(iii)⇒(Ci) $x^{\sqcup\prime} = x^{\sqcup\prime\cup\sqcup} = x^{\sqcup\prime\cup\prime\prime\sqcup} = ({\sim}{-}x)^{\prime\sqcup} \leq x^{\prime\sqcup}$, where the last inequality follows from $x \leq {\sim}{-}x$ and the fact that $'$ is order reversing and $^{\sqcup}$ is order preserving. Likewise, we obtain $x^{\cup\prime} \leq x^{\prime\cup}$. Furthermore, we have $x^{\prime\cup} = x^{\cup\sqcup\prime\cup} \leq x^{\cup\prime\sqcup\cup} = x^{\cup\prime}$, where the inequality follows from $x^{\sqcup\prime} \leq x^{\prime\sqcup}$ and the order-preservation of $^{\cup}$. Consequently, $x^{\cup\prime} = x^{\prime\cup}$ and the same holds for the other converse operation.

(Ci)⇒(In). We always have $x \leq {\sim}{-}x$. Hence, $({\sim}{-}x)' \leq x$, so ${\sim}{-}x' \leq x'$, for all x. Consequently, ${\sim}{-}x \leq x$, for all x. This establishes half of (In); the other half follows by symmetry.

(Ci)⇒(iii) We have shown that (In) follows. Therefore, we have $x^{\cup\sqcup} = [-(({\sim}x)')]' = -{\sim}x'' = x$; likewise we obtain the other half of (iii).

Assume that $\mathbf{A}$ is Boolean and involutive. We have ${\sim}x \wedge {\sim}(x') = {\sim}(x \vee x') = {\sim}\top = \bot$, so ${\sim}(x') \leq ({\sim}x)'$. Moreover, $({\sim}x)' \wedge ({\sim}x')' = ({\sim}(x \wedge x'))' = ({\sim}\bot)' = \bot$, so $({\sim}x)' \leq ({\sim}(x'))'' = {\sim}(x')$. $\square$

A 6-element counterexample shows that (In) is not equivalent to (Ci), even in the commutative case. The next result is obtained by adding (Ci) to Theorem 2.

Corollary 26. *CiDmFL$'$-algebras are term equivalent to algebras of the form* $\mathbf{A} = (A, \wedge, \vee, \cdot, ', ^{\cup}, ^{\sqcup}, 1)$*, such that* $(A, \wedge, \vee, ')$ *is a De Morgan lattice,* $(A, \cdot, 1)$ *is a monoid, and for all* $x, y, z \in A$, $x^{\cup\sqcup} = x = x^{\sqcup\cup}$ *and*

$$xy \leq z' \iff (z^{\prime\sqcup\prime}x)^{\cup} \leq y' \iff (yz^{\prime\cup\prime})^{\sqcup} \leq x'.$$

The above conjugation property can also be replaced by the following identities:

(i) $x(y \vee z) = xy \vee xz, \quad (x \vee y)z = xz \vee yz,$

(ii) $(x \vee y)^{\cup} = x^{\cup} \vee y^{\cup}$,
(iii) $((xy)^{\sqcup\prime}x)^{\cup} \leq y'$, $\quad (x(yx)^{\cup\prime})^{\sqcup} \leq y'$.

Note that any two of (Ci), (Di), (Cy) imply the third.

Lemma 27. *CiDmDpRL'-algebras are cyclic, hence they are cyclic qRAs.*

Proof. That they are RL′-algebras follows from Lemma 10. Moreover, we have $(-x)' \cdot x' = -(x') \cdot x' = (0/x')x' \leq 0$. Therefore, $1 = 0' \leq ((-x)' \cdot x')' = -x + x = (-x)/({\sim}x)$, hence ${\sim}x \leq -x$. By symmetry we obtain the reverse inequality. □

A *skew relation algebra* is defined to be a BInFL′-algebra. The variety of skew relation algebras is denoted by sRA. By Lemma 25 (Ci) holds, so the next result follows directly from Corollary 26.

Corollary 28. *Skew relation algebras are term equivalent to algebras of the form* $\mathbf{A} = (A, \wedge, \vee, \perp, \top, \cdot, 1, ', ^{\cup}, ^{\sqcup})$, *such that* $(A, \wedge, \vee, ', \perp, \top)$ *is a Boolean algebra,* $(A, \cdot, 1)$ *is a monoid and for all* $x, y, z \in A$

(i) $x^{\cup\sqcup} = x = x^{\sqcup\cup}$,
(ii) $x(y \vee z) = xy \vee xz$, $\quad (x \vee y)z = xz \vee yz$, $\quad (x \vee y)^{\cup} = x^{\cup} \vee y^{\cup}$,
(iii) $((xy)^{\sqcup\prime}x)^{\cup} \leq y'$ *and* $(x(yx)^{\cup\prime})^{\sqcup} \leq y'$.

