# Searching for finite models
# (or: some of what I did since ADAM 2006)


Peter Jipsen

Chapman University


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# Outline

- Theory Posets
- Searching for finite structures
- Finite lattices
- Relation algebras as expansions of FL-algebras
- Subalgebras of complex algebras of $Z_n$
- Representations of finite relation algebras

# Theory Posets

Theory Posets

Shortest pointed groupoid equation that has no nontrivial finite model

Dudek found this in 1980

Austin [1965] found one with no constant:
$(((yy)y)x)(((yy)((yy)y))z) = x$

These are now called Austin identities

Kisielewicz 1990 found an Austin identity with 7 variables:
$(((yy)y)x)(yz) = x$

Kisielewicz 1997 proved this is the shortest one

Open problem: is $y(y(y(x(zy)))) = x$ an Austin identity?

# Searching for finite structures

Advantages and disadvantages of using SAGE

Programming in Python

Good control over input, output

Interpreted, slow

Error prone

# Lattices

Finite lattices

Enumerate them 1, 2, 3, 4,... in increasing size.

How to get good diagrams?

Lat9mp.txt

# Relation algebras

Relation algebras as expansions of FL-algebras

Subalgebras of complex algebras of $Z_n$

why does CmZnsubalgs2.in get stuck at 11?

Representations of finite relation algebras

gets stuck on sets of size 13

# Tarski's variety of relation algebras

RA is the variety of algebras $(A, +, 0, \cdot, 1, ^-, ;, e, ^\smile)$ such that

- $(A, +, 0, \cdot, 1, ^-)$ is a Boolean algebra
- $(A, ;, e)$ is a monoid
- converse $^\smile$ is an *involution*: $x^{\smile\smile} = x$ and $(x;y)^\smile = y^\smile;x^\smile$
- ; and $^\smile$ distribute over $+$
- $x^\smile;(x;y)^- \leq y^-$

The last three are equivalent to:

$$(x;y) \cdot z = 0 \;\Leftrightarrow\; (x^\smile;z) \cdot y = 0 \;\Leftrightarrow\; (z;y^\smile) \cdot x = 0$$

# Concrete relation algebras

The variety RRA of *representable relation algebras* is generated by the *square relation algebras*

$$\mathrm{Re}(U) = (\mathcal{P}(U^2), \cup, \emptyset, \cap, U^2, ^-, \circ, id_U, ^{-1})$$

where $U$ is any set (closing under SP suffices)

Monk [1964] proved RRA is a nonfinitely axiomatizable subvariety of RA

Jónsson [1991] proved RRA cannot be axiomatized with finitely many variables

Hirsch and Hodkinson [2001] proved that it is undecidable whether a finite relation algebra is in RRA

Yet this decision has been made for many specific finite relation algebras

# Group relation algebras

If $(G, \circ, e, ^{-1})$ is a group, then the *complex algebra* of $G$ is

$$\mathrm{Cm}\,G = (\mathcal{P}(G), \cup, \emptyset, \cap, G, ^{-}, \circ, \{e\}, ^{-1})$$

is a *group relation algebra* with $X \circ Y = \{x \circ y : x \in X, y \in Y\}$ and $X^{-1} = \{x^{-1} : x \in X\}$.

This algebra is in RRA since it is embedded in $\mathrm{Re}(G)$ via the Cayley map $g \mapsto \{(x, xg) : x \in G\}$ (extended by distributivity to subsets of $G$).

McKenzie [1970] proved that the variety GRA, generated by all complex algebras of groups, is nonfinitely axiomatizable relative to RRA.

A relation algebra is *integral* if $e$ is an atom.

Of the 115 integral RAs of size $\leq 16$, most representable ones are in GRA.

