**Appendix**

# A

# *Varieties of Lattices*

*Peter Jipsen and Henry Rose*[1]

In this appendix, we discuss some of the more recent results and give a general overview of what is currently known about lattice varieties. Of course, it is impossible to give a comprehensive account. Often we only cite recent or survey papers, which themselves have many more references. We would like to apologize in advance for any errors, omissions, or miscrediting of results.

For proofs of the results mentioned here, we refer the reader to the original papers. Details of many of the results from before 1992 can also be found in our monograph, P. Jipsen and H. Rose [A21].

## 1. The lattice $\Lambda$

Recall from Section V.2 that the lattice $\Lambda$ of all lattice varieties is a dually algebraic, distributive lattice that has the variety $\mathbf{L}$ of all lattices at the top, the variety $\mathbf{T}$ of all trivial lattices at the bottom, and the variety $\mathbf{D} = \mathbf{Var}(\mathfrak{C}_2)$ of all distributive lattices as the unique atom. To conclude that $\mathbf{L}$ is join-irreducible and has no coatoms, B. Jónsson [1967] argued as follows: Let $\mathbf{V}$, $\mathbf{W}$ be proper subvarieties of $\mathbf{L}$ and choose lattices $K \notin \mathbf{V}$, $L \notin \mathbf{W}$. Using Ph. M. Whitman's [1946] result that every lattice can be embedded in a partition lattice, one obtains a subdirectly irreducible lattice $S$ that extends $K \times L$. Since $S \notin \mathbf{Si}(\mathbf{V}) \cup \mathbf{Si}(\mathbf{W}) = \mathbf{Si}(\mathbf{V} \vee \mathbf{W})$, it follows that $\mathbf{V} \vee \mathbf{W}$ is a proper subvariety as well, hence $\mathbf{L}$ is join-irreducible. By R. A. Dean [1956], $\mathbf{L}$ is generated by its finite members, so we

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may assume that $K$ is finite. The distributivity of $\Lambda$ and Jónsson's Lemma imply that the interval from $\mathbf{V}$ to $\mathbf{V} \vee \mathbf{Var}(K)$ is finite, so every proper subvariety has at least one cover in $\Lambda$, and $\mathbf{L}$ has no co-atoms since $\mathbf{V} < \mathbf{V} \vee \mathbf{Var}(K) < \mathbf{L}$ (by join-irreducibility).

A substantial amount of research has been done on the structure near the bottom of $\Lambda$. One of the aims was to investigate this lattice by finding all varieties of a given finite height. By Jónsson's Lemma (Theorem V.1.9), a finite lattice generates a variety of finite height. The converse assertion, called the Finite Height Conjecture, was a longstanding open problem. Finally, J. B. Nation [A35] found a counterexample, which we discuss after presenting results about the known coverings near the bottom of $\Lambda$.

Specific lattices are labeled by German capital letter and the varieties they generate are referred to by the corresponding boldface letter (for example, $\mathbf{N}_5 = \mathbf{Var}(\mathfrak{N}_5)$). We say that a variety $\mathbf{V}$ is *strongly covered* by a collection $\mathcal{C}$ of varieties, if every variety that properly contains $\mathbf{V}$ also contains at least one member of $\mathcal{C}$.

The first few levels above the trivial variety are described in Sections V.2 and V.3 (see Figure V.2.1). B. Jónsson [1968] showed that for any variety $\mathbf{V}$ of modular lattices, $\mathfrak{M}_{3^2} \notin \mathbf{V}$ if and only if every subdirectly irreducible member of $\mathbf{V}$ has length $\leq 2$ (see also G. Grätzer [1966]). From this result, he deduced the following general form of Theorem V.3.6.

**Theorem 1** For $n \geq 3$, the covers of $\mathbf{M}_n$ are $\mathbf{M}_{n+1}$, $\mathbf{M}_n \vee \mathbf{M}_{3^2}$ and $\mathbf{M}_n \vee \mathbf{N}_5$. The variety $\mathbf{M}_\omega$ is strongly covered by $\mathbf{M}_\omega \vee \mathbf{M}_{3^2}$ and $\mathbf{M}_\omega \vee \mathbf{N}_5$.

Here $\mathbf{M}_\omega$ is the variety generated by $\mathfrak{M}_\omega$, the countable lattice of length 2.

Let $\mathfrak{M}_{3^n}$, $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$ be the lattices in Figures 1, V.3.5, V.3.4 and suppose that $M$ is a subdirectly irreducible modular lattice. The main technical result of D. X. Hong [1972] is that if $\mathfrak{M}_{3^n}, \mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3 \notin \mathbf{HS}\{M\}$, then $M$ has length at most $n$. This is a typical *exclusion result* which is very useful when it comes to finding covers of varieties.

Let $\mathbf{M}_w^l$ be the variety generated by all modular lattices of length at most $l$ and of width at most $w$ ($1 \leq l$, $w \leq \infty$). For example, $\mathbf{M}_\infty^2 = \mathbf{M}_\omega$ and $\mathbf{M}_\infty^3$ is the variety generated by all subspace lattices of projective planes (see the proof of Theorem IV.5.23). With this notation, Hong's result implies that for any variety $\mathbf{V}$ of modular lattices, $\mathfrak{M}_{3^3}, \mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3 \notin \mathbf{V}$ if and only if $\mathbf{V} \subseteq \mathbf{M}_\infty^3$. It follows immediately that $\mathbf{M}_\infty^3$ has exactly five covers in $\Lambda$, given by $\mathbf{M}_\infty^3 \vee \mathbf{V}$ where $\mathbf{V} \in \{\mathbf{M}_{3^3}, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{N}_5\}$.

It is easy to check that the varieties $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{M}_{3^3}$, $\mathbf{F}_2$ (generated by the corresponding lattices in Figures 1, V.3.5, V.3.4, V.3.7, IV.3.4b, respectively) each cover the variety $\mathbf{M}_{3^2}$. Using the above exclusion result and some added detail, D. X. Hong [A18] proves that they are the only join-irreducible covers. More generally, he shows the following.

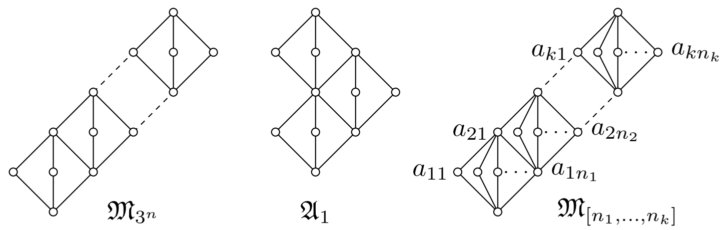


Figure 1:

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**Theorem 2**

(i) For $n \geq 2$, the covers of $\mathbf{M}_{3^n}$ are $\mathbf{M}_{3^{n+1}}$ and $\mathbf{M}_{3^n} \vee \mathbf{V}$, where $\mathbf{V} \in \{\mathbf{M}_4, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{F}_2, \mathbf{N}_5\}$.

