A Survey of Generalized Basic Logic Algebras

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ABSTRACT. Petr Hájek identified the logic **BL**, that was later shown to be the logic of continuous t-norms on the unit interval, and defined the corresponding algebraic models, BL-algebras, in the context of residuated lattices. The defining characteristics of BL-algebras are representability and divisibility. In this short note we survey recent developments in the study of divisible residuated lattices and attribute the inspiration for this investigation to Petr Hájek.

1 Introduction

Petr Hájek's book helped place fuzzy logic on firm mathematical ground. In particular, in view of the truth-functionality of the logic, he developed the algebraic theory of the corresponding models, which he situated within the study of residuated lattices. Moreover, he identified precisely the logic of continuous t-norms, by introducing Basic Logic **BL**.

Basic Logic generalizes three important and natural fuzzy logics, namely Łukasiewicz logic, Product logic and Gödel logic, and allows for the study of their common features. It turns out that BL-algebras, the algebraic models of **BL**, are in a sense made up of these three types of models. Arbitrary BL-algebras are subdirect products of BL-chains, which in turn are ordinal sums of MV-chains, Product-chains and Gödel-chains (though the latter are implicit within the ordinal sum construction).

Within commutative integral bounded residuated lattices, BL-algebras are exactly the representable and divisible ones. Representability as a subdirect product of chains (a.k.a. semilinearity) is often considered synonymous to fuzziness of a logic. It corresponds to the total order on the unit interval of the standard t-norms, and presents a natural view of fuzzy logic as pertaining to (linearly-ordered) degrees of truth.

On the other hand, divisibility renders the meet operation definable in terms of multiplication and its residual(s). It corresponds to the property of having a natural ordering in semigroups. Divisibility has also appeared in the study of complementary semigroups, in the work of Bosbach [Bo82], and in the study of hoops, as introduced by Büchi and Owens [BO]. This shows that it is a natural condition, appearing not only in logic, but also in algebra. While residuation corresponds to the left-continuity of a t-norm on the unit interval, basic logic captures exactly the semantics of continuous t-norms on the unit interval, as was shown later.

On a personal note, we would like to mention that Petr Hájek's book has been influential in developing some of the theory of residuated lattices and also connecting it with the study of logical systems. The book had just appeared shortly before a seminar on residuated lattices started at Vanderbilt University, organized by Constantine Tsinakis and attended by both authors of this article. The naturality of the definition of divisibility and its encompassing nature became immediately clear and *generalized BL-algebras* (or GBL-algebras) were born. These are residuated lattices (not necessarily commutative, integral, contractive or bounded) that satisfy divisibility: if $x \leq y$, there exist z, w with $x = zy = yw$. It turns out that the representable commutative bounded GBL-algebras are exactly the BL-algebras. In other words, GBL-algebras are a generalization of BL-algebras that focuses on and retains the divisibility property. Examples include lattice-ordered groups, their negative cones, (generalized) MV-algebras and Heyting algebras.

Despite their generality, GBL-algebras decompose into direct products of lattice-ordered groups and integral GBL-algebras, so it is these two subvarieties that are of main interest. From [Mu86] it follows that MV-algebras are certain intervals in abelian ℓ-groups and in [GT05] it is shown that GMV-algebras are certain convex sublattices in ℓ-groups. Also, BL-chains (the building blocks of BL-algebras) are essentially ordinal sums of parts of ℓ-groups (MV-chains and product chains). It is interesting that recent work has shown that algebras in many natural classes of GBL-algebras (including commutative, as well as k-potent) are made from parts of ℓ-groups put (in the form of a poset product) into a Heyting algebra grid. The poset product decomposition does not work for all GBL-algebras, but it is an open problem to find BL-algebras that are not locally parts of ℓ-groups, if such algebras exist.

2 BL-algebras as residuated lattices

2.1 Residuated lattices

A *residuated lattice* is a structure $(L, \wedge, \vee, \cdot, \backslash, /, 1)$, where $(L, \wedge, \vee)$ is a lattice, $(L, \cdot, 1)$ is a monoid and the law of residuation holds; i.e., for all $a, b, c \in L$,

$$a \cdot b \leq c \quad \Leftrightarrow \quad b \leq a \backslash c \quad \Leftrightarrow \quad a \leq c/b.$$

Sometimes the expression $x \rightarrow y$ is used for $x \backslash y$, while $y \leftarrow x$ (or $x \rightsquigarrow y$) is used for y/x. The corresponding operations are called the residuals of multiplication. *FL-algebras* are expansions $(L, \wedge, \vee, \cdot, \backslash, /, 1, 0)$ of residuated lattices with an additional constant operation 0.

Residuated lattices and FL-algebras are called *commutative* if multiplication is commutative, *integral* if 1 is the greatest element, *representable* (or *semilinear*) if they are subdirect products of chains, and *divisible* if they satisfy:

$$\text{If } x \leq y, \text{ there exist } z, w \text{ such that } x = zy = yw.$$

In commutative residuated lattices we have $x\backslash y = y/x$, and we denote the common value by $x \to y$.

