Reducts and expansions of residuated lattices

Peter Jipsen

School of Computational Sciences and
Center of Excellence in Computation, Algebra and Topology (CECAT)
Chapman University

August 1, 2014

16th Latin American Symposium on Mathematical Logic

Outline

- Nonclassical propositional logics and residuated lattices
- Reducts of residuated lattices
- Expansions of residuated lattices
- Proof theory and residuated frames
- Interpolation and amalgamation
- Checking the amalgamation property automatically

Classical propositional logics

Classical propositional logic combines **propositions** (or variables) $x_1, x_2, \ldots$ using **and**: $\wedge$, **or**: $\vee$, and **not**: $\neg$

The set of all formulas constructed this way is an **absolutely free algebra** *Fm*

Semantics are given by truth tables, i.e. mappings from $x_1, x_2, \ldots$ to the 2-element Boolean algebra $\mathbf{2}$

Any such map extends to a unique homomorphism $h : Fm \to \mathbf{2}$

A formula φ is **satisfiable** $\iff$ $h(\varphi) = 1$ for **some** h

A formula φ is a **tautology** $\iff$ $h(\varphi) = 1$ for **all** h

$\iff$ the equation $\varphi = 1$ holds in **all Boolean algebras**

Classical propositional logic **corresponds** to Boolean algebras

Nonclassical propositional logics

For many applications, **classical logic** is **unneccessarily strong**

Intuitionistic propositional logic does not derive $\varphi \vee \neg\varphi$

Good for **algorithmic reasoning** and **type theory**

Intuitionistic logic **corresponds** to Heyting algebras

Relevance logic does not derive $\psi \to (\varphi \to \psi)$

Considers $\varphi \to \psi$ true only if φ is used in the derivation of ψ

Substructural logic generalizes **many** such weaker logics

It uses a (possibly) **noncommutative dynamic conjunction** (**fusion**) which is associative but lacks some of the structural laws, e.g., **contraction** $\frac{\varphi\cdot\varphi\Rightarrow\psi}{\varphi\Rightarrow\psi}$ or **weakening** $\frac{\varphi\Rightarrow\psi}{\varphi,\theta\Rightarrow\psi}$

Substructural logics – Residuated lattices

Substructural logics **correspond** to residuated lattices

A **residuated lattice** $(A, \vee, \wedge, \cdot, 1, \backslash, /)$ is an algebra where $(A, \vee, \wedge)$ is a **lattice**, $(A, \cdot, 1)$ is a **monoid** and for all $x, y, z \in A$

$$x \cdot y \leq z \iff y \leq x \backslash z \iff x \leq z / y$$

FL = Full Lambek calculus = the starting point for **substructural logics**

An **FL-algebra** is a residuated lattices **with a new constant** 0

Extensions of substructural logic correspond to **subvarieties** of FL-algebras

Residuated lattices and **FL-algebras** generalize many algebras related to **logic**, e. g. **Boolean algebras**, **Heyting algebras**, **MV-algebras**, **Gödel algebras**, **Product algebras**, Hajek's **basic logic algebras**, **linear logic algebras**, **lattice-ordered groups**, ...

Hiroakira Ono

(California, September 2006)

[1985] *Logics without the contraction rule*

(with Y. Komori)

Provides a **framework** for studying many substructural

logics, relating sequent calculi with semantics

The name **substructural logics** was suggested

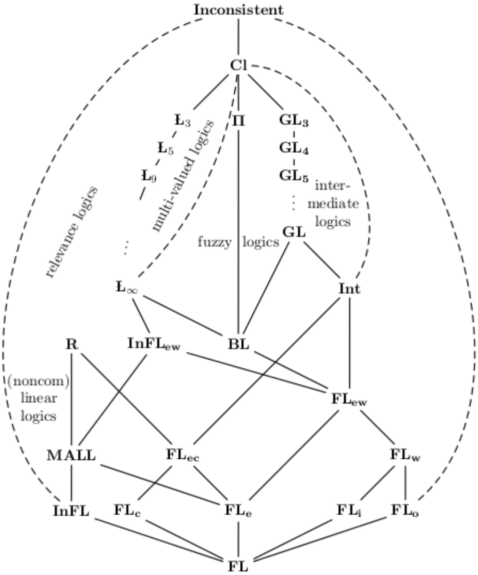
by K. Dozen, October 1990

[2007] *Residuated Lattices: An algebraic glimpse*

at substructural logics (with Galatos, J., Kowalski)

