Computational mathematics research via the integration of computer algebra systems with theorem provers and model finders

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Outline

- Background
- Direct decompositions of ℓ-groupoids
- CAS and TP/MC
- Sage

Definition

ℓ-groupoids, unital ℓ-groupoids, ℓ-monoids and ℓ-groups are defined as groupoids, unital groupoids ($ex = x = xe$), monoids and groups that are expanded with lattice operations and satisfy the identities

$x(y \vee z) = xy \vee xz$ and $(x \vee y)z = xz \vee yz$.

They are *bounded* if there are constants $0, 1$ denoting the bottom and top element of the lattice reduct.

Mostly we consider *integral* bounded unital ℓ-groupoids (or *ibu*ℓ-groupoids for short), i.e. they have the top element 1 as the unit.

A *residuated* ℓ-groupoid (or $r\ell$-groupoid) is an ℓ-groupoid for which the residuals $\backslash, /$ exist relative to the groupoid operation, i.e.,

$$x \cdot y \le z \quad \text{iff} \quad x \le z/y \quad \text{iff} \quad y \le x\backslash z$$

A *FL_w-algebra* is a residuated integral bounded ℓ-monoid.

An element c in an *ibuℓ*-groupoid $\mathbf{A}$ is *complemented* if there exists $c' \in A$ such that $c \wedge c' = 0$ and $c \vee c' = 1$.

The *Boolean center* of $\mathbf{A}$ is the set $B(\mathbf{A})$ of all complemented elements.

The next results generalize similar results for MV-algebras [Cignoli, D'Ottaviano and Mundici 2000] and BL-algebras [Di Nola, Georgescu and Leustan 2000]. With the help of Prover9 [McCune 2008] it was shown that associativity is not needed for some of these results. The first part of the following lemma is essentially from [Birkhoff 1967].

Lemma

Let $\mathbf{A}$ *be an ibu*ℓ*-groupoid and let* $c \in B(\mathbf{A})$. *Then*

1. $x \wedge c = xc = cx$ *for all* $x \in A$, *hence the Boolean center is a Boolean sublattice of central idempotent elements.*
2. *If* $\mathbf{A}$ *is a residuated ibu*ℓ*-groupoid then* $B(\mathbf{A})$ *is also closed under the residuals, the complement of* c *is* $-c = 0/c = c\backslash 0$ *and* $c\backslash x = x/c = -c \vee x$ *for all* $c \in B(\mathbf{A})$ *and* $x \in A$.

Proof.

(i) Suppose A is an *ibuℓ*-groupoid and $c \wedge d = 0$, $c \vee d = 1$. By integrality

$$xc \leq x \wedge c = (x \wedge c)(c \vee d) = (x \wedge c)c \vee (x \wedge c)d \leq xc \vee 0 = xc,$$

and similarly $cx \leq c \wedge x \leq cx$.

Suppose we also have $a \wedge b = 0$, $a \vee b = 1$. To see that $B(\mathbf{A})$ is a sublattice of $\mathbf{A}$, it suffices to show that $a \vee c$ and $b \wedge d$ are complements:

$(a \vee c) \wedge (b \wedge d) = (a \vee c)bd = abd \vee cbd = 0$ and

$(a \vee c) \vee (b \wedge d) = a \vee c \vee bd = a \vee c \vee bc \vee bd = a \vee c \vee b(c \vee d) = a \vee c \vee b = 1.$

Now $B(\mathbf{A})$ is complemented by definition, and it is a distributive lattice since $\cdot$ distributes over $\vee$, hence it is a Boolean lattice.

(ii) For complements c, d and any $x \in A$ we have

$c \backslash x = (c \vee d)(c \backslash x) = c(c \backslash x) \vee d(c \backslash x) \leq x \vee d.$

On the other hand $c(x \vee d) = cx \vee cd \leq x$ implies $x \vee d \leq c \backslash x$.

Hence $c \backslash x = d \vee x$, and for $x = 0$ we obtain $-c = c \backslash 0 = d$. Therefore $c \backslash x = -c \vee x$ for all $x \in A$.

The results for $/$ follow similarly. □

For an $ib(r)u\ell$-groupoid $\mathbf{A}$ and an element $c \in B(\mathbf{A})$, define

the *relativized subalgebra* $\mathbf{A}c = {\downarrow}c$ with unit $1^{\mathbf{A}c} = c$, operations $\wedge, \vee, \cdot$ restricted from $\mathbf{A}$,

and $a \backslash b = (a \backslash^{\mathbf{A}} b) \wedge c$, $a/b = (a/^{\mathbf{A}} b) \wedge c$ for all $a, b \in {\downarrow}c$.

Lemma

For any $ib(r)u\ell$-groupoid $\mathbf{A}$ and any $c \in B(\mathbf{A})$, the relativized subalgebra $\mathbf{A}c$ is an $ib(r)u\ell$-groupoid.
If $\mathbf{A}$ is an FL_w-algebra then the map $f : \mathbf{A} \to \mathbf{A}c$ given by $f(a) = ac$ is a homomorphism,
hence $\mathbf{A}c$ satisfies all identities that hold in $\mathbf{A}$.

Proof.

By (i) of the preceding lemma, $\mathbf{A}c$ has c as a unit and is closed under $\wedge, \vee, \cdot$, hence it is an *ibuℓ*-groupoid.

If $\mathbf{A}$ has residuals then for all $a, b, x \in \mathbf{A}c$ we have

$$ax \leq b \quad \text{iff} \quad x \leq^{\mathbf{A}} a\backslash^{\mathbf{A}} b \text{ and } x \leq^{\mathbf{A}} c,$$

whence $a\backslash b = (a\backslash^{\mathbf{A}} b) \wedge c$, and similarly $a/b = (a/^{\mathbf{A}} b) \wedge c$.

Now $f(1) = 1c = 1^{\mathbf{A}c}$, $(a \wedge b)c = a \wedge b \wedge c = ac \wedge bc$ and $(a \vee b)c = ac \vee bc$ hence f preserves $\wedge, \vee$.

If $\cdot$ is associative then $(ab)c = abcc = (ac)(bc)$. In any residuated lattice $x\backslash y \leq xz\backslash yz$, hence $f(a\backslash^{\mathbf{A}} b) \leq f(a)\backslash f(b)$.

For the opposite inequality, we have $ac(ac\backslash bc) \leq bc \leq b$ and therefore $c(ac\backslash bc) \leq a\backslash b$. □

Theorem

If $\mathbf{A}$ is an FL_w-algebra and if $c, d \in B(\mathbf{A})$ are complements then $\mathbf{A} \cong \mathbf{A}c \times \mathbf{A}d$.

Proof.

Consider the map $h : \mathbf{A} \to \mathbf{A}c \times \mathbf{A}d$ defined by $h(a) = (a \wedge c, a \wedge d)$.
The preceding two lemmas show that h is a homomorphism, and h has an inverse
given by $(x, y) \mapsto x \vee y$ since $ac \vee ad = a(c \vee d) = a$ and
for $x \leq c$, $y \leq d$ we have
$((x \vee y)c, (x \vee y)d) = (xc \vee yc, xd \vee yd) = (x, y)$. □

Conversely, any direct decomposition of an $ib(r)u\ell$-groupoid is obtained in this way, since the elements $(0, 1)$, $(1, 0)$ are complements.

Corollary

An FL_w-algebra is directly indecomposable iff its Boolean center contains only the elements $\{0, 1\}$.

References

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