An *FL_o-algebra*, also known as a *bounded residuated lattice*, is an FL-algebra in which $0 \leq x$ for all x. In this case 0 is also denoted by $\perp$. Moreover, it turns out that $\top = 0/0 = 0\backslash 0$ is the top element. Integral FL_o-algebras are also known as FL_w-algebras, and in the presence of commutativity as FL_{ew}-algebras.[1] For more on residuated lattices see [GJKO].

2.2 BL-algebras

It turns out that *BL-algebras*, as defined by P. Hájek, are exactly the representable and divisible algebras within the class of integral commutative bounded residuated lattices (i.e., within FL_{ew}-algebras).

THEOREM 1 [BT03] [JT02] *A residuated lattice is representable iff it satisfies* $[z\backslash((x \vee y)/x)z \wedge 1] \vee [w((x \vee y)/y)/w \wedge 1] = 1$.

The next result then follows easily.

COROLLARY 2 *An integal, commutative residuated lattice is representable iff it satisfies* $(x \to y) \vee (y \to x) = 1$ *(prelinearity).*

P. Hájek defined BL-algebras using the prelinearity condition, which captures representability in the integral, commutative case. He also used the simplified form of divisibility $x \wedge y = x(x \to y)$, which we will see is equivalent to the general form in the integral commutative case; actually integrality follows from this particular form of divisibility, by setting $x = 1$. The variety (i.e. equational class) of BL-algebras is denoted by BL.

Representable Heyting algebras, known as *Gödel algebras*, are exactly the idempotent ($x^2 = x$) BL-algebras. Moreover, Chang's MV-algebras are exactly the involutive ($(x \to 0) \to 0 = x$) BL-algebras. For additional subvarieties of FL_w see Table 1 and 3, as well as Figure 1.

[1] The subscripts e, w are from the names of the rules *exchange* and *weakening* in proof theory that correspond to commutativity and integrality.

divisible	$x(x\backslash(x \wedge y)) = x \wedge y = ((x \wedge y)/x)x$
prelinear	$x\backslash y \vee y\backslash x = 1 = x/y \vee y/x$
linear or **chain**	$x \leq y$ or $y \leq x$
integral	$x \leq 1$
bottom	$0 \leq x$
commutative	$xy = yx$
idempotent	$xx = x$

Table 1. Some residuated lattice axioms

2.3 Algebraization

P. Hájek defined basic logic **BL** via a Hilbert-style system, whose sole inference rule is modus ponens; see Table 2.

(sf)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$	(suffixing)
(int)	$(\varphi \cdot \psi) \to \varphi$	(integrality)
(com)	$(\varphi \cdot \psi) \to (\psi \cdot \varphi)$	(commutativity)
(conj)	$(\varphi \cdot (\varphi \to \psi)) \to (\psi \cdot (\psi \to \varphi))$	(conjunction)
$(\cdot\!\to)$	$((\varphi \cdot \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$	
$(\to\!\cdot)$	$(\varphi \to (\psi \to \chi)) \to ((\varphi \cdot \psi) \to \chi)$	
$(\to$pl)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$	(arrow prelinearity)
(bot)	$0 \to \varphi$	
	$\varphi \wedge \psi := \varphi \cdot (\varphi \to \psi)$	(conjunction definition)
	$\varphi \vee \psi := [(\varphi \to \psi) \to \psi] \wedge [(\psi \to \varphi) \to \varphi]$	(disjunction definition)
	$\neg\varphi := \varphi \to 0$	(negation definition)
	$1 := 0 \to 0$	(unit definition)

$$\frac{\varphi \quad \varphi \to \psi}{\psi}\ (\text{mp})$$

Table 2. Hajek's basic logic.

Using this system, one defines as usual the notion of proof from assumptions. If ψ is a propositional formula in the language of **BL**, and Φ is a set of such formulas, then $\Phi \vdash_{\mathbf{BL}} \psi$ denotes that ψ is provable in **BL** from (non-logical, i.e., no substitution instances are allowed) assumptions Φ.