Logic	Algebra	Axioms	w/o 0
Full Lambek Calculus	FL-algebras	Lattice+Monoid+$\backslash, /, 0$	RL
Intuition. Linear Logic	FL_e-algebras	FL + $xy{=}yx$	CRL
FL+weak.+exchange	FL_{ew}-algebras	FL_e + $0{\wedge}x{=}0, 1{\vee}x{=}1$	CIRL
Monoidal t-norm logic	MTL-algebras	FL_{ew} + $x/y \vee y/x{=}1$	$\mathsf{CIRL}^{\mathcal{L}}$
Hajek's Basic Logic	BL-algebras	MTL + $x{\wedge}y{=}(x/y)y$	BH
Łukasiewicz Logic	MV-algebras	BL + $\neg\neg x{=}x$	WH
Intuitionistic Logic	Heyting algebra	FL_{ew} + $x{\wedge}y{=}xy$	GHA
Classical Logic	Boolean algebra	HA + $\neg\neg x{=}x$	GBA

Some propositional logics extending FL



Recent members to the substructural family

Spinks and Verhoff [2008] Constructive logic with strong negation is a substructural logic, I, II

Busaniche and Cignoli [2009] Residuated lattices as an algebraic semantics for paraconsistent Nelson logic

Define a **paraconsistent residuated lattice** to be a **commutative distributive residuated lattice with involution** $\sim x = x\backslash 1$ such that $\sim\sim x = x$

$(x \wedge 1) \cdot (y \wedge 1) = (x \cdot y) \wedge 1$ and $(x \wedge 1) \cdot (x \wedge 1) = x \wedge 1$

Nelson paraconsistent RLs are a further subvariety given by

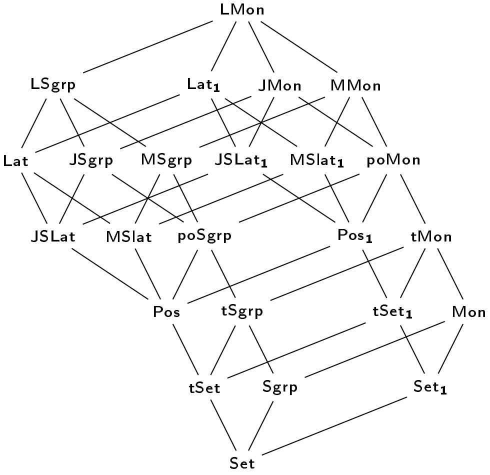
$$((x \wedge 1) \to y) \wedge ((\sim y \wedge 1) \to \sim x) = x \to y$$

$\Rightarrow$ results about residuated lattices are also true for these algebras

Reducts of Residuated Lattices

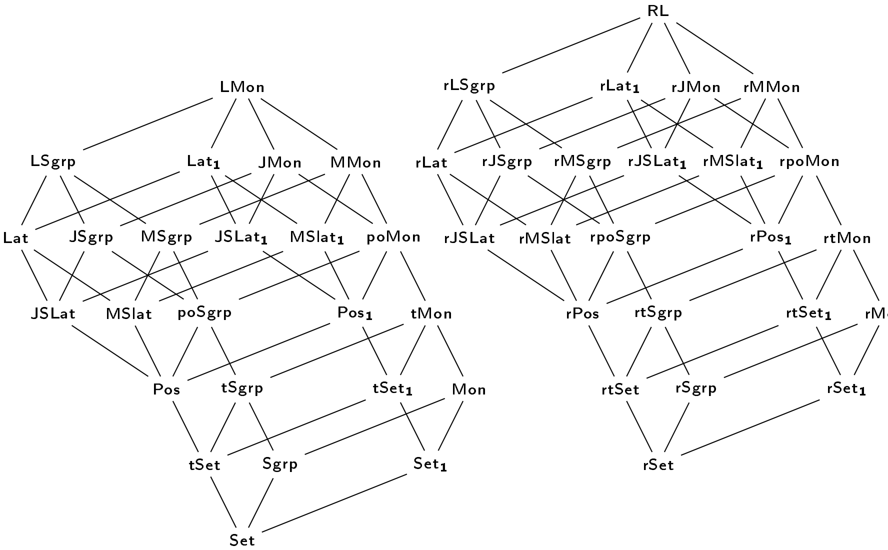
The signature of RL is $\{\vee, \wedge, \cdot, 1, \backslash, /\}$