Corollary 29. *Relation algebras are term equivalent to skew relation algebras that satisfy* (Dp): $(xy)' = x' + y'$, *namely Boolean qRAs.*

An element a of a residuated lattice is called *invertible* if there is an element b (called an inverse of a) such that $ab = 1 = ba$. The following lemma shows that invertible elements have unique inverses that we will denote by a^{-1}.

Lemma 30. *Let* $\mathbf{A}$ *be a residuated lattice (expansion) and* a *an invertible element.*

(1) *a has a unique inverse $a^{-1} = a/1 = 1\backslash a$. Also, $a\backslash x = a^{-1}x$ and $x/a = xa^{-1}$, for all $x \in A$.*
(2) *$(x \wedge y)a = xa \wedge ya$ and $a(x \wedge y) = ax \wedge ay$, for all $x, y \in A$.*
(3) *If $\mathbf{A}$ is a BFL′-algebra and a is invertible, then for all $x \in A$, $(ax)' = ax'$, $(xa)' = x'a$, and $a^{\cup} = a^{\sqcup} = a^{-1}$.*

Proof. (1) Let b be an inverse of a. For all $x, z \in A$, we have $z \leq xb$ iff $za \leq x$ iff $z \leq x/a$; so $x/a = xb$, and likewise $a\backslash x = bx$. In particular, $b = a/1 = 1\backslash a$.

(2) For all x, y, z, we have $z \leq xa$ iff $za^{-1} \leq x$, the forward direction following from the order preservation of multiplication by a^{-1} and the reverse by a. Consequently, we have $z \leq xa \wedge ya$ iff $z \leq xa, ya$ iff $za^{-1} \leq x, y$ iff $za^{-1} \leq x \wedge y$ iff $z \leq (x \wedge y)a$. Therefore, $(x \wedge y)a = xa \wedge ya$.

(3) Using distributivity and complementation, we have $z \leq (xa)'$ iff $xa \wedge z = \perp$ iff $xa \wedge za^{-1}a = \perp$ iff $(x \wedge za^{-1})a = \perp$ iff $x \wedge za^{-1} = \perp$ iff $za^{-1} \leq x'$ iff $z \leq x'a$. Therefore, $(xa)' = x'a$, for all x. Now we have $a^{\cup} = (a\backslash 1')' = (a^{-1}1')' = a^{-1}1'' = a^{-1}$, where the third equality follows from the fact that a^{-1} is invertible. Similarly $a^{\sqcup} = a^{-1}$. □

Corollary 31. *An element a is invertible iff it satisfies $a(a\backslash 1) = 1 = (a\backslash 1)a$, or equivalently $a(1/a) = 1 = (1/a)a$.*

Examples of skew relation algebras. Given a set X and a bijection π on X we define the algebra $Re(X,\pi) = (\mathcal{P}(X^2), \cup, \cap, \circ, {}^{\cup}, {}^{\sqcup}, id_X)$, where $\circ$ is relational composition, $R^{\cup} = \{(y, \pi(x)) : (x,y) \in R\}$ and $R^{\sqcup} = \{(\pi^{-1}(y), x) : (x,y) \in R\}$. It is easy to check that $Re(X,\pi)$ is a skew relation algebra. For example, we can take $X = \mathbb{Z}$ and $\pi(n) = n+1$, or $X = \mathbb{Z}_k$ and $\pi(n) = n +_k 1$.

Given a relation algebra $\mathbf{A} = (A, \wedge, \vee, {}', \bot, \top, \cdot, 1, {}^{\smile})$ and an element $\pi \in A$ that satisfies the identities $\pi\pi^{\smile} = 1 = \pi^{\smile}\pi$ (an invertible element), we define the algebra $\mathbf{A}_\pi = (A, \wedge, \vee, {}', \bot, \top, \cdot, 1, {}^{\cup}, {}^{\sqcup})$, where $x^{\cup} = x^{\smile}\pi$ and $x^{\sqcup} = \pi x^{\smile}$.

A *π-skew relation algebra* is a skew relation algebra that for $\pi = 1^{\cup}$ satisfies

(π_1) $\pi(\pi\backslash 1) = 1 = (\pi\backslash 1)\pi$ (π is invertible),
(π_2) $(xy)^{\cup} = y^{\cup}\pi^{-1}x^{\cup}$, $(xy)^{\sqcup} = y^{\sqcup}\pi^{-1}x^{\sqcup}$, and
(π_3) $x^{\cup}\pi^{-1} = \pi^{-1}x^{\sqcup}$.