The following results states that $\vdash_{\mathbf{BL}}$ is algebraizable (in the sense of Blok and Piggozzi [BP89]) with respect to the variety BL of BL-algebras. It

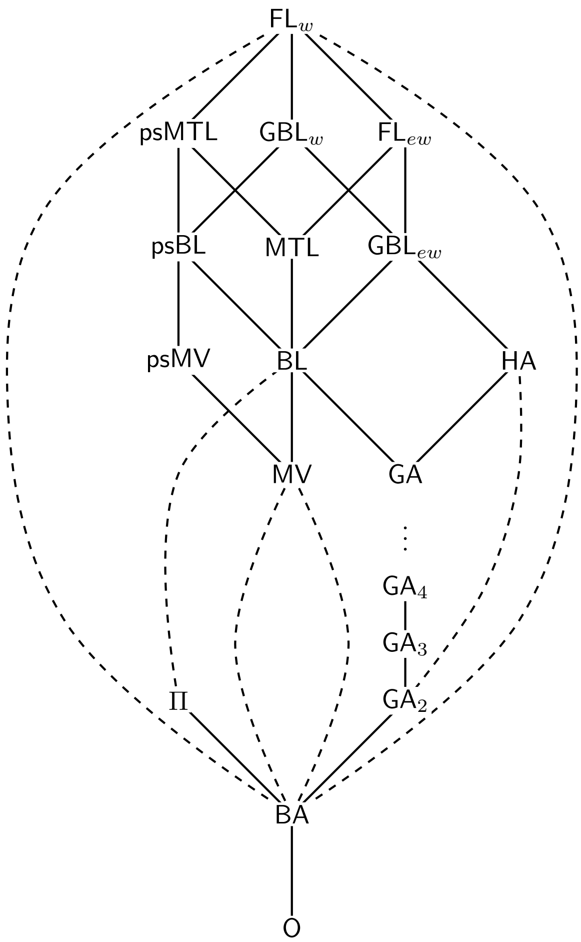

Figure 1. Some subvarieties of FL_w ordered by inclusion

FL_w	**FL-algebras with weakening** = integral residuated lattices with bottom
FL_{ew}	**FL_w-algebras with exchange** = commutative integral residuated lattices with bottom
GBL_w	**GBL-algebras with weakening** = divisible integral residuated lattices with bottom
GBL_{ew}	**GBL_w-algebras with exchange** = commutative GBL_w-algebras
psMTL	**pseudo monoidal t-norm algebras** = integral residuated lattices with bottom and prelinearity
MTL	**monoidal t-norm algebras** = psMTL with commutativity
psBL	**pseudo BL-algebras** = psMTL with divisibility
BL	**basic logic algebras** = MTL with divisibility
HA	**Heyting algebras** = residuated lattices with bottom and $x \wedge y = xy$
psMV	**pseudo MV-algebras** = psBL with $x \vee y = x/(y\backslash x) = (x/y)\backslash x$
MV	**MV-algebras or Łukasiewicz algebras** = BL-algebras that satisfy $\neg\neg x = x$ = commutative pseudo MV-algebras
MV_n	**MV-algebras generated by $n+1$-chains** = subdirect products of the $n+1$-element MV-chain
GA	**Gödel algebras or linear Heyting algebras** = BL-algebras that are idempotent = Heyting algebras with prelinearity
GA_n	**Gödel algebras generated by $n+1$-chains** = subdirect products of the $n+1$-element Heyting chain
Π	**product algebras** = BL-algebras that satisfy $\neg\neg x \leq (x \to xy) \to y(\neg\neg y)$
BA	**Boolean algebras** = Heyting algebras that satisfy $\neg\neg x = x$ = MV-algebras that are idempotent

Table 3. Some subvarieties of FL_w

follows directly from the more general algebraization of substructural logics by residuated lattices given in [GO06].

As usual, propositional formulas in **BL** are identified with terms of BL. Recall that for a set $E \cup \{s = t\}$ of equations, $E \models_{\mathsf{BL}} s = t$ ($s = t$ is a *semantial consequence* of E with respect to BL) denotes that for every BL-algebra $\mathbf{B}$ and for every assignment f into $\mathbf{B}$ (i.e., for every homomorphism from the formula/term algebra into $\mathbf{B}$), if $f(u) = f(v)$ for all $(u = v) \in E$, then $f(s) = f(t)$.

THEOREM 3 *For every set $\Phi \cup \{\psi\}$ of propositional formulas and every set $E \cup \{s = t\}$ of equations,*

- $\Phi \vdash_{\mathbf{BL}} \psi$ *iff* $\{\varphi = 1 : \varphi \in \Phi\} \models_{\mathsf{BL}} \psi = 1$
- $E \models_{\mathsf{BL}} s = t$ *iff* $\{(u \to v) \wedge (v \to u) : (u = v) \in E\} \vdash_{\mathbf{BL}} (s \to t) \wedge (t \to s)$
- $\varphi \dashv\vdash (\varphi \to 1) \wedge (1 \to \varphi)$
- $s = t \;=\!\mid\models_{\mathsf{BL}} (s \to t) \wedge (t \to s) = 1$

As a corollary we obtain that the lattice of axiomatic extensions of **BL** is dually isomorphic to the subvariety lattice of BL.