Consider all 16 subsets of $\{\vee, \wedge, \cdot, 1\}$ and add $\backslash, /$



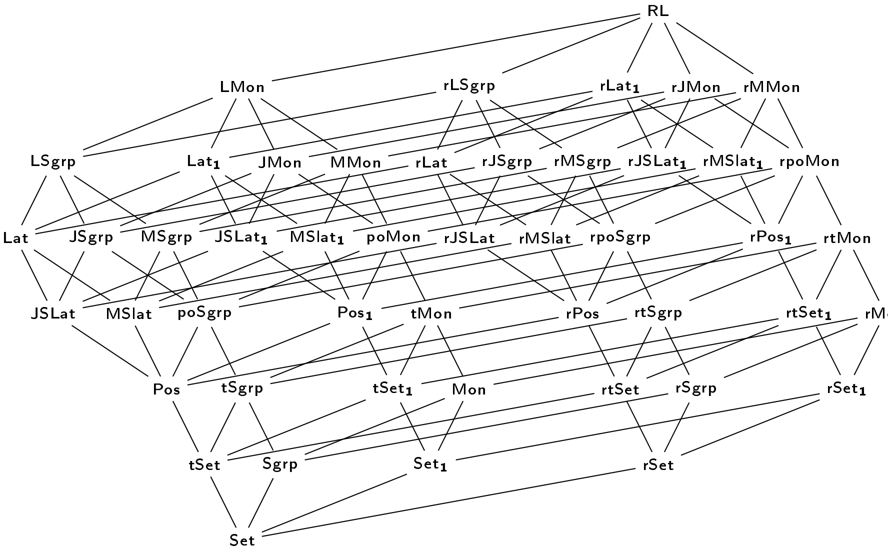
Reducts of Residuated Lattices

Now add $\backslash, /$



Reducts of Residuated Lattices

Now add $\backslash, /$



Subreducts of Residuated Lattices

The classes on the previous slide are usually bigger than the **actual reduct classes** (except for Set, Set_1 and RL)

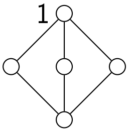
Which **lattices/monoids** are **reducts** of residuated lattices? (open)

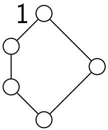
Every lattice **with an atom** can be a residuated lattice

$\implies$ All lattices are $\{\vee, \wedge\}$-**subreducts** of RL

Which members of $\mathbf{Lat}_1$ can be residuated lattices?

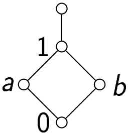
[Ward and Dilworth 1939]: Let X be a subset of a residuated lattice and assume the elements in X pairwise join to 1. Then X generates a **distributive** sublattice.

So 

and 

cannot be subreducts of a residuated lattice

A further restriction on subreducts



[J. 2014]: In any lattice-ordered monoid with 0, if $a \vee b = 1$ and $a \wedge b = 0$ (bottom) for **incomparable** a, b then all elements above 1 are **join-reducible**.

Proof: From $a, b \leq 1$, it follows that $0 \leq ab = ba \leq a \wedge b = 0$.

Assume $x \geq 1$ and, to the contrary, that x is **join-irreducible**.

Then $x = x1 = x(a \vee b) = xa \vee xb$, so $x = xa$ or $x = xb$.

If $x = xa$ then $b = 1b \leq xb = (xa)b = x(ab) = x0 = 0 \leq a$,

contradicting the **incomparability** of a, b.

The case $x = xb$ is the same with a, b **interchanged**. $\square$

Reduct example: Residuated join semilattices

Consider the variety of **residuated join-semilattices with 0**

In computer science they are **residuated idempotent semirings**

Often they are expanded with a Kleene-* (abstract reflexive transitive closure)

$\implies$ **Residuated Kleene algebras**, also called **action algebras** by Vaughn Pratt

Unlike Kleene algebras, **action algebras** are a **variety**

Applications to semantics of **programs** and **specifications**

Model: **Binary relations** on a set closed under $\cup, ;, id, \backslash, /, \emptyset, ^*$

Expansions of Residuated Lattices

Can add an unlimited number of operations

In practice: $0, \bot, \top, !, ?, ^*, \Diamond, \Box, +, \to$

Adding 0 is most common, producing **FL-algebras**

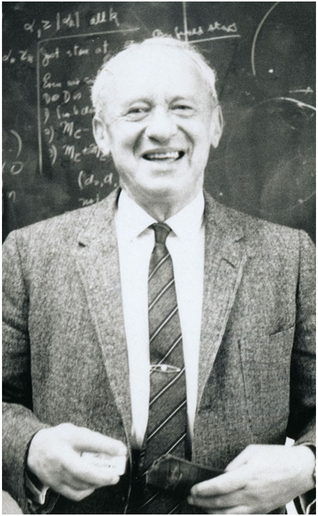
$\implies$ **linear negations**: $\sim x = 0\backslash x$ and $-x = x/0$

Involutive FL-algebras are defined by $\sim -x = x = -\sim x$

Cyclic FL-algebras are defined by $\sim x = -x$

To handle general expansions consider the following

Algebraic logic



Alfred Tarski

(May 1967, visiting at U. of Michigan)