(ii) Let $\mathbf{V}$ be a variety generated by a finite collection of finite modular lattices of length $\leq 3$ and let $\mathbf{W}$ be a variety generated by a finite collection of lattices of the form $\mathfrak{M}_{[n_1,\ldots,n_k]}$ (see Figure 1). Then each of the following varieties is strongly covered by finitely many varieties that can be effectively found:

$$\mathbf{V} \vee \mathbf{W}, \quad \mathbf{M}_\infty^2 \vee \mathbf{V} \vee \mathbf{W}, \quad \mathbf{M}_\infty^3 \vee \mathbf{V} \vee \mathbf{W}.$$

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This result gives a fairly good description of the bottom of Λ on the modular side.

**Problem 1.** *Find the covers of* $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$.

**Problem 2.** *Does the Finite Height Conjecture hold for modular varieties? Does it hold for the variety of modular 2-distributive lattices?*

**Problem 3.** *Does the variety of modular lattices or the variety of arguesian lattices have any dual covers?*

For nonmodular varieties, B. Jónsson and I. Rival [A25] proved that R. N. McKenzie's [1972] list of 15 covers of $\mathbf{N}_5$ is complete. The lattices which generate these covers are called $\mathfrak{L}_1, \ldots, \mathfrak{L}_{15}$ and are shown in Figures V.2.3–10 in the order

| | | |
|---|---|---|
| $\mathfrak{L}_5$ ($\mathfrak{L}_4$ is dual), | $\mathfrak{L}_3$, | $\mathfrak{L}_7$ ($\mathfrak{L}_8$ is dual), |
| $\mathfrak{L}_9$ ($\mathfrak{L}_{10}$ is dual), | $\mathfrak{L}_{13}$ ($\mathfrak{L}_{14}$ is dual), | $\mathfrak{L}_{15}$, |
| $\mathfrak{L}_{11}$ ($\mathfrak{L}_{12}$ is dual), | $\mathfrak{L}_1$ ($\mathfrak{L}_2$ is dual), | $\mathfrak{L}_6$. |

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**Theorem 3** The covers of $\mathbf{N}_5$ are $\mathbf{M}_3 \vee \mathbf{N}_5$, $\mathbf{L}_1$, ..., $\mathbf{L}_{15}$.

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**Theorem 4** The covers of $\mathbf{N}_5$ are $\mathbf{M}_3 \vee \mathbf{N}$, $\mathbf{L}_1$, ..., $\mathbf{L}_{15}$.

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The above result makes use of the semidistributive implications ($\mathrm{SD}_\vee$) and ($\mathrm{SD}_\wedge$) (see Section VI.1). A variety of lattices is said to be *semidistributive*, if every member satisfies both laws. The *standard meet-sequence terms* $y_n$, $z_n$, for variables $x$, $y$, $z$ are defined by

$$\begin{aligned} y_0 &= y, \\ z_0 &= z, \\ y_{n+1} &= y \wedge (x \vee z_n), \\ z_{n+1} &= z \wedge (x \vee y_n). \end{aligned}$$

The key exclusion result by B. Jónsson and I. Rival [A25] is the following.

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**Theorem 5** For any variety $\mathbf{V}$, the following are equivalent.

(i) $\mathbf{V}$ is semidistributive.

(ii) $\mathfrak{M}_3, \mathfrak{L}_1, \mathfrak{L}_2, \mathfrak{L}_3, \mathfrak{L}_4, \mathfrak{L}_5 \notin \mathbf{V}$.

(iii) For some $n$, the equation

($\mathrm{SD}_\vee^n$) $$x \vee (y \wedge z) = x \vee y_n$$

and its dual ($\mathrm{SD}_\wedge^n$) hold in $\mathbf{V}$.

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It follows from this result that semidistributivity is not an equational property.

The above equations define an increasing sequence of semidistributive varieties $\mathbf{SD}_n = \mathrm{Mod}((\mathrm{SD}_\vee^n), (\mathrm{SD}_\wedge^n))$. Obviously, $\mathbf{SD}_0 = \mathbf{T}$ and $\mathbf{SD}_1 = \mathbf{D}$. Lattices and subvarieties of $\mathbf{SD}_2$ are called *near distributive*. A useful characterization is given by the next exclusion result.

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**Theorem 6** A lattice variety $\mathbf{V}$ is neardistributive if and only if it is semidistributive and $\mathfrak{L}_{11}, \mathfrak{L}_{12} \notin \mathbf{V}$. (J. G. Lee [A28].)

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A lattice is said to be *almost distributive* if it is near distributive and satisfies the inequality

($\mathrm{AD}_\vee$) $$u \wedge (w \vee (v \wedge ((x \vee y) \wedge (x \vee z)))) \leq v \vee (u \wedge w),$$

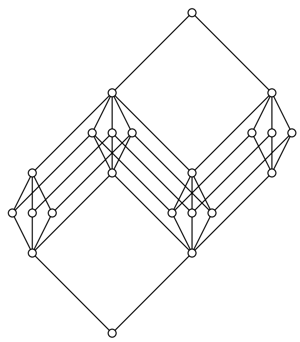

Figure 2: $(\mathfrak{C}_2 \times \mathfrak{C}_3) \star_C \mathfrak{M}_3$

where $w = x \vee (y \wedge (x \vee z))$, and it's dual ($\mathrm{AD}_\wedge$). The variety **AD** of all almost distributive lattices is studied by H. Rose [A43] and J. G. Lee [A28].

The main structural results about subdirectly irreducible almost distributive lattices require (a special case of) A. Day's doubling construction. The version described here is a generalization due to R. Freese, G. McNulty, and J. B. Nation [A15] which will also be used later in the description of inherently nonfinitely based lattices and Nation's counterexample to the Finite Height Conjecture. Given a lattice $L$, a convex subset $C$ of $L$ and a $\{0, 1\}$-lattice $K$, one defines a lattice $L \star_C K$, called the *inflation of $L$ at $C$ by $K$*, as follows. The underlying set is $(L - C) \cup (C \times K)$, and for elements $x$, $y$ in this set, put $x \leq y$ if

(i) $x, y \in L - C$ and $x \leq y$ holds in $L$,

(ii) $x, y \in C \times K$ and $x \leq y$ holds in $C \times K$,

(iii) $x \in L - C$, $y = \langle c, k\rangle \in C \times K$, and $x \leq c$ holds in $L$, or

(iv) $x = \langle c, k\rangle \in C \times K$, $y \in L - C$, and $c \leq y$ holds in $L$.

Day's original doubling construction is obtained when $K = \mathfrak{C}_2$, in which case $L \star_C \mathfrak{C}_2$ is denoted by $L[C]$, and when $C = \{c\}$ this is further simplified to $L[c]$. For example, if we take $L = \mathfrak{C}_2 \times \mathfrak{C}_3$ and $C = L - \{0, 1\}$ then $L \star_C \mathfrak{M}_3$ is the lattice in Figure 2, and $(\mathfrak{C}_3 \times \mathfrak{C}_3)[d]$ gives the lattice $\mathfrak{L}_{15}$ (Figure V.2.8). The doubling construction for single elements was actually used in the context of transferable lattices before Day's construction (see Appendix **??**.1.3).