3 Fuzzy logics and triangular norms: retaining representability

Truth in a fuzzy logic comes in degrees. In a so-called standard model, truth values are taken to be real numbers from the interval $[0, 1]$, while logical connectives of arity n are interpreted as functions from $[0, 1]^n$ to $[0, 1]$. In particular, multiplication in a standard model is assumed to be continuous and a *triangular norm* (*t-norm*), namely a binary operation on the interval $[0, 1]$ that is associative, commutative, monotone and has 1 as its unit element. It is easy to see that any such continuous t-norm defines a BL-algebra structure on $[0, 1]$. In fact, **BL** is complete with respect to continuous t-norms, as proved in [CEGT], where it is shown algebraically that the variety of BL-algebras is generated by all continuous t-norms.

Łukasiewicz t-norm ($L(x, y) = \max\{x+y-1, 0\}$), product t-norm ($\Pi(x, y) = xy$), and Gödel t-norm ($G(x, y) = \min\{x, y\}$), define three special standard models of **BL**. The corresponding logics that extend **BL** are denoted by **Ł**, **GL** and **Π**.

Chang [Ch59] proved that the standard model given by $L(x, y)$ generates the variety of MV-algebras. Thus, **Ł** is precisely the infinite-valued Łukasiewicz logic and it is axiomatized relatively to **BL** by $\neg\neg\varphi \to \varphi$. Also, **GL** is Gödel logic, namely the superintuitionistic logic defined by adding to

BL the axiom $\varphi \to \varphi^2$, and it is the smallest superintuitionistic logic that is also a fuzzy logic. Finally, $\mathbf{\Pi}$ is *product logic* and is defined relatively to **BL** by $\neg\neg\varphi \to (((\varphi \to (\varphi \cdot \psi)) \to (\psi \cdot \neg\neg\psi))$; see [Ci01].

It is easy to see that, due to the completeness of $[0, 1]$, a t-norm is residuated iff it is left-continuous; divisibility provides right-continuity, in this context. It turns out that the variety of representable FL_{ew}-algebras is generated by left-continuous t-norms; see [JM02]. The corresponding logic is called *monoidal t-norm logic* or **MTL**. Uninorm logic, on the other hand, corresponds to representable FL_{eo}-algebras, i.e., integrality is not assumed. The introduction of **BL** lead to the study of such general fuzzy logics, where representability is the main defining property, while divisibility is dropped alltogether. In the rest of the survey, we focus on generalizations that retain the divisibility property.

4 Generalized BL-algebras: retaining divisibility

4.1 GBL-algebras

A *generalized BL-algebra* (or *GBL-algebra* for short) is defined to be a divisible residuated lattice. We begin with presenting some equivalent reformulations of divisibility:

$$\text{If } x \leq y \text{, there exist } z, w \text{ such that } x = zy = yw.$$

LEMMA 4 [GT05] *The following are equivalent for a residuate lattice* $\mathbf{L}$.

1. $\mathbf{L}$ *is a GBL-algebra.*
2. $\mathbf{L}$ *satisfies* $((x \wedge y)/y)y = x \wedge y = y(y\backslash(x \wedge y))$.
3. $\mathbf{L}$ *satisfies the identities* $(x/y \wedge 1)y = x \wedge y = y(y\backslash x \wedge 1)$.

Proof. (1) $\Leftrightarrow$ (2): Assume that $\mathbf{L}$ is a GBL-algebra and $x, y \in L$. Since $x \wedge y \leq y$, there exist z, w such that $x \wedge y = zy = yw$. Since $zy \leq x \wedge y$, we have $z \leq (x \wedge y)/y$, so $x \wedge y = zy \leq ((x \wedge y)/y)y \leq x \wedge y$, by residuation, i.e. $((x \wedge y)/y)y = x \wedge y$. Likewise, wo obtain the second equation. The other direction is obvious.

(2) $\Leftrightarrow$ (3): By basic properties of residuation, we get

$$y \wedge x = y(y\backslash(y \wedge x)) = y(y\backslash y \wedge y\backslash x) = y(1 \wedge y\backslash x).$$

Likewise, we get the opposite identity.

Conversely assume (3). Note that for every element $a \geq 1$, we have $1 = a(a\backslash 1 \wedge 1) \leq a(a\backslash 1) \leq 1$; so, $a(a\backslash 1) = 1$. For $a = x\backslash x$, we have $a^2 = a$, by properties of residuation and $1 \leq a$. Thus, $a = (1/a)a^2 = (1/a)a = 1$.

Consequently, $y\backslash y = 1$, for every $y \in L$. Using properties of residuation, we get

$$y(y\backslash(y \wedge x)) = y(y\backslash y \wedge y\backslash x) = y(1 \wedge y\backslash x) = y \wedge x.$$

Likewise, we obtain the other equation. ■

LEMMA 5 [GT05] *Every GBL-algebra has a distributive lattice reduct.*

Proof. Let $\mathbf{L}$ be a GBL-algebra and $x, y, z \in L$. By Lemma 4 and the fact that in residuated lattices multiplication distribures over joins, we have

$$\begin{aligned} x \wedge (y \vee z) &= [x/(y \vee z) \wedge 1](y \vee z) \\ &= [x/(y \vee z) \wedge 1]y \vee [x/(y \vee z) \wedge 1]z \\ &\leq (x/y \wedge 1)y \vee (x/z \wedge 1)z \\ &= (x \wedge y) \vee (x \wedge z), \end{aligned}$$

for all x, y, z. Thus, the lattice reduct of $\mathbf{L}$ is distributive. ■

Integral GBL-algebras, or *IGBL-algebras*, have a simpler axiomatization.