# Group representations of small RAs

Maddux [2006] lists all integral RAs with $\leq 16$ elements

Here are group representations for integral RAs of size $\leq 8$

$1_1 \cong \mathrm{Cm}\mathbb{Z}_1$ $\quad 1_2 \cong \mathrm{Cm}\mathbb{Z}_2$ $\quad 2_2 \cong \mathrm{Sg}^{\mathrm{Cm}\mathbb{Z}_3}(\{0\})$

$1_3 \cong \mathrm{Sg}^{\mathrm{Cm}\mathbb{Q}}(\{r : r > 0\})$ $\quad 2_3 \cong \mathrm{Cm}\mathbb{Z}_3$ $\quad 3_3 \cong \mathrm{Sg}^{\mathrm{Cm}\mathbb{Z}_7}(\{1,2,4\})$

$1_7 \cong \mathrm{Sg}^{\mathrm{Cm}\mathbb{Z}_4}(\{2\})$ $\quad 2_7 \cong \mathrm{Sg}^{\mathrm{Cm}\mathbb{Z}_6}(\{2,4\})$ $\quad 3_7 \cong \mathrm{Sg}^{\mathrm{Cm}\mathbb{Z}_6}(\{3\})$

$4_7 \cong \mathrm{Sg}^{\mathrm{Cm}\mathbb{Z}_9}(\{3,6\})$ $\quad 5_7 \cong \mathrm{Sg}^{\mathrm{Cm}\mathbb{Z}_5}(\{1,4\})$ $\quad 6_7 \cong \mathrm{Sg}^{\mathrm{Cm}\mathbb{Z}_8}(\{1,4,7\})$

$$7_7 \cong \mathrm{Sg}^{\mathrm{Cm}\mathbb{Z}_3^2}(\{1,2\} \times \{1,2\})$$

E.g. $3_7$

| $\circ$ | $e$ | $a$ | $b$ |
|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ |
| $a$ | $a$ | $e$ | $b$ |
| $b$ | $b$ | $b$ | $1$ |

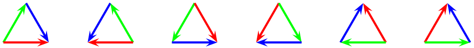
$G = \mathbb{Z}_6, \quad 1 = e + a + b$
$e = \{0\} = e^{-1}$
$a = \{3\} = a^{-1}$
$b = \{1,2,4,5\} = b^{-1}$

Much information in the table is redundant

Consider triples $(x, y, z)$ of atoms that satisfy $x;y \geq z$.

This condition is equivalent to $y^{\smile};x^{\smile} \geq z^{\smile}$, $x^{\smile};z \geq y$, $z^{\smile};x \geq y^{\smile}$, $y;z^{\smile} \geq x^{\smile}$, $z;y^{\smile} \geq x$, hence we group these triples into a *cycle*

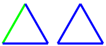
A cycle is represented compactly by a triangle of colored arrows (the converse elements are given by the reverse arrows)



For symmetric atoms a line is used instead of an arrow

| $1_3$ | $r$ | $r^{\smile}$ |
|---|---|---|
| $r$ | $r$ | $1$ |
| $r^{\smile}$ | $1$ | $r$ |

| $3_7$ | $a$ | $b$ |
|---|---|---|
| $a$ | $e$ | $b$ |
| $b$ | $b$ | $1$ |

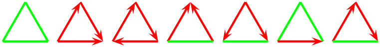


An integral relation algebra with identity atom $e$ contains an

*identity* cycle  if and only if $x = y$.

Hence algebras with the same number of atoms do not differ with respect to these cycles.

It follows that an algebra with a symmetric atom $a$ (green) and two nonsymmetric atoms $r, r^{\smile}$ (red arrow and reverse arrow) is determined by a subset of the following cycles:



Up to isomorphism there are 37 such integral RAs, numbered $1_{37}$–$37_{37}$

Listed by Maddux [2006] and Comer [1986] (different numbering)

# Hypergraphs of relation algebras

Since $e$ behaves the same for all integral relation algebras, this atom is omitted from the table, cycle list, and hypergraph.

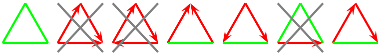
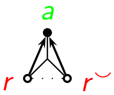
The other atoms of the algebra are given by vertices of a directed hypergraph.