For a variety **V**, let $\Lambda_{\mathbf{V}}$ be the lattice of subvarieties of **V**. If **V** is a lattice variety, then $\Lambda_{\mathbf{V}}$ is, of course, a principal ideal of $\Lambda$.

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**Theorem 7**

(i) A subdirectly irreducible lattice $L$ is almost distributive if and only if $L \cong D[d]$, for some distributive lattice $D$ and $d \in D$.

(ii) A lattice variety $\mathbf{V}$ is almost distributive if and only if it is semidistributive and $\mathfrak{L}_6, \ldots, \mathfrak{L}_{12} \notin \mathbf{V}$.

(iii) $\mathbf{AD}$ is locally finite (that is, every finitely generated member is finite), hence the Finite Height Conjecture holds for almost distributive varieties and $\mathbf{AD}$ is generated by its finite members.

(iv) The cardinality of $\Lambda_{\mathbf{AD}}$ is $2^{\aleph_0}$.

(v) There exists an infinite descending chain in $\Lambda_{\mathbf{AD}}$.

(vi) There exists an almost distributive variety with infinitely many covers in $\Lambda_{\mathbf{AD}}$ and one with infinitely many dual covers.

(H. Rose [A43], J. G. Lee [A28].)

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Judging from the above results and additional details by Rose and Lee, one might say that the structure of $\Lambda_{\mathbf{AD}}$ is fairly well understood.

**Problem 4.** *Is there a variety with uncountably many covers (or dual covers) in* $\Lambda$ *or* $\Lambda_{\mathbf{AD}}$*?*

**Problem 5.** *Does* $\mathbf{AD}$ *have any dual covers?*

We list below additional results about covers in $\Lambda$. In each case these results are established by long technical computations and the original papers contain further results that are of interest in their own right.

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**Theorem 8** For $i = 6, 7, 8, 9, 10, 13, 14, 15$ and $n \geq 0$, the variety $\mathbf{L}_i^{n+1}$ is the only join-irreducible cover of $\mathbf{L}_i^n$ (where $\mathbf{L}_i^0 = \mathbf{L}_i$, see Figure 3). (H. Rose [A43].)

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**Theorem 9** $\mathbf{L}_{12}$ has exactly two join-irreducible covers $\mathbf{L}_{12}^1$ and $\mathbf{G}^1$. For $n \geq 1$, $\mathbf{L}_{12}^{n+1}$ is the only join-irreducible cover of $\mathbf{L}_{12}^n$, and $\mathbf{G}^{n+1}$ is the only join-irreducible cover of $\mathbf{G}^n$. Above $\mathbf{L}_{11}$, the dual results hold (see Figure 3). (J. B. Nation [A32].)

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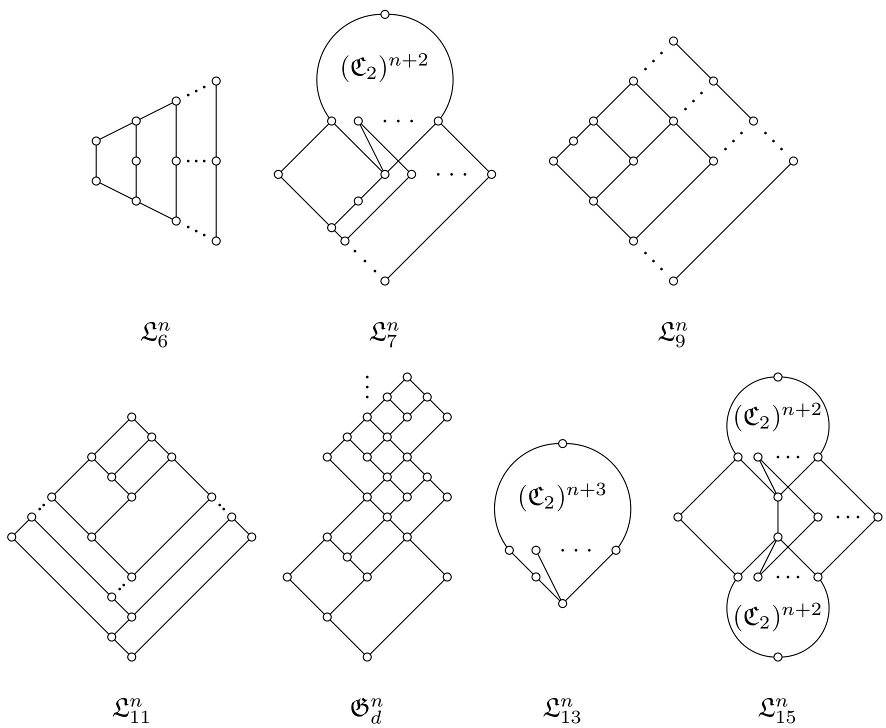


Figure 3: Sequences of lattices generating join-irreducible varieties. $\mathfrak{L}_8^n$, $\mathfrak{L}_{10}^n$, $\mathfrak{L}_{12}^n$, $\mathfrak{L}_{14}^n$ are dual to $\mathfrak{L}_7^n$, $\mathfrak{L}_9^n$, $\mathfrak{L}_{11}^n$, $\mathfrak{L}_{13}^n$ respectively. (Here $n$ is a superscript label, whereas $(\mathfrak{C}_2)^{n+2}$ is a power of $\mathfrak{C}_2$.)