LEMMA 6 [GT05] *IGBL-algebras are axiomatized, relative to residuated lattices, by the equations $(x/y)y = x \wedge y = y(y\backslash x)$.*

Proof. One direction holds by Lemma 4. For the converse, note that we show that the above identity implies integrality for $y = e$. ■

4.2 Lattice-ordered groups and their negative cones

Lattice ordered groups, or *ℓ-groups* are defined as algebras with a lattice and a group reduct such that the group multiplication is compatible with the order, see e.g. [AF88], [Gl99]. Equivalently, they can be viewed as residuated lattices that satisfy $x(x\backslash 1) = 1$. It is easy to see that ℓ-groups are examples of (non-integral) GBL-algebas.

Given a residuated lattice $\mathbf{L}$, its *negative cone* $\mathbf{L}^-$ is defined to be an algebra of the same type, with universe $L^- = \{x \in L : x \leq 1\}$, $x\backslash^{\mathbf{L}^-} y = x\backslash y \wedge 1$, $y/^{\mathbf{L}^-} x = y/x \wedge 1$ and where the other operations are the restrictions to L^- of the operations in $\mathbf{L}$. With this definition $\mathbf{L}^-$ is also a residuated lattice. The map $\mathbf{L} \mapsto \mathbf{L}^-$ preserves divisibility, hence negative cones of ℓ-groups are also examples of integral GBL-algebras.

Both ℓ-groups and their negative cones are cancellative residuated lattices, namely their multipliction is cancellative. Equivalently, they satisfy the identities $xy/y = x = y\backslash yx$.

THEOREM 7 [BCGJT] *The cancellative integral GBL-algebras are exactly the negative cones of ℓ-groups.*

Consequently, negative cones of ℓ-groups are equationally defined. As we mentioned, cancellative GBL-algebras in general (witout the asumption of integrality) include ℓ-groups as well. However, we will see that every cancellative GBL-algebra is the direct product of an ℓ-group and a negative cone of an ℓ-group. More generally, we will see that a similar decomposition exists for arbitrary GBL-algebras.

4.3 GMV-algebras

Recall that an *MV-algebra* is a commutative bounded residuated lattice that satisfies the identity $x \vee y = (x \to y) \to y$. This identity implies that the residuated lattice is also integral, divisible and representable, hence MV-algebras are examples of BL-algebras.

It turns out that MV-algrebras are intervals in abelian ℓ-groups. If $\mathbf{G} = (G, \wedge, \vee, \cdot, \backslash, /, 1)$ is an abelian ℓ-group and $a \leq 1$, then $\Gamma(\mathbf{G}, a) = ([a, 1], \wedge, \vee, \circ, \to, 1, a)$ is an MV-algebra, where $x \circ y = xy \vee a$, and $x \to y = x\backslash y \wedge 1$. (A version where $a \geq 1$ also yields an MV-algebra.) If $\mathbf{M}$ is an MV-algebra then there is an abelian ℓ-group $\mathbf{G}$ and an element $a \leq 1$ such that $\mathbf{M} \cong \Gamma(\mathbf{G}, a)$; see [Ch59], [Mu86].

Generalized MV-algebras, or *GMV-algebras*, are generalizations of MV-algebras in a similar way as GBL-algebras generalize BL-algebras, and are defined as residuated lattices that satisfy $x/((x \vee y)\backslash x) = x \vee y = (x/(x \vee y))\backslash x$. As with GBL-algebras, GMV-algebras have alternative characterizations.

LEMMA 8 [BCGJT] *A residuated lattice is a GMV algebra iff it satisfies* $x \leq y \Rightarrow y = x/(y\backslash x) = (x/y)\backslash x$.