An arrow points from vertex $a$ to $b$ if $a; a^{\smile} \geq b$

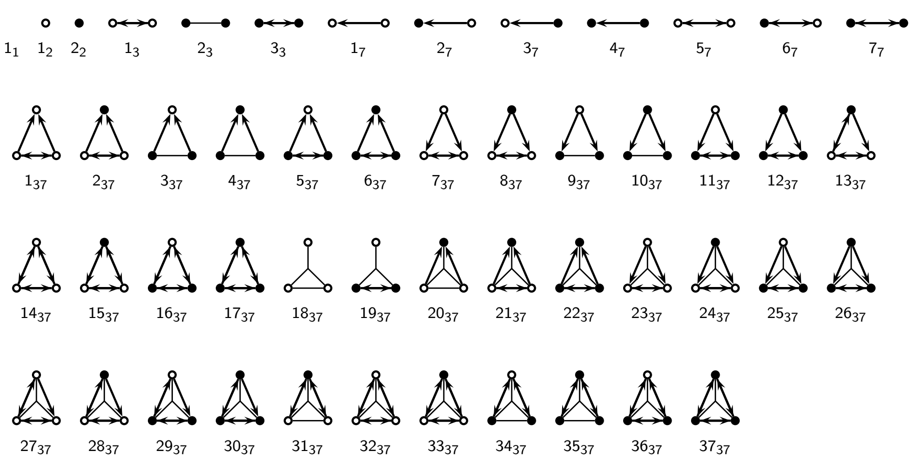
A vertex $a$ is colored **black** if $a; a \geq a^{\smile}$.

A 3-hyperedge (thin lines) connects 3 vertices $a$, $b$, $c$ if $a; b^{\smile} \geq c$.

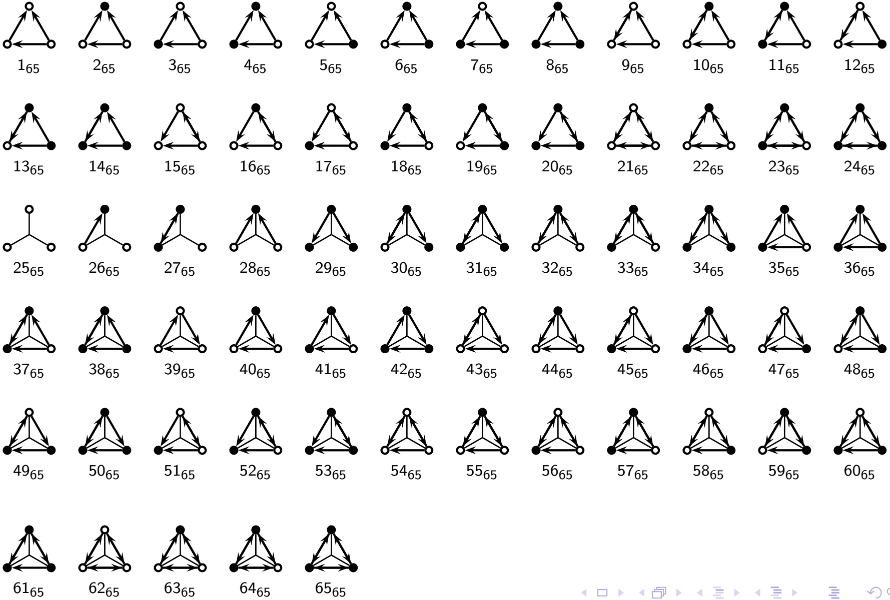
Two atoms connected by a dotted line represent a converse pair $r, r^{\smile}$.

E.g. $20_{37}$:  

# The first 50 integral relation algebras



# The 65 symmetric integral relation algebras of size 16



# Which of these algebras are in RRA or GRA?

The main methods for proving an algebra is representable are

- to construct the algebra from two smaller representable algebras by a *2-cycle extension* $\mathbf{A}[\mathbf{B}]$ (Comer [1983]) and
- a so-called *one-point extension* method, where it is shown that a (possibly infinite) representation can be built by adding one element at a time.

For integral algebras, the 2-cycle extension $\mathbf{A}[\mathbf{B}]$ has cycles defined by

- taking the union of the cycles in $\mathbf{A}$ and $\mathbf{B}$ and
- for all nonidentity atoms $a$ in $\mathbf{A}$ and $b$ in $\mathbf{B}$ add the cycle $b; b^{\smile} \geq a$

Graphically: add arrows from all vertices of $\mathbf{B}$ to all vertices of $\mathbf{A}$.