According to the MacTutor Archive, **Tarski** is recognised as one of the four greatest logicians of all time, the other three being **Aristotle**, **Frege**, and **Gödel**

Of these **Tarski** was the most prolific as a logician

His collected works, excluding the 20 books, runs to 2500 pages

Algebraic logic

Bjarni Jónsson

(AMS-MAA meeting in Madison, WI 1968)

Boolean Algebras with operators, Part I and Part II [1951/52] with **Alfred Tarski**

One of the cornerstones of algebraic logic

Constructs **canonical extensions** and provides **semantics** for multi-modal logics

Gives representation for **abstract relation algebras** by **atom structures**

Boolean algebras with operators

Let $\tau = \{f_i : i \in I\}$ be a set of operation symbols, each with a fixed finite arity

$\mathbf{BAO}_\tau$ is the class of algebras $(A, \vee, \wedge, \neg, \bot, \top, f_i\,(i \in I))$ such that $(A, \vee, \wedge, \neg, \bot, \top)$ is a **Boolean algebra** and the f_i are **operators** on A

i.e., $f_i(\ldots, x \vee y, \ldots) = f_i(\ldots, x, \ldots) \vee f_i(\ldots, y, \ldots)$

and $f_i(\ldots, \bot, \ldots) = \bot$ for all $i \in I$ (so the f_i are **strict**)

BAOs are the **algebraic semantics** of classical **multimodal logics**

Main result: every BAO $\mathbf{A}$ can be embedded in its **canonical extension** $\mathbf{A}^\sigma$, a **complete and atomic** Boolean algebra with operators

The **set of atoms** of this Boolean algebra is the **Kripke frame** of the multimodal logic

Example: Residuated Boolean monoids

A **residuated Boolean monoid** is an algebra $(A, \vee, \wedge, \neg, \bot, \top, \cdot, 1, \triangleright, \triangleleft)$ such that $(A, \vee, \wedge, \neg, \bot, \top)$ is a **Boolean algebra**, $(A, \cdot, 1)$ is a **monoid** and for all $x, y, z \in A$

$$(x \cdot y) \wedge z = \bot \iff (x \triangleright z) \wedge y = \bot \iff (z \triangleleft y) \wedge x = \bot$$

Rewrite this as

$$x \cdot y \leq z \iff y \leq \neg(x \triangleright \neg z) \iff x \leq \neg(\neg z \triangleleft y)$$

Define $x \backslash z = \neg(x \triangleright \neg z)$ and $z / y = \neg(\neg z \triangleleft y)$,

and forget $\neg, \bot, \top$ to get a (Boolean) **residuated lattice**

Jónsson and Tsinakis [1992]: **Relation algebras** are a subvariety of residuated Boolean monoids

$\Longrightarrow$ **Relation algebras** are expansions of **RL**

Distributive lattices with operators

Goldblatt [1989], **Gehrke and Jónsson** [1994] extended BAOs to **bounded distributive lattices** with **operators**

Operators are now defined to be **join-preserving** and **strict** or **meet-preserving** and **dually strict** in each argument

Examples: **Heyting algebras**, **MV-algebras**, **BL-algebras**, algebras of **relevance logics**, **distributive residuated lattices**,...

N. Martinez and H. Priestley [1998] develop a general duality for **implicative lattices** (bounded distributive lattices with an implication) that applies to **Gödel algebras**, **MV-algebras**, **lattice-ordered groups**, ...

Lattices with operators

Gehrke and Harding [2001] develop canonical extensions for **lattices with operators**

Gehrke [2006] defines **generalized Kripke frames** using (maximally disjoint) **filter–ideal pairs**

For the lattice reducts, this is based on G. Birkhoff's **polarities**, A. Urquhart's **lattice spaces** and the notion of **contexts** from R. Wille's **Formal Concept Analysis**

Expansions of residuated lattices **by operators** fit into this theory

However, integrating the **proof theory** of residuated lattices and their **reducts/expansions** requires further ideas

A glimpse of algebraic proof theory

Gentzen [1936] defined **sequent calculi**, including **LK** (for classical logic) and **LJ** (for intuistionistic logic)

For **proof search** and **proof normalization**, he proved that the **cut rule** can be **omitted** without affecting provability

Example: A simple **residuated unary sequent calculus**

Let $\mathbf{Lat}_{\Diamond eq}$ be the **equational theory** of **lattices** with a **residuated unary operator**

$(A, \vee, \wedge, \Diamond, \blacksquare)$ is a $\mathbf{Lat}_{\Diamond eq}$**-algebra** if $(A, \vee, \wedge)$ is a **lattice** and

$$\Diamond x \leq y \iff x \leq \blacksquare y \qquad \text{for all } x, y \in A$$