THEOREM 9 [BCGJT] *Every GMV-algebra is a GBL-algebra.*

Proof. We make use of the quasi-equational formulation from the preceding lemma. Assume $x \leq y$ and let $z = y(y\backslash x)$. Note that $z \leq x$ and $y\backslash z \leq x\backslash z$, hence

$$\begin{array}{rcll} x\backslash z & = & ((y\backslash z)/(x\backslash z))\backslash(y\backslash z) & \\ & = & (y\backslash(z/(x\backslash z)))\backslash(y\backslash z) & \text{since } (u\backslash v)/w = u\backslash(v/w) \\ & = & (y\backslash x)\backslash(y\backslash z) & \text{since } z \leq x \Rightarrow x = z/(x\backslash z) \\ & = & (y(y\backslash x)\backslash z & \text{since } u\backslash(v\backslash w) = vu\backslash w \\ & = & z\backslash z. & \end{array}$$

Therefore $x = z/(x\backslash z) = z/(z\backslash z) = z$, as required. The proof of $x = (x/y)y$ is similar. ■

LEMMA 10 [BCGJT]

(i) *Every integral GBL-algebra satisfies the identity* $(y/x)\backslash(x/y) = x/y$ *and its opposite.*

(ii) *Every integral GMV-algebra satisfies the identity* $x/y \vee y/x = e$ *and its opposite.*

(iii) *Every integral GMV-algebra satisfies the identities* $x/(y \wedge z) = x/y \vee x/z$, $(x \vee y)/z = x/z \vee y/z$ *and the opposite ones.*

(iv) *Every commutative integral GMV-algebra is representable.*

It will be shown in Section 5 that the assumption of integrality in condition (iv) is not needed. They do not have to be intervals, but they are convex sublattices in ℓ-groups. More details can be found in [GT05], while the standard reference for MV-algebras is [CDM00].

Ordinal sum constructions and decompositions have been used extensively for ordered algebraic structures, and we recall here the definition for integral residuated lattices. Usually the ordinal sum of two posets $\mathbf{A}_0, \mathbf{A}_1$ is defined as the disjoint union with all elements of A_0 less than all elements of A_1 (and if $\mathbf{A}_0$ has a top and $\mathbf{A}_1$ has a bottom, these two elements are often identified). However, for most decomposition results on integral residuated lattices a slightly different point of view is to *replace* the element 1 of $\mathbf{A}_0$ by the algebra $\mathbf{A}_1$. The precise definition for an arbitrary number of summands is as follows.

Let I be a linearly ordered set, and for $i \in I$ let $\{\mathbf{A}_i : i \in I\}$ be a family of integral residuated lattices such that for all $i \neq j$, $\mathbf{A}_i \cap \mathbf{A}_j = \{1\}$ and 1 is join irreducible in $\mathbf{A}_i$. Then the *ordinal sum* $\bigoplus_{i\in I} \mathbf{A}_i$ is defined on the set $\bigcup_{i\in I} A_i$ by

$$x \cdot y = \begin{cases} x \cdot_i y & \text{if } x, y \in A_i \text{ for some } i \in I \\ x & \text{if } x \in A_i \setminus \{1\} \text{ and } y \in A_j \text{ where } i < j \\ y & \text{if } y \in A_i \setminus \{1\} \text{ and } x \in A_j \text{ where } i < j. \end{cases}$$

The partial order on $\bigoplus_{i\in I} \mathbf{A}_i$ is the unique partial order $\leq$ such that 1 is the top element, the partial order $\leq_i$ on $\mathbf{A}_i$ is the restriction of $\leq$ to $\mathbf{A}_i$, and if $i < j$ then every element of $\mathbf{A}_i \setminus \{1\}$ precedes every element of $\mathbf{A}_j$. Finally, the lattice operations and the residuals are uniquely determined by $\leq$ and the monoid operation.

It not difficult to check that this construction again yields an integral residuated lattice, and that it preserves divisibility and prelinearity. If $I = \{0, 1\}$ with $0 < 1$ then the ordinal sum is simply denoted by $\mathbf{A}_0 \oplus \mathbf{A}_1$. The assumption that 1 is join-irreducible can be omitted if $\mathbf{A}_1$ has a least element m, since if $x \wedge_0 y = 1$ in $\mathbf{A}_0$ then $x \wedge y$ still exists in $\mathbf{A}_0 \oplus \mathbf{A}_1$

and has value m. If 1 is join-reducible in $\mathbf{A}_0$ and if $\mathbf{A}_1$ has no minimum then the ordinal sum cannot be defined as above. However an "extended" ordinal sum may be obtained by taking the ordinal sum of $(\mathbf{A}_0 \oplus \mathbf{2}) \oplus \mathbf{A}_1$, where $\mathbf{2}$ is the 2-element MV-algebra.

The following representation theorem was proved by Agliano and Montagna [AM03].

THEOREM 11 *Every linearly ordered commutative integral GBL-algebra* $\mathbf{A}$ *can be represented as an ordinal sum* $\bigoplus_{i\in I} \mathbf{A}_i$ *of linearly ordered commutative integral GMV-algebras. Moreover* $\mathbf{A}$ *is a BL-algebra iff* I *has a minimum* i_0 *and* $\mathbf{A}_{i_0}$ *is bounded.*

Thus ordinal sums are a fundamental construction for BL-algebras, and commutative integral GMV-algebras are the building blocks.

