

Relation algebras as expanded FL-algebras

Nikolaos Galatos and Peter Jipsen

University of Denver and Chapman University

June 5, 2010

Outline

- Relation algebras as residuated Boolean monoids
- Expansions of FL-algebras
- Quasi relation algebras
- Decidability
- Conclusion and open problems

Relation algebras

Definition (Tarski 1941)

Relation algebras are algebras $(A, \wedge, \vee, ', \perp, \top, \cdot, \smile, 1)$ such that

- $(A, \wedge, \vee, ', \perp, \top)$ is a Boolean algebra
- $(A, \cdot, 1)$ is a monoid and
- for all $x, y, z \in A$,
 $(x \vee y)z = xz \vee yz$ $(x + y)^\smile = x^\smile + y^\smile$
 $x^{\smile\smile} = x$ $(xy)^\smile = y^\smile x^\smile$ $x^\smile(xy)' \leq y'$

The five identities are equivalent to

$$xy \leq z' \iff x^\smile z \leq y' \iff zy^\smile \leq x'$$

so defining *conjugates* $x \triangleright z = x^\smile z$ and $z \triangleleft y = zy^\smile$ we have

$$xy \leq z' \iff x \triangleright z \leq y' \iff z \triangleleft y \leq x'$$

Residuated Boolean monoids

Definition (Birkhoff 1948, Jónsson 1991)

Residuated Boolean monoids are algebras $(A, \wedge, \vee, ', \perp, \top, \cdot, \triangleright, \triangleleft, 1)$ s. t.

- $(A, \wedge, \vee, ', \perp, \top)$ is a Boolean algebra
- $(A, \cdot, 1)$ is a monoid and
- for all $x, y, z \in A$, $xy \leq z' \iff x \triangleright z \leq y' \iff z \triangleleft y \leq x'$

Examples: For any monoid $\mathbf{M} = (M, *, e)$ the powerset monoid $\mathcal{P}(\mathbf{M}) = (\mathcal{P}(M), \cap, \cup, ', \emptyset, M, \cdot, \triangleright, \triangleleft, \{e\})$ is a residuated Boolean monoid

where $XY = \{x * y : x \in X, y \in Y\}$,
 $X \triangleright Y = \{z : x * z = y, x \in X, y \in Y\}$,
 $X \triangleleft Y = \{z : z * y = x, x \in X, y \in Y\}$

If $\mathbf{G} = (G, *, ^{-1})$ is a group, $\mathcal{P}(\mathbf{G})$ is a relation algebra, $X^\smile = \{x^{-1} : x \in X\}$

RM = the variety of residuated Boolean monoids

RA = the variety of relation algebras

Theorem (Jónsson and Tsinakis 1993)

RA is termequivalent to the subvariety of **RM** defined by $(x \triangleright y)z = x \triangleright (yz)$

The termequivalence is given by $x \triangleright y = x \smile y$, $x \triangleleft y = xy \smile$ and $x \smile = x \triangleright 1$

Aim to lift this result to residuated lattices and FL-algebras

RA and **RM** have undecidable equational theories

Want to find a larger variety "close to" RA that has a decidable equational theory, but ...

Kurucz, Nemeti, Sain and Simon [1993] proved that the variety of all Boolean algebras with an associative operator, as well as a "large number" of expanded subvarieties have undecidable equational theories

Residuals

The conjugation condition

$$xy \leq z' \iff x \triangleright z \leq y' \iff z \triangleleft y \leq x'$$

can be rewritten (replacing z by z') as

$$xy \leq z \iff y \leq (x \triangleright z')' \iff x \leq (z' \triangleleft y)'$$

so defining *residuals* $x \setminus z = (x \triangleright z')'$ and $z / y = (z' \triangleleft y)'$ get the equivalent *residuation property*

$$xy \leq z \iff y \leq x \setminus z \iff x \leq z / y$$

(this justifies the name *residuated* Boolean monoids)

FL-algebras

Definition (Ono 1990)

A *Full Lambek* (or *FL-*)*algebra* is of the form $(A, \wedge, \vee, \cdot, \backslash, /, 1, 0)$ where