Let $T = F_{\vee,\wedge,\Diamond,\blacksquare}(x_1, x_2, \ldots)$, $\quad W = F_{\hat{\Diamond}}(T)$, $\quad W' = U \times T$

$U = \{u \in F_{\hat{\Diamond}}(T \cup \{x_0\}) : u \text{ contains } \textbf{exactly one } x_0\}$

The Gentzen system $\mathbf{Lat}_\Diamond$

A **Horn formula** $\varphi_1 \& \cdots \& \varphi_n \to \psi$ is written $\frac{\varphi_1 \quad \cdots \quad \varphi_n}{\psi}$

Let $a, b, c \in T$, $\quad t \in W \quad$ and $\quad u \in U$

Lat$_\Diamond$:

$$\frac{}{a \Rightarrow a} \qquad \frac{t \Rightarrow a}{t \Rightarrow a \vee b} \qquad \frac{t \Rightarrow b}{t \Rightarrow a \vee b} \qquad \frac{u(a) \Rightarrow c \quad u(b) \Rightarrow c}{u(a \vee b) \Rightarrow c}$$

$$\frac{t \Rightarrow a \quad a \Rightarrow b}{t \Rightarrow b}(\text{cut}) \qquad \frac{u(a) \Rightarrow c}{u(a \wedge b) \Rightarrow c} \qquad \frac{u(b) \Rightarrow c}{u(a \wedge b) \Rightarrow c} \qquad \frac{t \Rightarrow a \quad t \Rightarrow b}{t \Rightarrow a \wedge b}$$

$$\frac{u(\hat{\Diamond} a) \Rightarrow c}{u(\Diamond a) \Rightarrow c} \qquad \frac{t \Rightarrow a}{\hat{\Diamond} t \Rightarrow \Diamond a} \qquad \frac{\hat{\Diamond} t \Rightarrow a}{t \Rightarrow \blacksquare a} \qquad \frac{u(a) \Rightarrow b}{u(\hat{\Diamond} \blacksquare a) \Rightarrow b}$$

Example of a *cut-free* **Lat**$_\Diamond$ proof

$$\dfrac{\dfrac{\dfrac{\dfrac{x \Rightarrow x}{\hat{\Diamond}\blacksquare x \Rightarrow x}}{\hat{\Diamond}(\blacksquare x \wedge \blacksquare y) \Rightarrow x} \qquad \dfrac{\dfrac{y \Rightarrow y}{\hat{\Diamond}\blacksquare y \Rightarrow y}}{\hat{\Diamond}(\blacksquare x \wedge \blacksquare y) \Rightarrow y}}{\hat{\Diamond}(\blacksquare x \wedge \blacksquare y) \Rightarrow x \wedge y}}{\blacksquare x \wedge \blacksquare y \Rightarrow \blacksquare (x \wedge y)}$$

Semantics of sequent calculi: Residuated frames

Let $\mathbf{Lat}_{\Diamond cf}$ be the sequent calculus $\mathbf{Lat}_{\Diamond}$ **without** the **cut rule**

Define a binary relation $N \subseteq W \times W'$ by

$$wN(u,a) \iff u(w) \Rightarrow a \quad \text{is provable in } \mathbf{Lat}_{\Diamond cf}$$

Define the **accessibility** relation $R \subseteq W^2$ by

$$v\, R\, w \iff v = \hat{\Diamond} w$$

Then (W, W', N, R) is a **residuated (modal) frame**

(A **general** residuated frame is $(W, W', N, R_i(i \in I)))$

Algebraic cut-admissibility

Theorem [Galatos, J. 2013]. The following are equivalent:

1. $t \Rightarrow a$ is provable in $\mathbf{Lat}_{\Diamond}$
2. $t \leq a$ holds in $\mathbf{Lat}_{\Diamond eq}$
3. $t \Rightarrow a$ is provable in $\mathbf{Lat}_{\Diamond cf}$

Proof (outline): (3⇒1) is obvious. (1⇒2) Assume $t \Rightarrow a$ is provable **with cut**. Show that **all sequent rules** hold as quasiequations in $\mathbf{Lat}_{\Diamond eq}$ (where $\Rightarrow, \hat{\Diamond}$ are **replaced by** $\leq, \Diamond$)