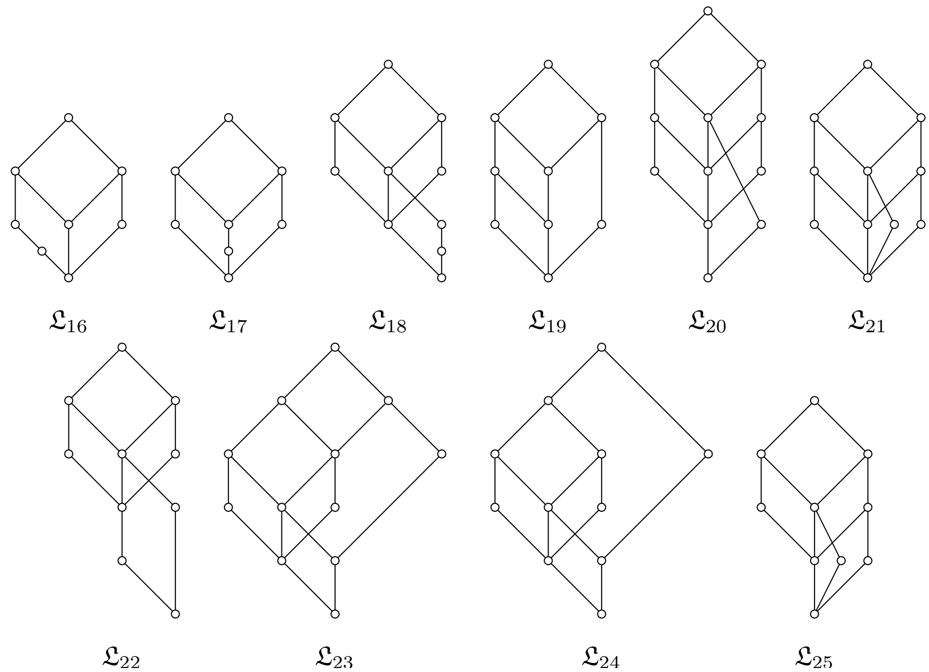


Figure 4: Lattices that generate covers of $\mathbf{L}_1$

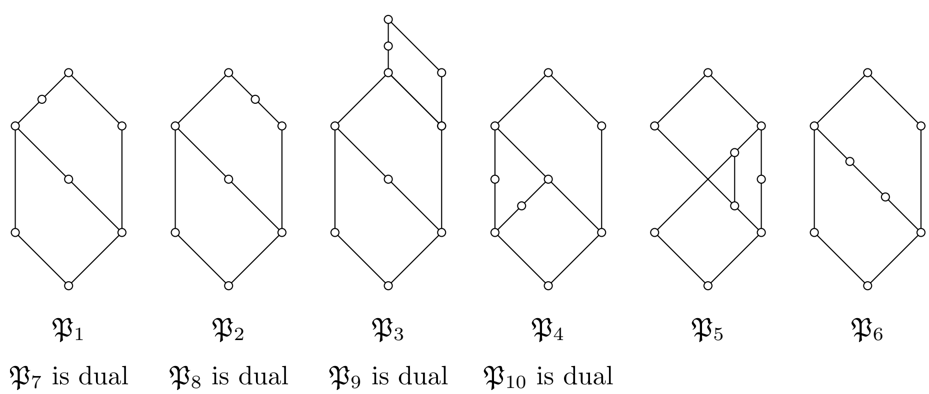


Figure 5: Lattices that generate covers of $\mathbf{L}_3$

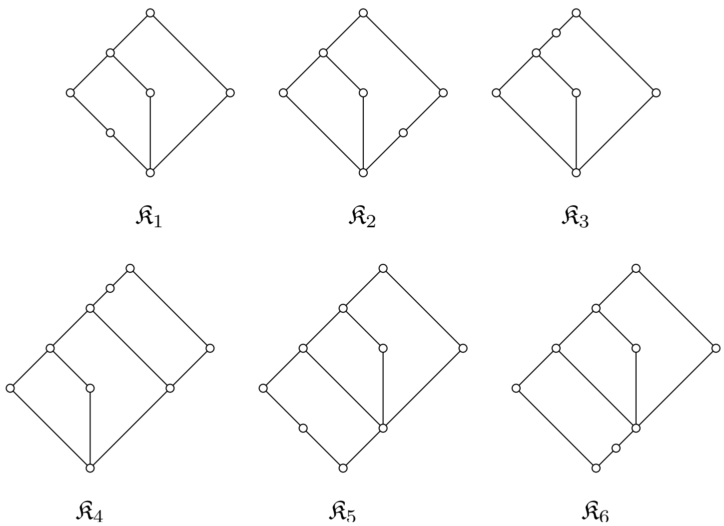


Figure 6: Lattices that generate covers of $\mathbf{L}_4$

**Theorem 10** The join-irreducible covers of $\mathbf{L}_1$ are $\mathbf{L}_{16}, \ldots, \mathbf{L}_{25}$. The covers of $\mathbf{L}_2$ are dual (see Figure 4). (J. B. Nation [A33].)

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An approach to finding covers in $\Lambda$ has been developed by J. B. Nation [A34] (see also A. Day and J. B. Nation [A9]).

C. Y. Wong [A48] investigates weakened forms of distributivity similar to semidistributivity to find the covers of $\mathbf{L}_3$, $\mathbf{L}_4$ and $\mathbf{L}_5$. A lattice is said to be *weakly distributive* if it satisfies the following implications:

$$(\mathrm{WD}_\vee) \qquad x \wedge y = x \wedge z \quad \text{implies that} \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z),$$

$$(\mathrm{WD}_\wedge) \qquad x \vee y = x \vee z \quad \text{implies that} \quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

A variety of lattices is said to be *weakly distributive*, if this is true for every member. This property can also be characterized by an exclusion result.

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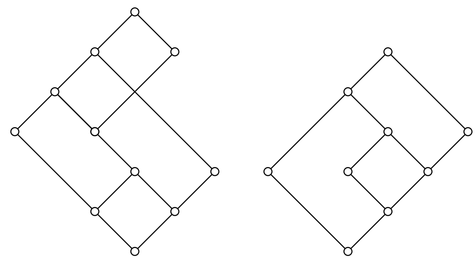
**Theorem 11** For any variety $\mathbf{V}$, the following are equivalent.

(i) $\mathbf{V}$ is weakly distributive.

(ii) $\mathfrak{M}_3, \mathfrak{L}_1, \mathfrak{L}_2, \mathfrak{L}_4, \mathfrak{L}_5, \mathfrak{L}_{11}, \mathfrak{L}_{12}, \mathfrak{L}_{13}, \mathfrak{L}_{14}, \mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4, \mathfrak{P}_4, \mathfrak{P}_5, \mathfrak{P}_{10} \notin \mathbf{V}$ (see remark before Theorem 3 and Figures 7, 5).

(iii) For some $n$, the equation $x \wedge (y_n \vee z_n) \leq (x \wedge y) \vee (x \wedge z)$ and its dual hold in $\mathbf{V}$ ($y_n$, $z_n$ are the standard meet sequence terms defined on page 4).



$\mathfrak{T}_1$ $\mathfrak{T}_3$

$\mathfrak{T}_2$ is dual $\mathfrak{T}_4$ is dual

Figure 7:

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Note that $\mathbf{L}_4$ is not weakly distributive, but does satisfy (WD$_\wedge$). Wong shows that (WD$_\wedge$) cannot be characterized by an exclusion result, i.e., there is no finite list of finite subdirectly irreducible lattices such that a variety satisfies (WD$_\wedge$) if and only if it contains none of these lattices. He then goes on to prove that (WD$_\wedge$) is *weakly finitely definable with respect to* $\mathfrak{L}_4$ which means that there is a finite list of finite subdirectly irreducible lattices not in $\mathbf{L}_4$ such that if (WD$_\wedge$) fails in a variety then it contains one of these lattices. Using this result together with the approach from J. B. Nation [A34] and (lots of) additional details, he succeeds in proving the following.

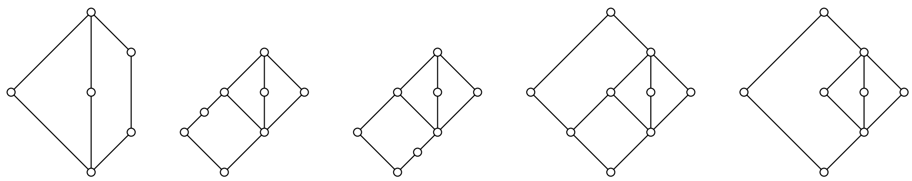
**Theorem 12** The join-irreducible covers of $\mathbf{L}_3$ are $\mathbf{P}_1, \ldots, \mathbf{P}_{10}$ (see Figure 5). The join-irreducible covers of $\mathbf{L}_4$ are $\mathbf{K}_1, \ldots, \mathbf{K}_6$ (see Figure 6). The covers of $\mathbf{L}_5$ are dual. (C. Y. Wong [A48].)