- (A, \wedge, \vee) is a lattice
- $(A, \cdot, 1)$ is a monoid
- 0 is a constant (with no properties assumed about it) and
- the *residuation property* holds, i. e., for all $x, y, z \in A$

$$x \cdot y \leq z \iff x \leq z/y \iff y \leq x \backslash z$$

Examples: Complementation free reducts of residuated Boolean monoids

Symmetric ($x^\smile = x$) *relation algebras* with $0 = 1'$, $x \backslash y = (xy')'$ and $x/y = (x'y)'$

In this case $x' = x \backslash 0 = 0/x$, but for RA in general $x \backslash 0 = (x^\smile 1'')' = x^\smile'$ so complementation is not recovered by this term

In an FL-algebra there are two *linear negations*

$$-x = 0/x \qquad \sim x = x \setminus 0$$

but they need not coincide or be involutive

To interpret relation algebras into FL-algebras we expand FL-algebras with a unary operation:

Definition

An *FL'-algebra* is an expansion of an FL-algebra with a unary operation $'$ that satisfies $x'' = x$. Also define the following terms:

- *converses* $x^{\cup} = (\sim x)'$ and $x^{\sqcup} = (-x)'$,
- *conjugates* $x \triangleright y = (x \setminus y)'$ and $y \triangleleft x = (y' / x)'$

and consider the identities

$$(In) \quad \sim -x = x = -\sim x \quad (\text{involutive law})$$

$$(Cy) \quad \sim x = -x \quad (\text{cyclic law})$$

$$(Dm) \quad (x \wedge y)' = x' \vee y' \quad (\text{De Morgan, equivalent to } (x \vee y)' = x' \wedge y')$$

Properties of FL'-algebras

Proposition

In an FL'-algebra the following properties hold:

- 1 $(xy) \triangleright z = y \triangleright (x \triangleright z)$ and $z \triangleleft (yx) = (z \triangleleft x) \triangleleft y$
- 2 $(xy)^\cup = y \triangleright x^\cup$ and $(xy)^\sqcup = y^\sqcup \triangleleft x$
- 3 $1 \triangleright x = x$ and $x \triangleleft 1 = x$
- 4 $\sim x = -x$ iff $x^\cup = x^\sqcup$ (cyclic/balanced)

If (Dm) $(x \wedge y)' = x' \vee y'$ is assumed then we also have

- $xy \leq z' \Leftrightarrow x \triangleright z \leq y' \Leftrightarrow z \triangleleft y \leq x'$ (conjugation)
- $(x \vee y)^\cup = x^\cup \vee y^\cup$ and $(x \vee y)^\sqcup = x^\sqcup \vee y^\sqcup$
- $(x \vee y) \triangleright z = (x \triangleright z) \vee (y \triangleright z)$ and $z \triangleleft (x \vee y) = (z \triangleleft x) \vee (z \triangleleft y)$
- $(x \vee y) \triangleleft z = (x \triangleleft z) \vee (y \triangleleft z)$ and $z \triangleright (x \vee y) = (z \triangleright x) \vee (z \triangleright y)$

RL'-algebras

FL-algebras are a subvariety of FL'-algebras if we define $x' = x$

Residuated lattices (**RL**) are a subvariety of **FL** if we define $0 = 1$

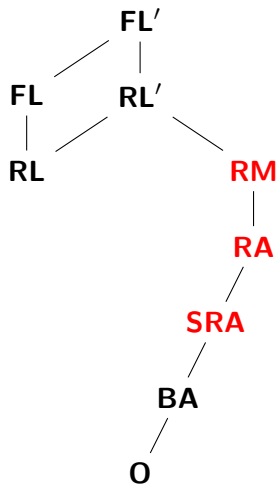
RL' is the subvariety of **FL'** defined by $1' = 0$

Lemma

In an RL'-algebra the following properties hold:

- $x \triangleright 1 = x^{\cup}$ and $1 \triangleleft x = x^{\sqcup}$
- $1^{\cup} = 1^{\sqcup} = 1$

Some subvarieties of \mathbf{FL}'



When negation commutes with ' $'$

Proposition

In an FL' -algebra the following are equivalent:

- (Ci) $\sim(x') = (\sim x)'$ and $-(x') = (-x)'$ (commuting involution)
- (ii) $x^{\cup'} = x'^{\cup}$ and $x^{\sqcup'} = x'^{\sqcup}$ (commuting converses involution)
- (iii) $x^{\cup\sqcup} = x = x^{\sqcup\cup}$ (converse involutive)
- (iv) $-x^{\cup} = x' = \sim x^{\sqcup}$

