Decompositions of ordered algebraic structures

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Outline

- Background
- Direct decompositions of integral bounded ℓ-groupoids
- Decompositions via poset products
- Sage demo
A **groupoid** is a set with a binary operation $x \cdot y$ (written $xy$)

It is **unital** if it has a constant $e$ that is a unit (i.e., $xe = ex = x$)

A **monoid** is an associative unital groupoid

A **lattice-ordered** groupoid or $\ell$-**groupoid** is a groupoid expanded with lattice operations $\lor, \land$ that satisfy the identities

$$x(y \lor z) = xy \lor xz \quad \text{and} \quad (x \lor y)z = xz \lor yz$$

*Unital* $\ell$-**groupoids**, $\ell$-**monoids** and $\ell$-**groups** are defined similarly

They are **bounded** if there are constants 0, 1 denoting the bottom and top element of the lattice reduct.

Mostly we consider *integral* bounded $\ell$-groupoids (or $ib\ell$-groupoids for short), i.e. they have the top element 1 as the unit.
A **residuated** $\ell$-groupoid (or $r\ell$-groupoid) is an $\ell$-groupoid for which the residuals $\setminus,$ $/$ exist relative to the groupoid operation, i.e.,

\[
x \cdot y \leq z \iff x \leq z/y \iff y \leq x\setminus z
\]

An $ibr\ell$-monoid is also called an $FL_w$-algebra

An element $c$ in an $ib\ell$-groupoid $A$ is **complemented** if there exists $c' \in A$ such that $c \land c' = 0$ and $c \lor c' = 1$.

The **Boolean center** of $A$ is the set $B(A)$ of all complemented elements.

The **direct decomposition** results below generalize similar results for MV-algebras [Cignoli, D’Ottaviano and Mundici 2000] and BL-algebras [Di Nola, Georgescu and Leustan 2000]. With the help of Prover9 [McCune 2008] it was shown that associativity is not needed for some of these results. The first lemma is essentially due to [Birkhoff 1967].
Lemma

Let \( A \) be an \( ib\ell \)-groupoid and let \( c \in B(A) \). Then \( x \land c = xc = cx \) for all \( x \in A \), and the Boolean center is a Boolean sublattice of central idempotent elements.

Proof.

Suppose \( A \) is an \( ib\ell \)-groupoid and \( c \land d = 0 \), \( c \lor d = 1 \). By integrality

\[
x c \leq x \land c = (x \land c)(c \lor d) = (x \land c)c \lor (x \land c)d \leq xc \lor 0 = xc,
\]

and similarly \( cx \leq c \land x \leq cx \).

Suppose we also have \( a \land b = 0 \), \( a \lor b = 1 \). To see that \( B(A) \) is a sublattice of \( A \), it suffices to show that \( a \lor c \) and \( b \land d \) are complements:

\[
(a \lor c) \land (b \land d) = (a \lor c)bd = abd \lor cbd = 0 \text{ and }
\]

\[
(a \lor c) \lor (b \land d) = a \lor c \lor bd = a \lor c \lor bc \lor bd = a \lor c \lor b(c \lor d) = a \lor c \lor b = 1.
\]

Now \( B(A) \) is complemented by definition, and it is a distributive lattice since \( \cdot \) distributes over \( \lor \), hence it is a Boolean lattice.
Lemma

If $\mathbf{A}$ is a residuated ibl-groupoid then $B(\mathbf{A})$ is also closed under the residuals, the complement of $c$ is $-c = 0/c = c\setminus 0$ and $c\setminus x = x/c = -c \lor x$ for all $c \in B(\mathbf{A})$ and $x \in A$.

Proof.

For complements $c, d$ and any $x \in A$ we have

$$c\setminus x = (c \lor d)(c\setminus x) = c(c\setminus x) \lor d(c\setminus x) \leq x \lor d$$

On the other hand $c(x \lor d) = cx \lor cd \leq x$ implies $x \lor d \leq c\setminus x$.

Hence $c\setminus x = d \lor x$, and for $x = 0$ we obtain $-c = c\setminus 0 = d$.

Therefore $c\setminus x = -c \lor x$ for all $x \in A$.

The results for $/$ follow similarly.
For an \(ib(r)\ell\)-groupoid \(A\) and an element \(c \in B(A)\), define

the relativized subalgebra \(A c = \downarrow c\) with unit \(1^{Ac} = c\), operations \(\wedge, \vee, \cdot\)

restricted from \(A\),

and \(a \backslash b = (a \backslash^A b) \wedge c, \quad a / b = (a /^A b) \wedge c\) for all \(a, b \in \downarrow c\).

**Lemma**

For any \(ib(r)\ell\)-groupoid \(A\) and any \(c \in B(A)\), the relativized subalgebra \(A c\) is an \(ib(r)\ell\)-groupoid.

**Proof.**

By the first lemma, \(x \wedge c = xc = cx\), so \(Ac\) has \(c\) as a unit and is closed under \(\wedge, \vee, \cdot\), hence it is an \(ibl\)-groupoid.