# Representations from 2-cycle products

**Theorem (Comer (1983))**

$\mathbf{A}, \mathbf{B} \in$ RRA *(or* GRA*) if and only if* $\mathbf{A}[\mathbf{B}] \in$ RRA *(or* GRA*)*

Using the earlier group representations for $1_1$, $1_2$, $2_2$, $1_3$–$3_3$, $5_7$–$7_7$, and 2-cycle extensions one can construct representations for the algebras $1_7$–$4_7$, $1_{37}$–$12_{37}$, $1_{65}$–$20_{65}$.

A *bidirectional 2-cycle extension* $\mathbf{A} \star \mathbf{B}$ has cycles defined by

- taking the union of the cycles in $\mathbf{A}$ and $\mathbf{B}$ and
- for all nonidentity atoms $x, y \in \mathbf{A} \cup \mathbf{B}$ add the cycle $x; x^{\smile} \geq y$

Graphically: add arrows between all vertices of $\mathbf{A}$ and $\mathbf{B}$ and make all dots black.

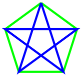
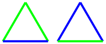
It is used to show that $15_{37}$, $17_{37}$, $24_{65}$ are representable.

# Representations as edge-colored digraphs

A representation of an integral relation algebra is an embedding into Re($X$) for some $X$.

Equivalently, if the algebra has atoms $r_0, r_1, \ldots$, a representation is a complete edge-colored digraph $(X, R_0, R_1, \ldots)$ with vertices $i, j \in X$ labeled by $r_m$ if $(i, j) \in R_m$ such that

- excluded cycles do not appear in the graph, and
- for all vertices $i, j$, if the edge $(i, j)$ is labeled by $z$ and $x;y \geq z$ then there exists a vertex $k$ such that $(i, k)$ is labeled $x$ and $(k, j)$ is labeled $y$,
  i.e. each cycle must appear on each matching edge in the graph.

E.g.  represents $5_7$: 

# Representations by one-point extensions

A relation algebra has a *flexible atom* $a$ if $a \le x;y$ for all $x, y \notin \{0, e\}$

In the hypergraph $a$ (and $a^{\smile}$) have all possible edges and hyperedges entering and leaving.

The one-point extension method applies to all algebras that have a flexible atom, hence $31_{37}$, $33_{37}$, $35_{37}$, $36_{37}$, $37_{37}$, $32_{65}$, $33_{65}$, $34_{65}$, $55_{65}$, $57_{65}$, $59_{65}$, $61_{65}$, $63_{65}$, $64_{65}$, $65_{65}$ are representable.

It also applies to several algebras that do not have a flexible atom, including $13_{37}$, $23_{37}$, $30_{37}$, $30_{65}$, $31_{65}$, $52_{65}$.

# Additional finite group representations

$18_{37} \cong \mathrm{Cm}\mathbb{Z}_4$

$19_{37} \cong \mathrm{Sg}^{\mathrm{Cm}Q_8}(\{i, j, k\})$

$20_{37} \cong \mathrm{Sg}^{\mathrm{Cm}\mathbb{Z}_{12}}(\{1, 7, 9\})$

$22_{37} \cong \mathrm{Sg}^{\mathrm{Cm}(\mathbb{Z}_{16})}(\{1, 3, 5, 6, 7, 14\})$

$25_{65} \cong \mathrm{Cm}\mathbb{Z}_2^2$

$26_{65} \cong \mathrm{Sg}^{\mathrm{Cm}\mathbb{Z}_6}(\{1, 5\})$

$27_{65} \cong \mathrm{Sg}^{\mathrm{Cm}\mathbb{Z}_{10}}(\{1, 2, 8, 9\})$

$28_{65} \cong \mathrm{Sg}^{\mathrm{Cm}D_{12}}(\{b, ab, a^3 b\})$ where $D_{12} = \langle a, b \mid a^6 = b^2 = e, ba = a^5 b\rangle$