Moreover, each of these properties implies the following identity:

$$(In) \quad \sim -x = x = -\sim x$$

Quasi relation algebras

Define the term $x + y = \sim(-y \cdot -x)$ ($= -(\sim y \cdot \sim x)$ if (In) is assumed)

Proposition

In every InFL'-algebra the following are equivalent and they imply $0 = 1'$

- 1 $(xy)^\cup = y^\cup x^\cup$
- 2 $(xy)^\sqcup = y^\sqcup x^\sqcup$
- 3 $x \triangleright y = x^\cup y$
- 4 $y \triangleleft x = yx^\sqcup$
- 5 $(xy)' = x' + y'$

A *quasi relation algebra* (qRA) is a CiDmFL'-algebra that satisfies $(xy)' = x' + y'$

Lemma

Every qRA is cyclic, i.e., satisfies $\sim x = -x$

Examples of quasi relation algebras

Let $G = \text{Aut}(C)$ be the ℓ -group of all order-automorphisms of a chain C , and assume that C has a dual automorphism $\partial : C \rightarrow C$

G is a cyclic involutive FL-algebra with $\sim x = -x = x^{-1}$, $x + y = xy$, and $0 = 1$

For $g \in G$, define $g'(x) = g(x^\partial)^\partial$. Then $g'' = g$, $1' = 1$

$$\begin{aligned} y = g^{-1'}(x) &\Leftrightarrow y = g^{-1}(x^\partial)^\partial &\Leftrightarrow y^\partial = g^{-1}(x^\partial) \\ g(y^\partial)^\partial = x &\Leftrightarrow g'(y) = x &\Leftrightarrow y = g'^{-1}(x) \end{aligned}$$

$$(g \vee h)'(x) = (g(x^\partial) \vee h(x^\partial))^\partial = g(x^\partial)^\partial \wedge h(x^\partial)^\partial = (g' \wedge h')(x) \text{ and}$$

$$(gh)'(x) = (g(h(x^\partial)))^\partial = g(h(x^\partial)^\partial)^\partial = (g'h')(x) = (g' + h')(x).$$

Hence G expanded with $'$ is a quasi relation algebra.

For InFL-algebra $(A, \wedge, \vee, \cdot, \sim, -, 1, 0)$ define $\mathbf{A}^\partial = (A, \vee, \wedge, +, -, \sim, 0, 1)$

\mathbf{A}^∂ is also an InFL-algebra called the *dual* of \mathbf{A}

Define $F : \mathbf{InFL} \rightarrow \mathbf{InFL}'$ by $F(\mathbf{A}) = \mathbf{A} \times \mathbf{A}^\partial$ expanded with $(a, b)' = (b, a)$

For a homomorphism $h : \mathbf{A} \rightarrow \mathbf{B}$ define $F(h) : F(\mathbf{A}) \rightarrow F(\mathbf{B})$ by $F(h)(a, b) = (h(a), h(b))$.

Theorem (generalization of Brzozowski 2001)

F is a functor from \mathbf{InFL} to \mathbf{InRL}' , and the restriction to cyclic InFL-algebras maps into \mathbf{qRA} .

If G is the reduct functor from \mathbf{InRL}' to \mathbf{InFL} then for any $\mathbf{qRA} \mathbf{C}$, the map $\sigma_{\mathbf{C}} : \mathbf{C} \rightarrow FG(\mathbf{C})$ given by $\sigma_{\mathbf{C}}(a) = (a, a')$ is an embedding.

Corollary

The equational theory of \mathbf{qRA} is a conservative extension of that of \mathbf{CyInFL} ; i.e., every equation over the language of \mathbf{CyInFL} that holds in \mathbf{qRA} , already holds in \mathbf{CyInFL} .