(2⇒3) Assume $t \leq a$ holds in $\mathbf{Lat}_{\Diamond eq}$ and define an algebra $\mathbf{W}^+ = (C[\mathcal{P}(W)], \cup, \cap, \Diamond, \blacksquare)$ using the **closed sets** of the **polarity** (W, W', N) and

$$\Diamond X = C(\{v : vRw \text{ for some } w \in X\})$$

$$\blacksquare X = \{x \in W : \Diamond\{w\} \subseteq X\}.$$

Proof outline (continued)

Then $\mathbf{W}^+$ is a $\mathbf{Lat}_{\Diamond eq}$-algebra, hence satisfies $t \leq a$

Let $f : T \to \mathbf{W}^+$ be a **homomorphism**

Extend to $\bar{f} : W \to \mathbf{W}^+$, so $t \leq a$ implies $\bar{f}(t) \subseteq \bar{f}(a)$

Define $\{b\}^{\triangleleft} = \{w \in W : wN(x_0, b)\}$

Prove by **induction** that $b \in \bar{f}(b) \subseteq \{b\}^{\triangleleft}$ for all $b \in T$

Then $t \in \bar{f}(t) \subseteq \bar{f}(a) \subseteq \{a\}^{\triangleleft}$, hence $tN(x_0, a)$

Therefore $\quad t \Rightarrow a \quad$ holds in $\mathbf{Lat}_{\Diamond cf}$ $\square$

Other Expansions: Heyting algebras with operators

This is an **interesting expansion** of residuated lattices

Close to BAO but **better** behaved; more **expressive** than DLO

Sequent calculi and **residuated frames** work

Example 1: **Bunched implication logic**

The algebraic models are $(A, \vee, \wedge, \to, \bot, *, 1, -\!\!*)$ where $(A, \vee, \wedge, \to, \bot)$ is a **Heyting algebra**, $(A, *, 1)$ is a **commutative monoid** and

$$x * y \leq z \iff y \leq x -\!\!* z$$

Equational theory is **decidable** (false if $\neg\neg x = x$)

Applications in computer science; basis of **separation logic**

Example 2: Heyting relation algebras

A **Heyting relation algebra** has the form $(A, \vee, \wedge, \to, \bot, ;, 1, \backslash, /, \sim)$ where $(A, \vee, \wedge, \to, \bot)$ is a **Heyting algebra** and $(A, \vee, \wedge, \to, \bot, ;, 1, \backslash, /, \sim)$ is a **cyclic involutive residuated lattice**

Hence $(A, \vee, \wedge, \to, \bot, \sim)$ is a **symmetric Heyting algebra** in the sense of **A. Monteiro**

Connection to **relation algebras**: Let $(P, \sqsubseteq)$ be a preorder

$R \subseteq P^2$ is a **weakening relation** if $\sqsubseteq; R; \sqsubseteq\, = R$

The set $W(P)$ of **all weaking relations** is closed under $\bigcup, \bigcap, ;$

$\sqsubseteq$ is the **identity element** w.r.t. composition

$\backslash, /$ and $\to$ **exist** since ; and $\cap$ **distribute** over $\bigcup$

Currently developing the **proof theory** for these algebras

The Amalgamation Property

Let $\mathcal{K}$ be a class of **mathematical structures** (e. g. sets, groups, residuated lattices, ...) with **homomorphisms** as maps

$\mathcal{K}$ has the **amalgamation property (AP)** if

for all $A, B, C \in \mathcal{K}$ and all **injective** $f : A \hookrightarrow B$, $g : A \hookrightarrow C$

there exists $D \in \mathcal{K}$ and **injective** $h : B \hookrightarrow D$, $k : C \hookrightarrow D$ such that

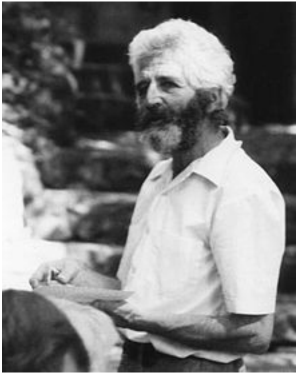
$h \circ f = k \circ g$

$$\begin{tikzcd} & B \arrow[dr, hook, dashed, "h"] & \\ A \arrow[ur, hook, "f"] \arrow[dr, hook, "g"'] & & D \\ & C \arrow[ur, hook, dashed, "k"'] & \end{tikzcd}$$

$\mathcal{K}$ has the **strong amalgamation property** (**SAP**) if,

in addition, $h[f[A]] = h[B] \cap k[C]$

Connections with logic



Bill Craig

(Berkeley, CA 1977)

Craig interpolation theorem [1957]

If $\phi \implies \psi$ is true in first order logic

then there exists θ containing only

the relation symbols in both ϕ, ψ

such that $\phi \implies \theta$ and $\theta \implies \psi$

Also true for many other logics, including classical propositional logic and intuitionistic propositional logic