For the variety $\mathbf{M}_3 \vee \mathbf{N}_5$, only the finitely generated covers are known at this point.

**Theorem 13** The finitely generated join-irreducible covers of $\mathbf{M}_3 \vee \mathbf{N}_5$ are $\mathbf{V}_1, \ldots, \mathbf{V}_8$ (see Figure 8). (W. Ruckelshausen [A44].)

**Problem 6.** *Does* $\mathbf{M}_3 \vee \mathbf{N}_5$ *have any nonfinitely generated covers?*

All the preceding results support the Finite Height Conjecture in that every finitely generated lattice variety of height at most 4 has only finitely many finitely generated covers (see Figure 9). Recently, however, J. B. Nation [A35] showed that the conjecture fails for lattices in general. Consider the lattice $\mathfrak{J}$ in Figure 10,



$\mathfrak{V}_1$ $\mathfrak{V}_2$ $\mathfrak{V}_4$ $\mathfrak{V}_6$ $\mathfrak{V}_8$

$\mathfrak{V}_3$ is dual $\mathfrak{V}_5$ is dual $\mathfrak{V}_7$ is dual

Figure 8: Lattices that generate finitely generated covers of $\mathbf{M}_3 \vee \mathbf{N}_5$

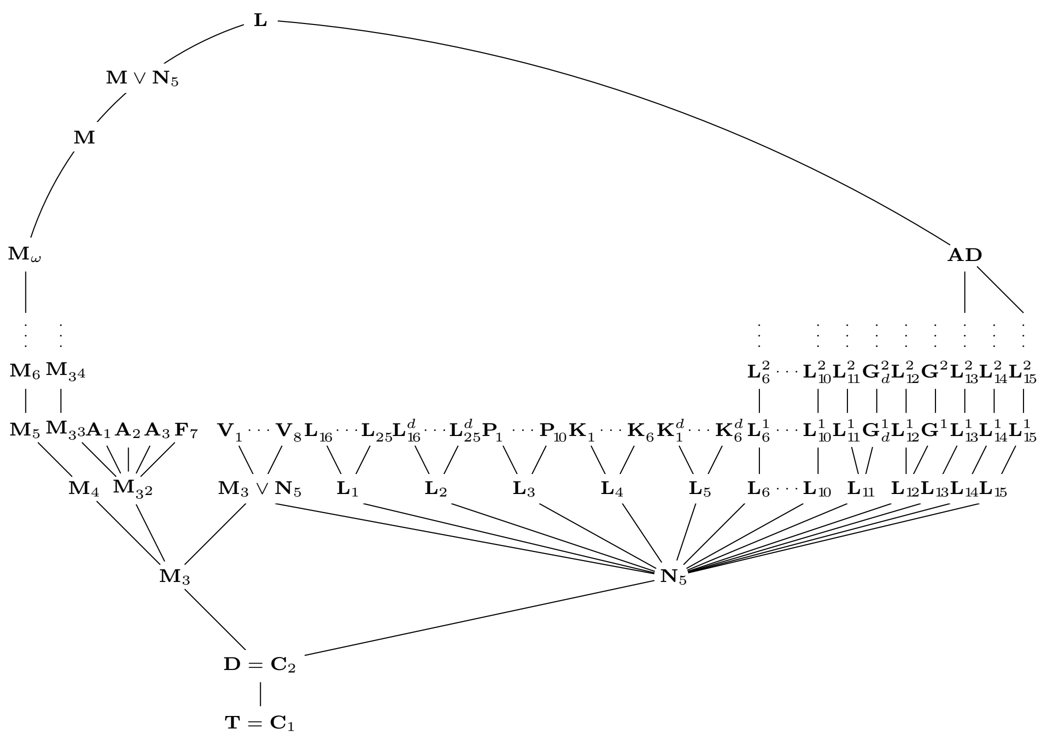



Figure 9: Known join-irreducible covers near the bottom of Λ

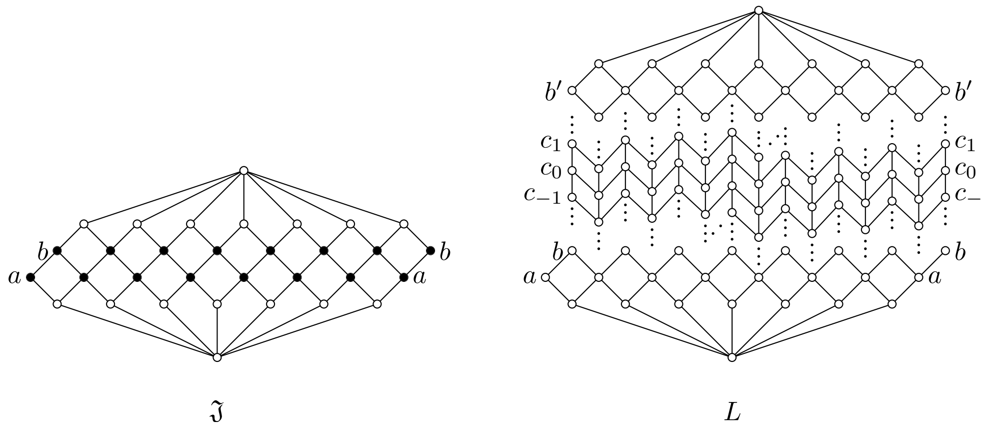



Figure 10: Nation's counterexample to the Finite Height Conjecture

with the convex subset $C$ given by the elements of height 2 and 3. Note that the elements of equal height on the left and the right of $C$ should be identified, so $C$ has actually 14 elements.

Let $\mathfrak{B}$ be a chain isomorphic to the integers with top and bottom elements added. The subdirectly irreducible lattice $L$ is obtained by replacing each element in $C$ with a copy of $\mathfrak{B}$, and defining the ordering as indicated in the figure.

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**Theorem 14** The infinite subdirectly irreducible lattice $L$ in Figure 10 generates a variety of finite height.

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This beautiful counterexample shows that even varieties of finite height are highly nontrivial, and has already inspired new results about inherently nonfinitely based varieties (see Section 3). Note that the lattice has a cyclic automorphism of order 7 and is best visualized by rolling the page up into a cylinder. In essence, $L$ is a slightly modified version of $\mathfrak{J} \star_C \mathfrak{B}$, so that the prime quotients in the 14 chains in $L$ transpose in a spiral up and down this lattice and are all collapsed by the smallest nontrivial congruence $\mu$ on $L$. The lattice $L/\mu$ is isomorphic to $\mathfrak{J} \star_C \mathfrak{C}_3$, which is a subdirect product of two copies of the finite subdirectly irreducible lattice $F = \mathfrak{J} \star_C \mathfrak{C}_2$. The proof proceeds by showing that every finitely generated lattice in $\mathbf{SiVar}(L)$ is in $\mathbf{H}\,\mathbf{S}(L)$, and that $\mathbf{H}\,\mathbf{S}(L) - \mathbf{I}(L) \subseteq \mathbf{Var}(F)$, whence $\mathbf{Var}(L)$ is a cover of the variety $\mathbf{Var}(F)$.

