Some results obtained with the Universal Algebra Calculator, theorem provers and constraint satisfiers in a Sage package

Peter Jipsen

Chapman University, Orange, California

March 19, 2011

Sage (sagemath.org) is a free open-source computer algebra system

It combines many research tools with a convenient browser interface

E.g. includes GAP (for Groups, Algorithms and Programs)

Maxima for symbolic algebra

Networkx for graph theory

R for statistics, etc ...

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Can calculate Con(A), Sub(A), A^n , $F_{V(A)}(n)$, ...

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Filtered the list to get only subdirectly irreducible lattices

Task: compute (the bottom of) the poset of join-irreducible subvarieties ${\cal V}({\cal A})$ of $\Lambda_{\rm Lat}$

Since Λ_{Lat} is distributive, the downsets of this poset form Λ_{Lat} (f.g. part)

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For finite s.i. lattices A, B we have $V(A) \subseteq V(B)$ iff $A \in HS(B)$

So how does one compute HS(B)?

Use the UA Calculator to find Sub(B)

Eliminate isomorphic copies to get S(B)

For each $C \in S(B)$ test if $A \in H(C)$, but how?

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- $X = \{x_0, \dots, x_{n-1}\}$ is a set of variables
- *D* is a finite set, the *domain* for the variables

• $C = \{(v_0, R_0), \dots, (v_{n-1}, R_{n-1})\}$ is a set of *constraints*, i.e. $v_i \in X^{k_i}$ and $R_i \subseteq D^{k_i}$ for some $k_i > 0$

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The generalized CSP is a pair (\mathbf{U}, \mathbf{V}) of finite relational structures of the same type with finitely many relation symbols

A solution of the generalized CSP is a homomorphism $h: \mathbf{U} \to \mathbf{V}$

Lemma

The CSP and the generalized CSP are equivalent

Given
$$(X, D, C)$$
, let $\mathbf{U} = (X, \{v_0\}, \dots, \{v_{m-1}\})$ and $\mathbf{V} = (D, R_0, \dots, R_{m-1})$

Conversely, given a pair (\mathbf{U}, \mathbf{V}) , both of type R_0, \ldots, R_{t-1}

let
$$X = U$$
, $D = V$ and $C = \{(v, R_i^{V}) | v \in R_i^{U}, i = 0, ..., t - 1\}$

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Minion is a recent open-source CSP solver

Input syntax is similar to the traditional (X, D, C) version

Wrote short Sage/Python routines that convert a pair of finite algebras into a Minion file, call Minion and read the output file back into Sage

Implemented commands: Hom(A, B), End(A),

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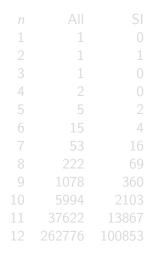
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Heitzig and Reinhold [2002] enumerated all lattices up to n = 18

Peter Jipsen (Chapman U.)

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п	All	SI
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2	1	1
3	1	0
4	2	0
5	5	2
6	15	4
7	53	16
8	222	69
9	1078	360
10	5994	2103
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There are 2555 subdirectly irreducible lattices of size ≤ 10

The poset of subvarieties that they generate has height 7

The bottom (level 0) is the variety of distributive lattices

Level 1 has two varieties generated by M_3 and N_5 found by Dedekind, 1900 Level 2 has 25 varieties:

2 modular M_4 and $M_{3,3}$, found by Grätzer 1966, Jónsson 1968

15 nonmodular L_1, \ldots, L_{15} covering N_5 , found by McKenzie 1972

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enumerated finite complemented lattices with up to 8 join-irreducibles (includes the BA with 256 elements)

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Waldmeister is an equational theorem prover

Both were used to investigate ℓ -pregroups, i.e. lattice-ordered monoids with two unary operations x^l, x^r that satisfy $x^l x \leq 1 \leq xx^l$ and $xx^r \leq 1 \leq x^r x$

An ℓ -pregroup is *periodic* if it satisfies the identity $x^{l^n} = x^{r^n}$ for some positive integer *n*

Theorem (N. Galatos, P. J. 2010)

Periodic *l*-pregroups have distributive lattice reducts

Parts of the proof were found with the equational theorem prover Waldmeister (273 lemmas, > 140 pages pdf)

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sagemath.org

uacalc.org

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prover9.org

Thank You

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UACalc, Prover9 and CSP in Sage

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