Let $\mathcal{K}$ be a class of algebras of an algebraizable logic $\mathcal{L}$

Then $\mathcal{K}$ has the (strong/super) **amalgamation property** iff $\mathcal{L}$ satisfies the **Craig interpolation property**

A sample of what is known

These categories **have** the **strong amalgamation property**:

Sets
Groups [Schreier 1927]
Sets with any binary operation [Jónsson 1956]
Variety of all algebras of a fixed signature
Partially ordered sets [Jónsson 1956]
Lattices [Jónsson 1956]

These categories **only** have the **amalgamation property**:

Distributive lattices [Pierce 1968]
Abelian lattice-ordered groups [Pierce 1972]

These categories **fail** to have the **amalgamation property**:

Semigroups [Kimura 1957]
Lattice-ordered groups [Pierce 1972]

Why AP fails for semigroups

Originally due to Kimura [1957], example by M. Sapir:

Let $A = \{0, a_1, a_2\}$, $B = \{0, a_1, a_2, b\}$ and $C = \{0, a_1, a_2, c\}$,

$._B$	0	a_1	a_2	b
0	0	0	0	0
a_1	0	0	0	a_2
a_2	0	0	0	0
b	0	0	0	0

and

$._C$	0	a_1	a_2	c
0	0	0	0	0
a_1	0	0	0	0
a_2	0	0	0	0
c	0	0	a_1	0

Note that A is a **subalgebra** of the **semigroups** B and C

Suppose D is an algebra s.t. B, C are **subalgebras** of D

Then $(c \cdot a_1) \cdot b = 0 \cdot b = 0$ whereas $c \cdot (a_1 \cdot b) = c \cdot a_2 = a_1$

Hence D **cannot be a semigroup** $\square$

Kiss, Márki, Pröhle and Tholen [1983] Categorical algebraic properties. A **compendium on amalgamation**, congruence extension, epimorphisms, residual smallness and injectivity

They summarize some general techniques for establishing these properties

They give a table with **known results for 100 categories**

For recent surveys on **amalgamation** for some varieties of residuated lattices:

Busaniche and Montagna [2011]: *Amalgamation, interpolation and Beth's property in* **BL** (Section 6 in Handbook of Mathematical Fuzzy Logic)

Metcalfe, Montagna and Tsinakis [2014]: *Amalgamation and interpolation in ordered algebras*, Journal of Algebra

How to prove/disprove the AP

Look at **three** examples:

1. Why does **SAP hold** for class of all **Boolean algebras**?
2. Why does **AP hold** for **distributive lattices**?
3. Why does **AP fail** for **distributive residuated lattices**?

1. Boolean algebras (**BA**) can be embedded in **complete and atomic** Boolean algebras (**caBA**)

$$
\begin{tikzcd}
 & B \arrow[r, hook] & B^\sigma \arrow[dr, hook, dashed, "h"] & \\
A \arrow[ur, hook, "f"] \arrow[r, hook] \arrow[dr, hook, "g"'] & A^\sigma \arrow[ur, hook, "f^\sigma"] \arrow[dr, hook, "g^\sigma"] & & D \\
 & C \arrow[r, hook] & C^\sigma \arrow[ur, hook, dashed, "k"'] &
\end{tikzcd}
$$

caBA is dually equivalent to **Set**

1. Amalgamation for BA

So we need to fill in the following dual diagram in **Set**

$$\begin{tikzcd}
 & Uf(B) \arrow[dr, two heads, "Uf(f)"] & \\
P \arrow[ur, dashed, two heads, "h"] \arrow[dr, dashed, two heads, "k"'] & & Uf(A) \\
 & Uf(C) \arrow[ur, two heads, "Uf(g)"'] &
\end{tikzcd}$$

Can take P to be the **pullback**, so
$P = \{(b, c) \in Uf(B) \times Uf(C) : Uf(f)(b) = Uf(g)(c)\}$

Then $h = \pi_1|_P$ and $k = \pi_2|_P$

h is **surjective** since for all $b \in Uf(B)$, there exists $c \in Uf(C)$ s.t. $Uf(f)(b) = Uf(g)(c)$ because $Uf(g)$ is **surjective**

Similarly k is **surjective**

2. Amalgamation for distributive lattices

An algebra is **strictly simple** if it has **no nontrivial** congruences or subalgebras

Theorem [J. and Rose 1989]: Let $\mathcal{V}$ be a **congruence distributive** variety whose members have **one-element** subalgebras, and assume that $\mathcal{V}$ is generated by a **finite strictly simple** algebra. Then $\mathcal{V}$ has the **amalgamation property**.