In an unpublished note, Nation points out that lattices other than $\mathfrak{J}$ can serve as the basis for the construction. For example the Boolean lattice $(\mathfrak{C}_2)^5$ gives a narrower (but less easily visualizable) example with only 10 chains.

J. B. Nation [A35] also shows that there is a variety of finite height that has countably infinitely many covers.

**Problem 7.** *Is every variety of finite height finitely based?*

**Problem 8.** *Is every variety of finite height generated by a lattice of finite width?*

**Problem 9.** *Is there an algorithm to find the covers of a finitely generated variety?*

## 2. Generating sets of varieties

It is well known that the variety of all lattices is generated by its finite members (R. A. Dean [1956]). Using the doubling construction and R. N. McKenzie's [1972] characterization of splitting lattices as finite subdirectly irreducible bounded lattices, A. Day [1977] was able to prove the following sharper version of Dean's result.

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**Theorem 15** The variety $\mathbf{L}$ of all lattices is generated by the class of all splitting lattices.

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The significance of this result is enhanced by the fact that it implies every finitely generated free lattice is weakly atomic (R. N. McKenzie [1972] and A. Kostinsky [1972] proved this condition equivalent to Day's theorem).

More recently, R. N. McKenzie [A30] showed that $\mathbf{L}$ is also generated by the collection of all finite minimal simple lattices. (A simple lattice $L$ is *minimal* if $L \ncong \mathfrak{C}_2$ and no simple lattice other than $\mathfrak{C}_2$ generates a proper subvariety of $\mathbf{Var}(L)$.)

For the variety of modular lattices, the situation is quite different.

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**Theorem 16**

(i) The variety $\mathbf{M}$ of all modular lattices is not generated by its finite members. (R. Freese [A10].)

(ii) Neither $\mathbf{M}$ nor the variety $\mathbf{A}$ of all arguesian lattices is generated by its members of finite length. (Ch. Herrmann [A17].)

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Using P. Pudlák and J. Tůma's [A42] result that every finite lattice can be embedded into a finite partition lattice, P. Bruyns and H. Rose [A4] show that every lattice is embeddable into an ultraproduct of finite partition lattices, hence $\mathbf{L} = \mathbf{S}\,\mathbf{P}_U(\{\mathrm{Part}\,n \mid n \in \omega\})$. Furthermore, since any lattice variety $\mathbf{V}$ satisfies the Embedding Property (see Section V.4), there exists a lattice $L \in \mathbf{V}$ such that every member of $\mathbf{V}$ is embeddable into an ultrapower of $\mathbf{L}$, that is, $\mathbf{V} = \mathbf{S}\,\mathbf{P}_U(L)$.

Such lattices $L$ are referred to as *ultra-universal* (see also C. Naturman and H. Rose [A36]).

R. N. McKenzie [1972] showed that splitting lattices in $\Lambda$ are finite. However, splitting lattices can be defined in any lattice of varieties.

**Problem 10.** *Is every splitting lattice in* $\Lambda_{\mathbf{M}}$ *finite?*

**Problem 11.** *Is* $\mathbf{M}$ *generated by all the splitting lattices in* $\Lambda_{\mathbf{M}}$*?*

If $L$ is a splitting lattice in $\Lambda$, then the largest variety that does not contain $L$ is called the *conjugate variety* of $L$.

**Problem 12.** *Is there a nontrivial conjugate variety in* $\Lambda$ *that is generated by its finite members?*

**Problem 13.** *Is there a conjugate variety* $\mathbf{V}$ *with infinite subdirectly irreducible members that are projective in* $\mathbf{V}$*?*

Note that if a variety $\mathbf{V}$ is generated by its finite members then every subdirectly irreducible projective member is finite. Thus a positive answer to the previous problem implies that $\mathbf{V}$ is not generated by its finite members.

## 3. Equational Bases

Recall that an algebra is said to be *finitely based* if the variety which it generates is determined by finitely many equations. Nonfinitely based lattices were constructed by K. A. Baker [1969a], [A1], R. Freese [1977], Ch. Herrmann [A17], R. N. McKenzie [1970] and R. Wille [1972]. One such lattice, due to McKenzie, is shown in Figure 11.

An algebra $A$ is said to be *inherently nonfinitely based* if $\mathbf{Var}(A)$ is locally finite, and any locally finite variety to which $A$ belongs is not finitely based. This concept was introduced independently by V. L. Murskiĭ [A31] and P. Perkins [A40]. Inspired by J. B. Nation's [A35] counterexample to the Finite Height Conjecture, R. Freese, G. McNulty, and J. B. Nation [A15] construct inherently nonfinitely based lattices. Here we only state a special case of their main result (see page 5 for the definition of $L \star_C K$).

---

**Theorem 17** Let $\mathfrak{L}_f$ be the lattice in Figure 11 and define $C = \mathfrak{L}_f - \{0,1\}$. Let $K$ be a $\{0,1\}$-lattice which belongs to a locally finite variety, and assume that $K$ has an automorphism with an infinite orbit. Then $\mathfrak{L}_f \star_C K$ is inherently nonfinitely based.

---

The two least complicated lattices $K$ with the required automorphism are $\mathfrak{M}_\omega$ and $\mathfrak{B}$ (a chain isomorphic to the integers with top and bottom elements added). The resulting lattices $\mathfrak{L}_f \star_C K$ are given in Figure 11. In the same paper, it is also shown that the lattice $\mathfrak{L}_f$ is not inherently nonfinitely based.

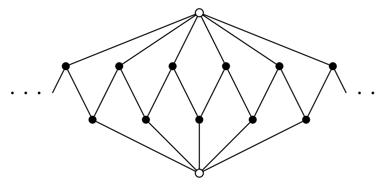

Figure 11: The lattice $\mathfrak{L}_f$

**Problem 14.** *Are there any modular lattices that are inherently nonfinitely based?*

Analogously to the varieties $\mathbf{M}_w^l$ one defines $\mathbf{V}_w^l$ to be the variety generated by all lattices of height $l$ and width $w$. We allow $l = \infty$ or $w = \infty$ in which case the respective parameter is not restricted. For $l, w < \infty$, the varieties $\mathbf{M}_w^l$ and $\mathbf{V}_w^l$ are finitely generated and hence finitely based.

Note that if a variety $\mathbf{V}$ is strongly covered by a finite set of varieties, then it is finitely based. Results about whether the varieties $\mathbf{M}_w^l$ and $\mathbf{V}_w^l$ are finitely based for $l = \infty$ or $w = \infty$, are as follows:

---

**Theorem 18**

(i) $\mathbf{M}_\infty^2$ $(= \mathbf{V}_\infty^2)$ and $\mathbf{M}_\infty^3$ are finitely based (see, Theorems 1 and 2), $\mathbf{M}_\infty^n$ is finitely based for all $n$ (K. Baker, see Ch. Herrmann [1973]).