Theorem 32. *π-skew relation algebras coincide (up to term equivalence) with algebras of the form $\mathbf{A}_\pi$, where $\mathbf{A}$ is a relation algebra and $\pi \in A$ with $\pi\pi^{\smile} = 1 = \pi^{\smile}\pi$.*

Proof. We will use Corollary 28. Suppose $\mathbf{A}$ is a relation algebra with an invertible element π. Clearly, we have that the appropriate reducts of $\mathbf{A}_\pi$ are a Boolean algebra and a monoid, and multiplication distributes over joins.

(i) We have $x^{\cup\sqcup} = \pi(x^{\smile}\pi)^{\smile} = \pi\pi^{\smile}x^{\smile\smile} = x$. Likewise $x^{\sqcup\cup} = x$.

(ii) $(x \vee y)^{\cup} = (x \vee y)^{\smile}\pi = x^{\smile}\pi \vee y^{\smile}\pi = x^{\cup} \vee y^{\cup}$.

(iii) $(y(xy)'^{\cup})^{\sqcup} = \pi(y(xy)'^{\smile}\pi)^{\smile} = \pi\pi^{\smile}(xy)'y^{\smile} = (xy)'y^{\smile} \le x'$, where the last inequality follows from the last identity in the axiomatization of relation algebras. Thus $\mathbf{A}_\pi$ is (term equivalent to) a skew relation algebra. It is a π-skew relation algebra since $(xy)^{\cup} = (xy)^{\smile}\pi = y^{\smile}x^{\smile}\pi = y^{\smile}\pi\pi^{\smile}x^{\smile}\pi = y^{\cup}\pi^{\smile}x^{\cup}$ and $x^{\cup}\pi^{\smile} = x^{\smile}\pi\pi^{\smile} = x^{\smile} = \pi^{\smile}\pi x^{\smile} = \pi^{\smile}x^{\sqcup}$.

Conversely, assume that we are given a π-skew relation algebra $\mathbf{A}_s$, so $\pi = 1^{\cup}$ and $(\pi_1), (\pi_2), (\pi_3)$ hold. We define $\mathbf{A} = (A, \wedge, \vee, {}', \bot, \top, \cdot, 1, {}^{\smile})$ where $x^{\smile} = x^{\cup}\pi^{-1}$, which also equals $\pi^{-1}x^{\sqcup}$ by (π_3). Note that again appropriate reducts of $\mathbf{A}$ are a Boolean algebra and a monoid, and $\pi^{\smile} = \pi^{-1}(1^{\cup})^{\sqcup} = \pi^{-1}$. Furthermore $x^{\cup} = \pi^{-1}x^{\sqcup}\pi$ and $(xy)^{\cup} = \pi^{-1}(xy)^{\sqcup}\pi = \pi^{-1}y^{\sqcup}\pi^{-1}x^{\sqcup}\pi = y^{\cup}\pi^{-1}x^{\cup}$. Similarly $(xy)^{\sqcup} = y^{\sqcup}\pi^{-1}x^{\sqcup}$.

From $x^{\cup}\pi^{\smile} = \pi^{\smile}x^{\sqcup}$, we have $x^{\cup} = \pi^{\smile}x^{\sqcup}\pi$, hence with the help of (π_2), $x = x^{\cup\sqcup} = (\pi^{\smile}x^{\sqcup}\pi)^{\sqcup} = \pi^{\sqcup}\pi^{\smile}x^{\sqcup\sqcup}\pi^{\smile}\pi^{\smile\sqcup}$. By choosing $x = \pi$ and recalling that $\pi^{\sqcup} = 1^{\cup\sqcup} = 1$ and $\pi^{\sqcup\sqcup} = 1^{\sqcup} = \pi$, we obtain $\pi = \pi^{\smile}\pi\pi^{\smile}\pi^{\smile\sqcup}$, hence $\pi = \pi^{\smile}\pi^{\smile\sqcup}$. It remains to check that $\mathbf{A}$ satisfies the identities (i)-(v) of relation algebras.

(i) Using (π_2), we obtain $x^{\smile\smile} = \pi^{\smile}(x^{\smile})^{\sqcup} = \pi^{\smile}(x^{\cup}\pi^{\smile})^{\sqcup} = \pi^{\smile}\pi^{\smile\sqcup}\pi^{\smile}x^{\cup\sqcup} = \pi\pi^{\smile}x = x$. Moreover, (ii) holds since $(xy)^{\smile} = (xy)^{\cup}\pi^{\smile} = y^{\cup}\pi^{\smile}x^{\cup}\pi^{\smile} = y^{\smile}x^{\smile}$, and (iii) is inherited from $\mathbf{A}_s$. For (iv) we have $(x \vee y)^{\smile} = (x \vee y)^{\cup}\pi^{\smile} = (x^{\cup} \vee y^{\cup})\pi^{\smile} = x^{\cup}\pi^{\smile} \vee y^{\cup}\pi^{\smile} = x^{\smile} \vee y^{\smile}$.