4.4 Pseudo BL-algebras and pseudo MV-algebras

Pseudo BL-algebras [DGI02] [DGI02b] are divisible *prelinear* integral bounded residuated lattices, i.e., prelinear integral bounded GBL-algebras. Similarly, pseudo MV-algebras are integral bounded GMV-algebras (in this case prelinearity holds automatically). Hence BL-algebras and MV-algebras are exactly the commutative pseudo BL-algebras and pseudo MV-algebras respectively, but the latter do not have to be commutative. They do not have to be representable either, as prelinearity is equivalent to representability only under the assumption of commutativity. By a fundamental result of [Dv02], all pseudo MV-algebras are obtained from intervals of ℓ-groups, as with MV-algebras in the commutative case. Hence any nonrepresentable ℓ-group provides examples of nonrepresentable pseudo MV-algebras. The tight categorical connections between GMV-algebras and ℓ-groups (with a strong order unit) have produced new results and interesting research directions in both areas [Ho05], [DH07], [DH09].

The relationship between noncommutative t-norms and pseudo BL-algebras is investigated in [FGI01]. In particular it is noted that any continuous t-norm must be commutative. Hájek [Ha03a] shows that noncommutative pseudo BL-algebras can be constructed on "non-standard" unit intervals, and in [Ha03b] it is shown that all BL-algebras embed in such pseudo BL-algebras.

As for Boolean algebras, Heyting algebras and MV-algebras, the constant 0 can be used to define unary negation operations $-x = 0/x$ and $\sim x = x\backslash 0$. For pseudo BL-algebras, these operations need not coincide. Pseudo MV-algebras are *involutive residuated lattices* which means the negations satisfy $\sim -x = x = -\sim x$. However, for pseudo BL-algebras this identity need not hold, and for quite some time it was an open problem whether the weaker identity $\sim\!-x = -\!\sim\! x$ might also fail. Recently it was noted in [DGK09] that an example from [JM06] shows the identity can indeed fail, and a construction is given to show that there are uncountably many varieties of pseudo BL-algebras in which it does not hold.

We conclude this brief section with an important result on the structure of representable pseudo BL-algebras. Dvurečenskij [Dv07] proved that the Agliano-Montagna decomposition result extends to the non-commutative case.

THEOREM 12 *Every linearly ordered integral GBL-algebra* $\mathbf{A}$ *can be represented as an ordinal sum* $\bigoplus_{i\in I}\mathbf{A}_i$ *of linearly ordered integral GMV-algebras. Moreover* $\mathbf{A}$ *is a pseudo BL-algebra iff* I *has a minimum* i_0 *and* A_{i_0} *is bounded.*

As a consequence it is shown that representable pseudo BL-algebras satisfy the identity $\sim\!-x = -\!\sim\! x$, and that countably complete representable pseudo BL-algebras are commutative. Further results and references about pseudo BL-algebras can be found in [Ha03], [Dv07], [Io08] and [DGK09].

4.5 Hoops and pseudo-hoops

Hoops, originally introduced in [BO] by Büchi and Owens under an equivalent definition, can be defined as algebras $\mathbf{A} = (A, \cdot, \to, 1)$, where $(A, \cdot, 1)$ is a commutative monoid and the following identities hold:

$$x \to x = 1 \qquad x(x \to y) = y(y \to x) \qquad (xy) \to z = y \to (x \to z)$$

It is easy to see that the relation defined by $a \le b$ iff $1 = a \to b$ is a partial order and that $\mathbf{A}$ is a hoop iff $(A, \cdot, \to, 1, \le)$ is an integral residuated partially ordered monoid that satisfies $x(x \to y) = y(y \to x)$. Actually, if $\mathbf{A}$ is a hoop, then $(A, \le)$ admits a meet operation defined by $x \wedge y = x(x \to y)$. Consequently, every hoop satisfies divisibility. It turns out that not all hoops have a lattice reduct; the ones that do are exactly the join-free reducts of commutative integral GBL-algebras. Also, among those, the ones that satisfy prelinearity are exactly the reducts of BL-algebras and are known as *basic hoops*. If a hoop satisfies

$$(x \to y) \to y = (y \to x) \to x$$

then it admits a join given by $x \vee y = (x \to y) \to y$. Such hoops are known as *Wajsberg hoops* and as *Łukasiewicz hoops* and they are term equivalent to commutative integral GMV-algebras.