(ii) $\mathbf{V}_\infty^3$ is finitely based (Ch. Herrmann [1973]), $\mathbf{V}_\infty^n$ is not finitely based for $n \geq 4$ (K. Baker [A1]).

(iii) $\mathbf{M}_1^\infty = \mathbf{M}_2^\infty = \mathbf{D}$, $\mathbf{M}_3^\infty = \mathbf{M}_3$ (see Theorem 1, since both its modular covers are generated by lattices of width 4), $\mathbf{M}_4^\infty$ is strongly covered by 10 varieties (each generated by $\mathbf{M}_4^\infty$ together with one of the lattices in Figures V.3.3, ..., V.3.10, IV.3.4b or $\mathfrak{N}_5$) and hence finitely based (R. Freese [1977]), $\mathbf{M}_n^\infty$ is not finitely based for $n \geq 5$ (K. A. Baker [A1]).

(iv) $\mathbf{V}_2^\infty = \mathbf{N}_5$ (O. T. Nelson [1968]), hence finitely based, $\mathbf{V}_n^\infty$ is not finitely based for $n \geq 3$ ($n \geq 4$ due to K. A. Baker [A1], $n = 3$ due to Y.-C. Hsueh [A19]).

---

B. Jónsson [1974] showed that the join of two finitely based lattice varieties need not be finitely based, and K. A. Baker [A1] did the same for two finitely based modular varieties. In view of these result, it is natural to look for sufficient conditions under which the join of two finitely based varieties remains finitely based.

---

**Theorem 19** Suppose that **V** and **W** are finitely based lattice varieties. If one of the following conditions holds, then $\mathbf{V} \vee \mathbf{W}$ is finitely based.

(i) **V** is modular and **W** is generated by a finite lattice that excludes $\mathfrak{M}_3$.

(ii) **V** and **W** are locally finite and the projective radius of $\mathbf{V} \cap \mathbf{W}$ is finite.

(iii) **V** and **W** are modular and **W** is generated by a lattice of finite length.

(iv) **V** is modular and **W** is generated by a finite lattice with finite projective radius.

(v) $\mathbf{V} \cap \mathbf{W} = \mathbf{D}$, the variety of all distributive lattices.

---

(i) and (ii) are due to J. G. Lee [A29], (iii) is due to Jónsson and the remaining statements are due to Y. Y. Kang [A26].

Note that it follows from part (i) above that $\mathbf{M} \vee \mathbf{N}_5$ is finitely based. B. Jónsson [A22] constructed an explicit basis for this variety of eight identities. The following problem was inspired by this result.

**Problem 15.** *Is the unique cover of a conjugate variety in* $\Lambda$ *always finitely based? (A. Day.)*

## 4. Amalgamation and absolute retracts

G. Grätzer, B. Jónsson, and H. Lakser [1973] showed that, besides the varieties **T** and **D**, no modular variety has the amalgamation property (see Section V.4 for a discussion). A. Day and J. Ježek [A8] finally extended this result to all lattice varieties.

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**Theorem 20** **T**, **D**, and **L** are the only lattice varieties with the amalgamation property.

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For other varieties of algebras the amalgamation property also turned out to be rarely satisfied. A comprehensive survey about amalgamation for various types of algebras can be found in E. W. Kiss, L. Márki, P. Pröhle, and W. Tholen [A27]. These results indicate that the concept of amalgamation does not mesh well with that of a variety. However the amalgamation class **Amal**(**V**) of a variety **V**, introduced by G. Grätzer and H. Lakser [1971], has proved to be very fruitful. M. Yasuhara [1974] showed that for any variety **V** of algebras, each member of **V** has an extension in **Amal**(**V**), hence **Amal**(**V**) is a proper class (Theorem V.4.10). At present the main directions of study are to characterize the amalgamation class of a given variety and to decide whether it is (strictly) elementary, i.e., if it can be defined by a (finite) collection of first order sentences. Although we do not know anything about a single member of **Amal**(**M**), significant progress has been made with residually small lattice varieties. This started with a characterization of the amalgamation class of finitely generated lattice varieties by B. Jónsson [A24], and was generalized by P. Jipsen and H. Rose [A20] (see also P. Ouwehand and H. Rose [A38]).

Many of the results below are valid for various congruence distributive varieties (not only lattice varieties), so we will state the more general results where applicable. A *retraction* of an embedding $f\colon A \to B$ is a homomorphism $g\colon B \to A$ such that $g \circ f = \mathrm{id}_A$. An algebra $A$ in a class $\mathbf{K}$ is said to be an *absolute retract in* $\mathbf{K}$ if for every embedding $f\colon A \hookrightarrow B \in \mathbf{K}$, there is a retraction. The class of all absolute retracts of $\mathbf{K}$ is denoted by $\mathbf{Ar}(\mathbf{K})$. The concept of absolute retract is of interest here since C. Bergman [A2] observed that for any variety $\mathbf{V}$ we have $\mathbf{Ar}(\mathbf{V}) \subseteq \mathbf{Amal}(\mathbf{V})$.

A variety is said to be *residually small*, if there is an upper bound on the cardinality of its subdirectly irreducible members. W. Taylor [A45] proved that a variety $\mathbf{V}$ is residually small if and only if $\mathbf{V} = \mathbf{S}\,\mathbf{Ar}(\mathbf{V})$.

---

**Theorem 21** Let $\mathbf{V}$ be a residually small congruence distributive variety in which every member has a one-element subalgebra. Then $A \in \mathbf{Amal}(\mathbf{V})$ if and only if for any embedding $f\colon A \hookrightarrow B \in \mathbf{V}$ and any homomorphism $h\colon A \to M \in \mathbf{Si}(\mathbf{Ar}(\mathbf{V}))$ there exists a homomorphism $g\colon B \to M$ such that $h = fg$.

---

The reverse implication is due to C. Bergman [A2] and the forward direction is from P. Jipsen and H. Rose [A20]. A useful corollary is that for finite algebras the condition in the preceding theorem can be checked.

**Corollary 22.** *Let* $\mathbf{V}$ *be a finitely generated congruence distributive variety in which every member has a one-element subalgebra. For finite algebras in* $\mathbf{V}$*, membership in* $\mathbf{Amal}(\mathbf{V})$ *is decidable. (B. Jónsson [A24], P. Jipsen and H. Rose [A20].)*

Since the amalgamation class of a variety is in general a proper subclass, it is interesting to ask whether it is an elementary class. Even for a finitely generated lattice variety this is a nontrivial problem.

---

**Theorem 23** The amalgamation class of any finitely generated nondistributive modular lattice variety is not elementary. (C. Bergman [A3].)

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**Problem 16.** *For which finitely generated varieties is the amalgamation class elementary?*

Recent progress on this problem has been made by P. Ouwehand and H. Rose [A39].

---

**Theorem 24** Let $\mathbf{V}$ be a finitely generated variety of lattices. Suppose that there is a lattice $L \in \mathbf{Amal}(\mathbf{V})$ with either a bottom or a top element, which does not have $\mathfrak{C}_2$ as homomorphic image, but some ultrapower $L^I/\mathcal{U}$ does have $\mathfrak{C}_2$ as homomorphic image. Then $L^I/\mathcal{U} \notin \mathbf{Amal}(\mathbf{V})$, and hence neither $\mathbf{Amal}(\mathbf{V})$ nor its complement are elementary.