$29_{65} \cong \mathrm{Sg}^{\mathrm{Cm}(\mathbb{Z}_3 \times \mathbb{Z}_3)}(\{(0, 1), (0, 2)\}, \{(1, 0), (2, 0)\})$

# Further representations

$39_{65} \cong \mathrm{Sg}^{\mathrm{Cm}\mathbb{Z}_7}(\{1,6\})$

$46_{65} \cong \mathrm{Sg}^{\mathrm{Cm}\mathbb{Z}_{20}}(\{5,6,14,15\})$

$53_{65} \cong \mathrm{Sg}^{\mathrm{Cm}\mathbb{Z}_2^4}(\{0001,0010,0011\},\{0101,0110,1010,1011,1110,1111\})$

$62_{65} \cong \mathrm{Sg}^{\mathrm{Cm}\mathbb{Z}_{13}}(\{1,5,8,12\},\{2,3,10,11\})$

Additional ad hoc infinite representations

$51_{65}$ by Comer [1986]

$56_{65}$ by Lukacs [1991]

These are all 71 representable integral relation algebras of 102 with size 16

# Checking if a finite relation algebra is not representable

**Theorem (Lyndon 1950, Maddux 1983)**

*There is an algorithm that halts if a given finite relation algebra is not representable*

Lyndon gives a recursive axiomatization for RRA

Maddux defines a sequence of varieties $\mathrm{RA}_n$ such that $\mathrm{RA} = \mathrm{RA}_4 \supset \mathrm{RA}_5 \supset \ldots \mathrm{RRA} = \bigcap_{n \geq 4} \mathrm{RA}_n$ and it is decidable if a finite algebra is in $\mathrm{RA}_n$

Implemented as a GAP program [Jipsen 1993]

# Relation algebras and logical games

Recall that a representation of an integral RA is a complete edge-coloured digraph such that each cycle from the algebra appears on each matching edge

[Hirsch Hodkinson 2002] express the construction of a representation as a two-player game, and the algebra is representable if the existential player has a winning strategy.

Given a partial representation, the universal player chooses an edge $(i, j)$ with label $z$ and two atoms $x$, $y$ such that $x;y \geq z$ in the algebra.

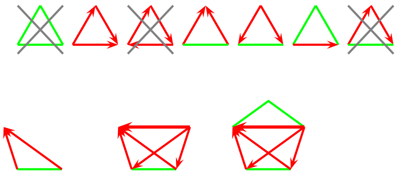
The existential player adds a point $k$ to the graph, labels $(i, k)$ with $x$, $(k, j)$ with $y$, and now has to find edge-labels for all the remaining edges $(k, m)$ such that no excluded cycle is forced to appear in the partial representation.

A "$n$-pebble" version of this game characterizes membership in $RA_n$

# Example of a nonrepresentation game

E.g. [McKenzie 1970] showed that the 4-atom algebra $14_{37}$ is not representable

$14_{37}$:



The resulting configuration is said to be *forbidden* from any representation, since the existential player cannot label all the edges using only the cycles in $14_{37}$

By construction this particular configuration is also *required* in any representation of this algebra, a contradiction. Hence no representation exists.

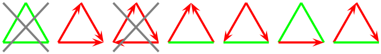
# Nonrepresentability results

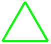
Similar games on 5 points show that $16_{37}$, $21_{37}$, $24_{37}$–$29_{37}$, $34_{37}$, $21_{65}$, $22_{65}$, $23_{65}$, $35_{65}$–$38_{65}$, $40_{65}$–$45_{65}$, $47_{65}$–$50_{65}$, $54_{65}$, $58_{65}$ are not in $\mathsf{RA}_5$

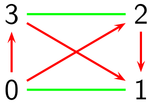
The last two algebras, $32_{37}$ and $60_{65}$, were found to be in $\mathsf{RA}_5 \setminus \mathsf{RA}_6$

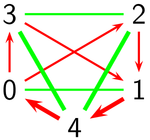
**Theorem**

$32_{37}$ *is not representable.*

$32_{37}$: 

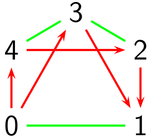
We show that there is no complete edge-labeled graph that contains all the required cycles on each matching edge and omits the green cycle  and the red cycle 

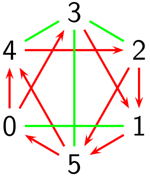



$C_1(0,1,2,3)$

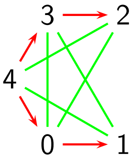
*Claim 1:* $C_1(0,1,2,3)$ is forbidden. Else  should occur on edge $(0,1)$. But then  occurs at $2,3,4$.