The variety of **distributive lattices** is generated by the **two-element lattice**, which is **strictly simple**, hence **AP holds**.

Corollary: The **Amalgamation Property holds** for all varieties of residuated lattices that are generated by a **finite strictly simple algebra**, e. g., the variety of **Sugihara algebras** $= V(\{-1, 0, 1\}, \vee, \wedge, \oplus, 0, \neg)$ and infinitely many other varieties

3. AP fails for distributive residuated lattices

To **disprove AP** or **SAP**, we essentially want to search for 3 **small** models A, B, C in $\mathcal{K}$ such that A is a **submodel** of both B and C

We use the **Mace4 model finder** from **Bill McCune [2009]** to enumerate nonisomorphic models $A_1, A_2, \ldots$ in a **finitely axiomatized** first-order theory Σ

For each A_i we construct the **diagram** Δ_i and use **Mace4** again to find all **nonisomorphic** models $B_1, B_2, \ldots$ of $\Delta_i \cup \Sigma \cup \{\neg(c_a = c_b) : a \neq b \in A_i\}$ with 1,2,... **more** elements than A_i

Note that **by construction**, each B_j has A_i as a **subalgebra**

Checking failure of AP

Iterate over **distinct** pairs of models B_j, B_k and construct the theory Γ that extends Σ with the **diagrams of these two models**, using only **one set of constants** for the overlapping submodel A_i

Add formulas to Γ that ensure all constants of B_j are **distinct**, and the same for B_k

Use **Mace4** to check for a **limited** time whether Γ is satisfiable in some **small** model

If not, use the **Prover9 automated theorem prover** (McCune [2009]) to search for a proof that Γ is **inconsistent**. If **yes**, then a **failure of AP** has been found

To check **SAP**, add formulas that ensure constants of **each pair** of models **cannot** be identified, and **also iterate** over pairs B_j, B_j

How to compute finite residuated lattices

First compute all **lattices** with n elements (up to isomorphism)

[J. and Lawless 2013]: For $n = \mathbf{19}$ there are **1 901 910 625 578**

Then compute all **lattice-ordered monoids** with **zero** ($\perp$) over each lattice

The residuals are **determined** by the monoid

There are **295292 residuated lattices** of size $n = \mathbf{8}$

[Belohlavek and Vychodil 2010]: For **commutative integral** residuated lattices there are **30 653 419** of size $n = \mathbf{12}$

Amalgamation for residuated lattices

Open problem: Does **AP** hold for all **residuated lattices**?

Commutative residuated lattices satisfy $x \cdot y = y \cdot x$

Kowalski, Takamura ['04] **AP** holds for commutative resid. lattices

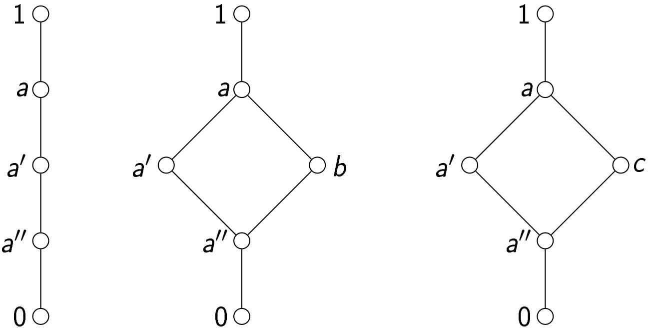
SAP fails for totally ordered (commutative integral) monoids

Distributive residuated lattices satisfy $x \wedge (y \wedge z) = (x \wedge y) \vee (x \wedge z)$

Theorem [J. 2014]: **AP fails** for any variety of **distributive residuated lattices** that includes two specific 6-element **commutative** distributive **integral** residuated lattices

In particular, **AP fails** for the varieties DRL, CDRL, IDRL, CDIRL and any varieties between these

Picture proof



$\Longrightarrow \quad b = c$

$$x \cdot y = y \cdot x = \begin{cases} y & \text{if } x = 1 \\ a'' & \text{if } {}^{x \in \{a,b,c\}}_{y \in \{a,a'\}} \\ 0 & \text{otherwise} \end{cases} \qquad b \cdot b = 0 \qquad c \cdot c = a''$$

Conclusion

Substructural logics and **residuated lattices** are an excellent **framework** for investigating and **comparing** propositional logics

By considering **reducts** and **expansions** (almost) all propositional logics are covered

Algebraic, **semantic** and **proof theoretic** techniques can often be adapted to the **reducts** and **expansions**

Interpolation for logics can be investigated algebraically via the **amalgamation property**

Using **computational tools**, many **minimal** failures of **AP** and **SAP** can be found automatically

Some References

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Thank You