---

The lattice $L$ is usually constructed by glueing countably many copies of a maximal subdirectly irreducible member on top of eachother (identifying the top of one member with the bottom of the next). Applications of this result include a simple proof of Theorem 23 as well as the result that any lattice variety generated by a finite simple lattice has a nonelementary amalgamation class. Further generalizations to nonfinitely generated varieties imply, for example, that $\mathbf{M}_\omega$ does not have an elementary amalgamation class.

**Problem 17.** *If* $\mathbf{Amal}(\mathbf{V})$ *is an elementary class, does it follow that it is a Horn class?*

P. Ouwehand and H. Rose [A38] show that if an elementary class $\mathbf{K}$ is closed under updirected unions, then it is closed under finite direct products if and only if it is closed under reduced products (and hence definable by Horn sentences). This result applies to elementary amalgamation classes since M. Yasuhara [1974] showed that they are closed under updirected unions. Hence the above problem is equivalent to asking if every elementary amalgamation class is closed under finite products.

**Problem 18.** *Is there a nonfinitely generated variety other than* $\mathbf{L}$ *whose amalgamation class is elementary? In particular, is* $\mathbf{Amal}(\mathbf{M})$ *an elementary class?*

**Absolute retracts.** We now consider the problem of how the class of all absolute retracts of a variety can be constructed from its subdirectly irreducible members. Even for congruence distributive varieties, the product of two absolute retracts need not be an absolute retract (W. Taylor [A46]), but fortunately lattices are well behaved.

---

**Theorem 25** Let $\mathbf{V}$ be a congruence distributive variety in which every member has a one-element subalgebra. Then the class of absolute retracts of $\mathbf{V}$ is closed under direct products and direct factors, that is, $\prod_{i\in I} A_i \in \mathbf{Ar}(\mathbf{V})$ iff $\{ A_i \mid i \in I \} \subseteq \mathbf{Ar}(\mathbf{V})$. (P. Jipsen and H. Rose [A20], P. Ouwehand and H. Rose [A38].)

---

In fact, P. Ouwehand and H. Rose [A38] show that for congruence distributive varieties, all finite absolute retracts can be obtained as products of subdirectly irreducible absolute retracts. The general case is more complicated and requires the concept of equational compactness (see also Section 1.9 of Appendix A). Here we only need the algebraic formulation: an algebra $A$ is *equationally compact* if for every diagonal embedding of $A$ into an ultrapower of $A$, there is a retraction. Clearly every finite algebra and every absolute retract with respect to some variety is equationally compact. Ouwehand and Rose also observe that equationally compact lattices are complete (a result implicit in B. Weglorz [A47]). Hence absolute retracts in a lattice variety are complete lattices.

Consider the following characterization:

(∗) An algebra $A$ is in $\mathbf{Ar}(\mathbf{V})$ if and only if $A$ is a product of equationally compact reduced powers of $\mathbf{Si}(\mathbf{Ar}(\mathbf{V}))$.

---

**Theorem 26** Let $\mathbf{V}$ be a finitely generated variety of lattices.

(i) Every equationally compact reduced power of a finite absolute retract in $\mathbf{V}$ is an absolute retract in $\mathbf{V}$ (hence the reverse implication of (∗) holds).

(ii) If none of the subdirectly irreducible absolute retracts in $\mathbf{V}$ are homomorphic images of each other then $\mathbf{V}$ satisfies (∗).

(iii) Assume every proper subvariety satisfies (∗). If $\mathbf{V}$ is the join of its proper subvarieties or contains only one subdirectly irreducible absolute retract, then $\mathbf{V}$ satisfies (∗).

(P. Ouwehand and H. Rose [A38].)

---

Note that the previous theorem is a generalization of the well known result that the absolute retracts in $\mathbf{D}$ are precisely the complete Boolean lattices (since every complete Boolean lattice is a reduced power of $\mathfrak{C}_2$, which is the only subdirectly irreducible in $\mathbf{D}$). All finite lattices in $\mathbf{Si}(\mathbf{M})$ are simple, hence (ii) implies that every finitely generated modular variety satisfies (∗). It follows from Theorem 7(i) that any homomorphic image of a lattice in $\mathbf{Si}(\mathbf{AD})$ is distributive, whence (∗) also holds for all finitely generated almost distributive varieties.

## 5. Congruence varieties

A congruence variety is a variety of lattices which is generated by the congruence lattices of some variety of algebras. An account of this area of research can be found in B. Jónsson's appendix to G. Grätzer [A16] (see also B. Jonsson [A23]). In this section, we mention some more recent results and some additional results not included there.

### 5.1 The nonmodular case: Polin's variety

Contrary to the belief of many researchers, S. V. Polin [A41] constructed a variety of algebras whose congruence variety is a proper nonmodular subvariety of **L**. In the reconstruction of Polin's proof (from sketchy notes) A. Day showed that there are infinitely many distinct nonmodular congruence varieties, each of which contains no nondistributive modular lattices. Since the join of congruence varieties is again a congruence variety, there are infinitely many nonmodular congruence varieties. Moreover, we have the following results.

---

**Theorem 27**

(i) Any nonmodular congruence variety contains the variety of all almost distributive lattices. (A. Day 1977].)

(ii) Polin's congruence variety is the unique minimal nonmodular congruence variety. (A. Day and R. Freese [A7].)

---

For further information about Polin's variety see R. Freese [A12].

---

**Theorem 28** Each minimal modular nondistributive congruence variety is determined by one of the varieties generated by all vector spaces of characteristic $p$ (a prime or 0). (R. Freese, Ch. Herrmann and A. P. Huhn [A13].)

---

Since **D** is meet-irreducible in the lattice of modular varieties, it follows from this result that the meet of two congruence varieties does not have to be a congruence variety.

**Corollary 29.** *The set of all congruence varieties is not a sublattice of* $\Lambda$.

**Problem 19.** *Is there a unique largest modular congruence variety?*

We now turn to the question of congruence identities. Among the most significant results are the following.

---

**Theorem 30**

(i) There is a lattice equation strictly weaker than the modular law such that any congruence variety which satisfies this law is a modular variety. (J. B. Nation [1974].)

(ii) Every modular congruence variety is arguesian. (R. Freese and B. Jónsson [A14].)

(iii) No modular nondistributive congruence variety is finitely based. (R. Freese [A11])

(iv) For each $n \geq 0$, the congruence lattice $\operatorname{Con} F_n$ of the free $n$-generated Polin algebra is a splitting lattice. Thus (by Theorem 27(ii)) a variety is congruence modular if and only if it satisfies the conjugate equation of one of these splitting lattices. (A. Day and R. Freese [A7].)

(v) It is decidable whether a lattice equation implies congruence modularity (or distributivity). (G. Czédli and R. Freese [A5].)

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**Problem 20.** *Is there a nondistributive congruence variety which is finitely based?*

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