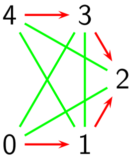




$C_2(0,1,2,3,4)$

*Claim 2:* $C_2(u,v,x,y,z)$ is forbidden. Else  should occur on edge $(0,1)$. But then 3—4 causes  to occur at $2,3,5$.



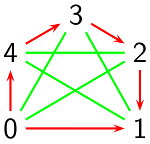

$C_3(0,1,2,3,4)$

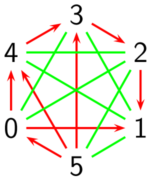



$C'_3(0,1,2,3,4)$

*Claim 3:* $C_3(0,1,2,3,4)$ and $C'_3(0,1,2,3,4)$ are forbidden. Suppose $C_3(0,1,2,3,4)$ occurs. If $2 \to 1$ then $C_2(4,1,2,0,3)$ occurs and if $1 \to 2$ then $C_2(4,2,1,3,0)$ occurs, both impossible by Claim 2. However $1 — 2$ is also impossible since 

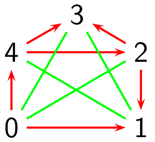
The proof for $C'_3(0,1,2,3,4)$ is analogous.

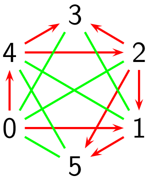


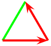

$C_4(0,1,2,3,4)$

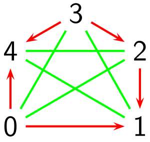
*Claim 4:* $C_4(0,1,2,3,4)$ is forbidden. Else  should occur on edge $(0,1)$. Then $5 \to 4$ and $5 \to 3$. Now $2 \to 5$ implies  on $5,3,2$, while $5 \to 2$ implies $C_1(5,1,2,0)$, and finally $2 — 5$ implies $C_2(5,1,2,0,3)$.

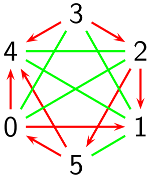



$C_5(0, 1, 2, 3, 4)$

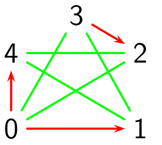
*Claim 5:* $C_5(0, 1, 2, 3, 4)$ is forbidden. Else  should occur on edge $(0, 1)$, hence $2 \longrightarrow 5$. Now $5 \longrightarrow 4$ implies  on $5, 4, 2$, while $4 \longrightarrow 5$ implies $C_1(0, 5, 1, 4)$, and finally $4 — 5$ implies $C_3(5, 1, 2, 0, 3)$.




$C_6(0, 1, 2, 3, 4)$

*Claim 6:* $C_6(0, 1, 2, 3, 4)$ is forbidden. Else  should occur on edge $(0, 1)$, hence $5 \rightarrow 4$ and $5 \rightarrow 3$. Now $5 \rightarrow 2$ implies $C_1(5, 1, 2, 0)$, while $2 — 5$ implies $C_3'(5, 1, 3, 2, 0)$, and finally $2 \rightarrow 5$ implies $C_5(2, 1, 5, 4, 0)$.



(a) (b) (c)

Note that (a) is a *required* configuration. The missing edges must be red arrows but, depending on how they are oriented, we obtain one of the forbidden configurations $C_3$, $C_4$ or $C_6$.

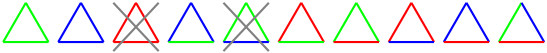
This completes the proof that $32_{37}$ is nonrepresentable.

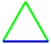
A similar but longer argument proves the same for $60_{65}$.