Finally (v) $x^{\smile}(xy)' = x^{\cup}\pi^{\smile}(xy)' = x^{\cup}\pi^{\smile}(xy)'^{\sqcup\cup} = ((xy)'^{\sqcup}x)^{\cup} \leq y'$. Hence $\mathbf{A}$ is a relation algebra and $\mathbf{A}_\pi = \mathbf{A}_s$. $\square$

We note that there are skew relation algebras that are not of the form $\mathbf{A}_\pi$, as is illustrated by Example 11. Furthermore, since skew relation algebras have Boolean reducts, it follows from the main result of [13] that the equational theory of sRA is undecidable.

References

[1] G. Birkhoff, "Lattice Theory", 3rd edition, American Mathematical Society, (1967).

[2] K. Blount and C. Tsinakis, *The structure of residuated lattices*, Internat. J. Algebra Comput., 13(4) (2003), 437-461.

[3] J. A. Brzozowski, *A characterization of DeMorgan algebras*, International Journal of Algebra and Computation, 11(5) (2001), 525–527.

[4] L. H. Chin and A. Tarski, *Distributive and modular laws in the arithmetic of relation algebras*, University of California Publications in Mathematics, New Series 1 (1951), 341–384.

[5] N. Galatos and P. Jipsen, *Residuated frames with applications to decidability*, *Transactions of the AMS*, to appear.

[6] N. Galatos, P. Jipsen, T. Kowalski and H. Ono, "Residuated lattices: an algebraic glimpse at substructural logics", Studies in Logic and the Foundations of Mathematics, Vol. 151, Elsevier, 2007.

[7] P. Jipsen, *Representable sequential algebras and observation spaces*, J. on Relational Methods in Computer Science, Vol. 1 (2004), 235–250.

[8] P. Jipsen and R. D. Maddux, *Nonrepresentable sequential algebras*, Logic Journal of the IPGL, 5(4) (1997), 565–574.

[9] B. Jónsson, *A survey of Boolean algebras with operators*, in "Algebras and Orders", ed. I. Rosenberg, G. Sabidussi, Springer, 1993, 239–286.

[10] B. Jónsson and A. Tarski, *Boolean algebras with operators, Part II*, American Journal of Mathematics, 74 (1952), 127–162.

[11] B. Jónsson and C. Tsinakis, *Relation algebras as residuated Boolean algebras*, Algebra Universalis, 30(4) (1993), 469–478.

[12] M. Kozak, *Cyclic involutive distributive full Lambek calculus is decidable*, J. Logic Comput., 21(2) (2011), 231-252.

[13] Á. Kurucz, I. Németi, I. Sain and A. Simon, *Undecidable varieties of semilattice-ordered semigroups, of Boolean algebras with operators, and logics extending Lambek calculus*, Logic Journal of IGPL, 1(1) (1993), 91–98.

[14] J. Lambek, *Type grammars revisited*, Lecture Notes in Computer Science 1582, Springer-Verlag (1999), 1-27.

[15] R. Maddux, *Some varieties containing relation algebras*, Transactions of the AMS, 272(2) (1982), 501–526.

[16] H. Ono, *Structural rules and a logical hierarchy*, Mathematical Logic (P. P. Petkov, ed.), Plenum, New York, 1990, 95-104.

[17] A. Tarski and S. Givant, "A formalization of set theory without variables", Providence, R.I., American Mathematical Society, 1987.

[18] B. von Karger and C. A. R. Hoare, *Sequential calculus*, Information Processing Letters, 53(3) (1995), 123–130.

[19] M. Ward and R. P. Dilworth, *Residuated lattices*, Transactions of the Amer. Math. Soc., 45(3) (1939), 335–354.

[20] A. Wille, *A Gentzen system for involutive residuated lattices*, Algebra Universalis, 54(4) (2005), 449–463.

[21] D. N. Yetter, *Quantales and (noncommutative) linear logic*, J. Symbolic Logic, 55(1) (1990), 41–64.

Nikolaos Galatos

Department of Mathematics, University of Denver, 2360 S. Gaylord St., Denver, CO 80208, USA
e-mail: ngalatos@du.edu

Peter Jipsen

Chapman University, Department of Mathematics and Computer Science, One University Drive, Orange, CA 92866, USA
e-mail: jipsen@chapman.edu