Lifting the Jónsson-Tsinakis result to qRAs

Theorem

qRA is termequivalent to the subvariety of **CiDmRL'** defined by
 $(x \triangleright y)z = x \triangleright (yz)$

The termequivalence is given by $x \triangleright y = x^{\cup}y$, $x \triangleleft y = xy^{\cup}$ and $x^{\cup} = x \triangleright 1$

We also note that to get from **qRA** to **RA** it suffices to add

distributivity: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and

complementation: $x \wedge x' = \perp$ ($= 1 \wedge 1'$) and $x \vee x' = \top$ ($= 1 \vee 1'$)

qRAs have a decidable equational theory

We make use of the following result:

Theorem (Yetter 1990, Wille 2005)

*The variety **CyInFL** has a decidable equational theory*

For an **InFL**-term t , we define the *dual* term t^∂ inductively by

$$\begin{array}{ll} x^\partial = x & (s \wedge t)^\partial = s^\partial \vee t^\partial \\ 0^\partial = 1 & (s \vee t)^\partial = s^\partial \wedge t^\partial \\ 1^\partial = 0 & (s \cdot t)^\partial = s^\partial + t^\partial \\ (\sim s)^\partial = -s^\partial & (s + t)^\partial = s^\partial \cdot t^\partial \\ (-s)^\partial = \sim s^\partial & \end{array}$$

We also define $(s = t)^\partial$ to be $s^\partial = t^\partial$.

Lemma

An equation ε is valid in **InFL** iff ε^∂ is also valid in **InFL**.

We fix a partition of the denumerable set of variables into two denumerable sets X and X^\bullet , and fix a bijection $x \mapsto x^\bullet$ from the first set to the second (hence $x^{\bullet\bullet}$ denotes x).

For a **qRA**-term t , we define the term t° inductively by

$$\begin{array}{ll}
 x^\circ = x & (s'')^\circ = s \\
 0^\circ = 0, \quad 1^\circ = 1, & ((s \wedge t)')^\circ = s'^\circ \vee t'^\circ, \\
 (0')^\circ = 1, \quad (1')^\circ = 0, & ((s \vee t)')^\circ = s'^\circ \wedge t'^\circ, \\
 (s \diamond t)^\circ = s^\circ \diamond t^\circ, \text{ for all } \diamond \in \{\wedge, \vee, \cdot, +\}, & ((s \cdot t)')^\circ = s'^\circ + t'^\circ, \\
 (\sim s)^\circ = \sim s^\circ, \quad (-s)^\circ = -s^\circ, & ((s + t)')^\circ = s'^\circ \cdot t'^\circ, \\
 ((\sim s)')^\circ = -(s'^\circ), \quad ((-s)')^\circ = -(s'^\circ), & (x')^\circ = x^\bullet
 \end{array}$$

Lemma

For every **qRA**-term t , $t^{\circ\partial}(x_1, \dots, x_n) = t'^{\circ}(x_1^{\bullet}, \dots, x_n^{\bullet})$.

For a substitution σ , we define a substitution σ° by $\sigma^{\circ}(x) = (\sigma(x))^{\circ}$, if $x \in X$, and $\sigma^{\circ}(x) = (\sigma(x)')^{\circ}$, if $x \in X^{\bullet}$.

Lemma

For every **qRA**-term t and **qRA**-substitution σ , $(\sigma(t))^{\circ} = \sigma^{\circ}(t^{\circ})$.

Theorem

An equation ε over X holds in **qRA** iff the equation ε° holds in **CyInFL**.

Corollary

The equational theory of **qRA** is decidable.

Conclusion

By expanding FL-algebras with a unary De Morgan operation one can interpret relation algebras with FL'-algebras

This leads to the variety of quasi relation algebras that has many properties in common with RA

In addition **qRA** has a decidable equational theory

Problem: Is **qRA** generated by its finite members?

Problem: Does the subvariety of distributive qRAs have a decidable equational theory?

References

- B. Jónsson, *A survey of Boolean algebras with operators*, in “Algebras and Orders”, ed. I. Rosenberg, G. Sabidussi, Springer, 1993, 239–286
- B. Jónsson and C. Tsinakis, *Relation algebras as residuated Boolean algebras*, Algebra Universalis 30 (1993), no. 4, 469–478
- Á. Kurucz, I. Németi, I. Sain and A. Simon, *Undecidable Varieties of Semilattice — ordered Semigroups, of Boolean Algebras with Operators, and logics extending Lambek Calculus*, Logic Journal of IGPL, (1993) 1(1), 91–98
- A. Wille, *A Gentzen system for involutive residuated lattices*, Algebra Universalis 54 (2005), no. 4, 449–463