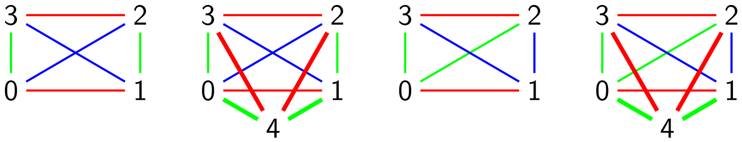
# $60_{65}$ is nonrepresentable

**Theorem**

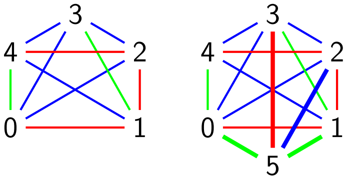
$60_{65}$ *is not representable.*

$60_{65}$: 

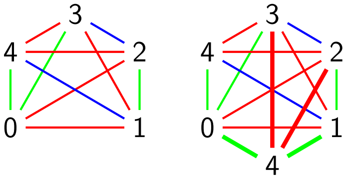
We show that there is no complete edge-labeled graph that contains all the required cycles on each matching edge and omits the red 1-cycle  and the green-blue 2-cycle 




$F_1(0, 1, 2, 3)$ $F_2(0, 1, 2, 3)$




$F_3(0, 1, 2, 3, 4)$




$F_4(0, 1, 2, 3, 4)$

# Part 2: RAs as expansions of involutive FL-algebras

Joint work with N. Galatos

A *FL-algebra* is a residuated lattice $(A, \wedge, \vee, \cdot, \backslash, /, 1)$ expanded with a constant 0 that can denote any element

The linear negations are defined as $\sim x = x\backslash 0$ and $-x = 0/x$

A FL-algebra is *involutive* if $\sim -x = x = -\sim x$ and *cyclic* if $\sim x = -x$

We repeat the definition of relation algebras in a signature close to residuated lattices as algebras of the form $\mathbf{A} = (A, \wedge, \vee, ', \bot, \top, \cdot, 1, ^\smile)$, such that $(A, \wedge, \vee, ', \bot, \top)$ is a Boolean algebra $(A, \cdot, 1)$ is a monoid and for all $a, b, c \in A$

1. $(a^\smile)^\smile = a, \quad (ab)^\smile = b^\smile a^\smile$
2. $a(b \vee c) = ab \vee ac, \quad (a \vee b)^\smile = a^\smile \vee b^\smile$ and
3. $a^\smile(ab)' \leq b'$

# FL$'$-algebras

Given a relation algebra $\mathbf{A}$, we define $a\backslash b = (a^{\smile} b')'$, $b/a = (b' a^{\smile})'$ and $0 = 1'$.

Then $(A, \wedge, \vee, \cdot, \backslash, /, 1, 0)$ is a cyclic involutive FL-algebra with $\sim x = -x = x'^{\smile} = x^{\smile\prime}$.

A *FL$'$-algebra* is an expansion of a FL-algebra with a unary operation $'$ that satisfies the conditions

- $x'' = x$
- $(x \vee y)' = x' \wedge y'$
- $\sim(x') = (\sim x)'$
- $-(x') = (-x)'$

## Lemma

*Every involutive $FL'$-algebra that satisfies $(xy)' = x' + y'$ is cyclic.*

## Proof.

Note that $x\backslash y = (\sim x + y)'' = (\sim x' \cdot y')'$, so
$1' = 1\backslash 1' = (\sim 1' \cdot 1'')' = \sim 1 = 0$. (Similarly $x/y = (x' \cdot -y')'$.)
Moreover, for every $x, y$, we have $\sim x \leq y$ iff $y' \leq \sim x'$ iff $x'y' \leq 1' = 0$ iff $(x+y)' \leq 0$ iff $1 \leq x + y$ iff $-x \leq w$. Therefore $\sim x = -x$, for all $x$. □

We also define two constants $\bot = 1 \wedge 1'$ and $\top = 1 \vee 1'$.

An involutive FL$'$-algebra is called *Boolean* if the reduct $(A, \wedge, \vee, ', \bot, \top)$ is a Boolean algebra, or equivalently if it is distributive and satisfies $x \vee x' = \top$.

# Characterizing relation algebras

## Lemma

*An involutive FL′-algebra is (term equivalent to) a relation algebra iff it is Boolean and satisfies* $(xy)' = x' + y'$.