If \(A\) has residuals then for all \(a, b, x \in Ac\) we have

\[ ax \leq b \iff x \leq^A a \backslash^A b \text{ and } x \leq^A c, \]

whence \(a \backslash b = (a \backslash^A b) \wedge c\), and similarly \(a / b = (a /^A b) \wedge c\). \(\square\)
Lemma

If $A$ is an $ib(r)\ell$-monoid and $c \in B(A)$ then the map $f : A \to Ac$ given by

$$f(a) = ac$$

is a surjective homomorphism,

hence $Ac$ satisfies all identities that hold in $A$.

Proof.

$$f(1) = 1c = 1^A,$$

$$(a \land b)c = a \land b \land c = ac \land bc$$

and

$$(a \lor b)c = ac \lor bc$$

hence $f$ preserves $\land$, $\lor$.

If $\cdot$ is associative then $(ab)c = abcc = (ac)(bc)$.

In any residuated lattice $x \backslash y \leq xz \backslash yz$, hence $f(a^A \backslash b) \leq f(a) \backslash f(b)$.

For the opposite inequality, we have $ac(ac \backslash bc) \leq bc \leq b$

and therefore $c(ac \backslash bc) \leq a \backslash b$. 

☐
Theorem

If $A$ is an $ib(r)$-$\ell$-monoid and if $c, d \in B(A)$ are complements then

$$A \cong Ac \times Ad$$

Proof.

Consider the map $h : A \rightarrow Ac \times Ad$ defined by $h(a) = (ac, ad)$.

The preceding lemma shows that $h$ is a homomorphism, and $h$ has an inverse given by $(x, y) \mapsto x \lor y$
since $ac \lor ad = a(c \lor d) = a1 = a$ and
for $x \leq c, y \leq d$ we have

$$((x \lor y)c, (x \lor y)d) = (xc \lor yc, xd \lor yd) = (x, y)$$
Conversely, any direct decomposition of an \(ib(r)\ell\)-groupoid is obtained in this way, since the elements \((0, 1), (1, 0)\) are complements.

**Corollary**

An \(ib(r)\ell\)-monoid is directly indecomposable iff its Boolean center contains only the elements \(\{0, 1\}\).

The structure of directly indecomposables can be further analyzed by using *subdirect decompositions*.

However, we now consider a *poset product* that can even be used to decompose subdirectly irreducible algebras.
The **poset product** uses a partial order on the index set to define a subset of the direct product.

Specifically, let $X = (X, \leq)$ be a poset, and assume $\{A_i : i \in X\}$ is a family of algebras that have two constant operations denoted 0, 1.

The poset product of $\{A_i : i \in X\}$ is

$$\prod X A_i = \{ f \in \prod_{i \in X} A_i : f(i) = 0 \text{ or } f(j) = 1 \text{ for all } i < j \text{ in } X \}$$

If $X$ is an **antichain** then the poset product is the same as the direct product.

If $X$ is a **chain** and the $A_i$ are ordered, then the poset product is the (amalgamated) ordinal sum of the $A_i$.
For an \( \ell \)-groupoid \( A \) define \( I_A = \{ c \in A : c \land x = cx = xc \text{ for all } x \in A \} \)

Note that \( \land \) distributes over \( \lor \) in \( I_A \), but \( I_A \) need not be a subalgebra of \( A \).

A \textit{GBL-algebra} is a \( r\ell \)-monoid that satisfies
\[ x \leq y \Rightarrow x = (x/y)y = y(y\backslash x) \]

[J., Montagna 2006] prove that for bounded GBL-algebras, \( I_A \) is a subalgebra, hence a Heyting algebra contained in \( A \), and \( B(A) \) is the subalgebra of complemented elements of \( I_A \).

For MV-algebras \( I_A = B(A) \)

\begin{lemma}
Let \( A \) be an \( ib(r)\ell \)-monoid and let \( a, b \in I_A \) with \( a \leq b \). Then the interval \( [a, b] = \{ x \in A : a \leq x \leq b \} \) is an \( ib(r)\ell \)-monoid, with \( 0 = a, 1 = b \), \( \land, \lor, \cdot \) inherited from \( A \), and \( x \backslash y = (x \backslash A y) \land b \), \( x/y = (x/ A y) \land b \). If \( A \) is a GBL-algebra, then so is \( [a, b] \).
\end{lemma}
We now generalize the poset sum decomposition result of [J., Montagna 2006] from finite GBL-algebras to certain \(ib(r)\ell\)-monoids

**Theorem**

Consider an \(ib(r)\ell\)-monoid \(A\) with a finite subalgebra \(C\) such that \(C \subseteq I_A\), and let \(X\) be the dual of the partially ordered set of completely join irreducible elements of \(C\).

For \(c \in X\), let \(c^*\) denote the unique lower cover of \(c\) in \(C\).

If \(Ac = \downarrow c^* \oplus [c^*, c]\) for all \(c \in X\) then \(A \cong \prod_X [c^*, c]\).

**Corollary**

Suppose \(A\) is a normal GBL-algebra such that \(I_A\) is finite.

Then \(A\) is isomorphic to a poset product of GMV-algebras.
References


(Brief) Sage demo of posets package