## Proof.

It is easy to check that every relation algebra satisfies the above properties. Conversely, assume that an involutive FL′-algebra satisfies the properties. By the preceding lemma the algebra is cyclic. Define $x^\cup = {\sim}(x')$. We have $(x \vee y)^\cup = {\sim}(x \vee y)' = {\sim}(x' \wedge y') = ({\sim}x' \wedge {\sim}y') = x^\cup \vee y^\cup$.
Using the commutation of $\sim$ and $'$, and cyclicity, we get
$x^{\cup\cup} = {\sim}(({\sim}x)'') = {\sim}{\sim}x = {\sim}{-}x = x$.
$(xy)^\cup = {\sim}(xy)' = {\sim}(x' + y') = {\sim}y' \cdot {\sim}x' = y^\cup x^\cup$.
To verify $x^\cup(xy)' \le y'$ we rewrite it by applying converse on both sides to get the equivalent equation $(xy)'^\cup x \le y'^\cup$, namely $-(xy)x \le -y$ or $(0/(xy))x \le 0/y$. This is equivalent to $0/(xy) \le 0/(xy)$, hence true. □

# Skew relation algebras

A *skew relation algebra* is defined to be an involutive FL$'$-algebra that is Boolean.

**Theorem**

*Skew relation algebras are term equivalent to algebras of the form* $\mathbf{A} = (A, \wedge, \vee, ', \bot, \top, \cdot, 1, ^{\cup}, ^{\sqcup})$, *such that* $(A, \wedge, \vee, ', \bot, \top)$ *is a Boolean algebra,* $(A, \cdot, 1)$ *is a monoid and for all* $a, b, c \in A$

- $(a^{\cup})^{\sqcup} = a = (a^{\sqcup})^{\cup}$ *and*
- $(a \cdot b) \wedge c = \bot \iff (b \cdot c^{\cup})^{\sqcup} \wedge a = \bot \iff (c^{\sqcup} \cdot a)^{\cup} \wedge b = \bot$

The term equivalence is via $x^{\cup} = {\sim}x'$ and $x^{\sqcup} = -x'$

# Representable $\pi$ relation algebras

Given a set $X$ and a bijection $\pi$ on $X$ we define the algebra $\mathrm{Re}(X,\pi) = (\mathcal{P}(X^2), \cup, \cap, \circ, ^{\cup}, ^{\sqcup}, id_X)$, where

$\circ$ is relational composition,

$R^{\cup} = \{(y, \pi(x)) : (x, y) \in R\}$ and

$R^{\sqcup} = \{(\pi^{-1}(y), x) : (x, y) \in R\}$

It is easy to check that $Re(X,\pi)$ is a skew relation algebra. Moreover, it satisfies $1^{\cup\cup} = 1$

For example, we can take $X = \mathbb{Z}$ and $\pi(n) = n + 1$, or $X = \mathbb{Z}_k$ and $\pi(n) = n +_k 1$.

# Relation algebras with an invertible constant

Given a relation algebra $\mathbf{A} = (A, \wedge, \vee, ', \bot, \top, \cdot, 1, ^\smile)$ and an element $\pi \in A$ that satisfies the identities $\pi\pi^\smile = 1 = \pi^\smile\pi$ (a bijective element), we define the algebra $\mathbf{A}_\pi = (A, \wedge, \vee, ', \bot, \top, \cdot, 1, ^\cup, ^\sqcup)$, where $x^\cup = x^\smile\pi$ and $x^\sqcup = \pi x^\smile$.

It is easy to see that $\mathbf{A}_\pi$ is (term equivalent to) a skew relation algebra.

*Problems*: Is every skew relation algebra of the form $\mathbf{A}_\pi$?

Find all minimal skew RAs (cf. Jónsson [1982], Jipsen & Lukács [1994]).

Is the equational theory of skew relation algebras decidable?

The variety of representable skew relation algebras is generated by the algebras $\mathrm{Re}(X, \pi)$ for any set $X$ and $\pi$ any permutation on $X$.

This variety is not finitely axiomatizable.

# Conclusion