More on hoops and Wajsberg hoops can be found in [AFM07], [BF00], [BP94] and their references. Pseudo-hoops are the non-commutative generalizations of hoops. Their basic properties are studied in [GLP05]. The

RRL	**representable residuated lattices** = residuated lattices that are subdirect products of residuated chains = residuated lattices with $1 \leq u\backslash((x \vee y)\backslash x)u \vee v((x \vee y)\backslash y)/v$
GBL	**generalized BL-algebras** = divisible residuated lattices
GMV	**generalized MV-algebras** = residuated lattices with $x \vee y = x/((x \vee y)\backslash x) = (x/(x \vee y))\backslash x$
Fleas	= integral residuated lattices with prelinearity
GBH	**generalized basic hoops** = divisible integral residuated lattices
BH	**basic hoops** = commutative prelinear generalized basic hoops
WH	**Wajsberg hoops** = commutative integral generalized MV-algebras
LG	**lattice-ordered groups or ℓ-groups** = residuated lattices that satisfy $1 = x(x\backslash 1)$
RLG	**representable ℓ-groups** = ℓ-groups with $1 \leq (1\backslash x)yx \vee 1\backslash y$
CLG	**commutative ℓ-groups**
LG^-	**negative cones of lattice-ordered groups** = cancellative integral generalized BL-algebras
RLG^-	**negative cones of representable ℓ-groups** = cancellative integral representable generalized BL-algebras
CLG^-	**negative cones of commutative ℓ-groups** = cancellative basic hoops
Br	**Brouwerian algebras** = residuated lattices with $x \wedge y = xy$
RBr	**representable Brouwerian algebras** = Brouwerian algebras that satisfy prelinearity = basic hoops that are idempotent
GBA	**generalized Boolean algebras** = Brouwerian algebras with $x \vee y = (x\backslash y)\backslash y$ = Wajsberg hoops that are idempotent

Table 4. Some subvarieties of RL

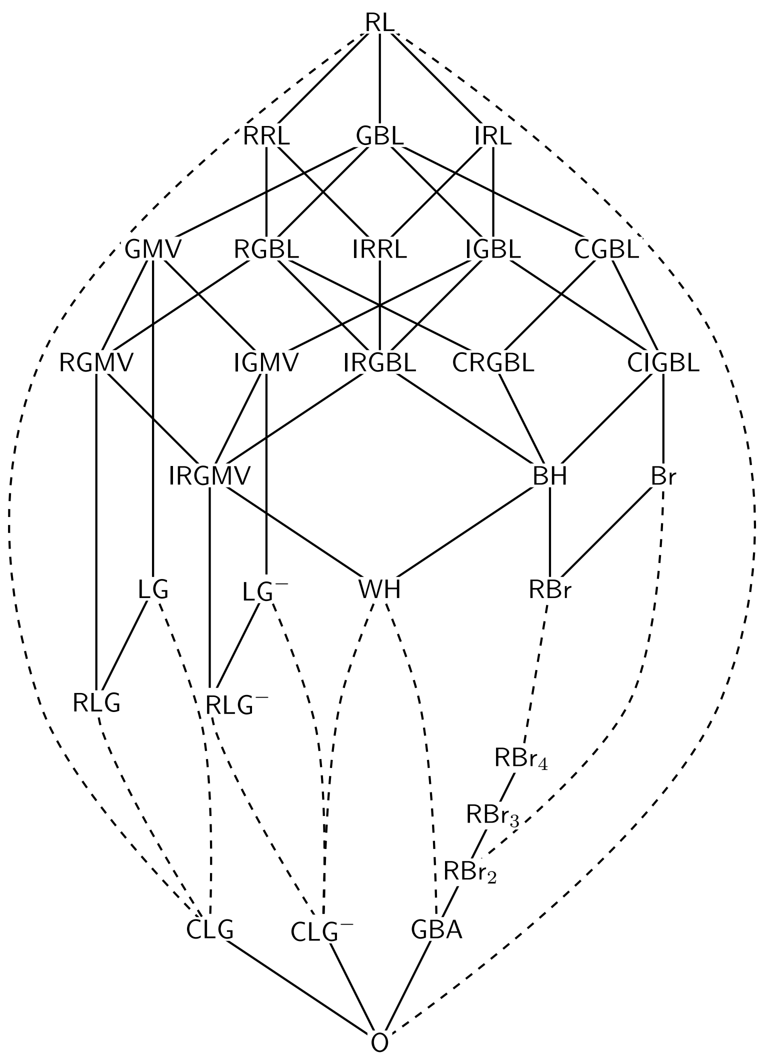

Figure 2. Some subvarieties of RL ordered by inclusion

join operation is definable in representable pseudo hoops, hence they are term-equivalent to integral representable GBL-algebras.

5 Decomposition of GBL-algebras

The structure of ℓ-groups has been studied extensively in the past decades. The structure of GMV-algebras has essentially been reduced to that of ℓ-groups (with a nucleus operation), and GMV-algebras are parts of ℓ-groups. The study of the structure of GBL-algebras has proven to be much more difficult. Nevertheless, the existing results indicate that ties to ℓ-groups exist. We first show that the study of GBL-algebras can be reduced to the integral case, by showing that every GBL-algebra is the direct product of an ℓ-group and an integral GBL-algebra. This result was proved in the dual setting of DRl-monoids in [Ko96] and independently in the setting of residuated lattices in [GT05]. With hindsight, the later result can, of course, be deduced by duality from the former. The presentation below follows [GT05].

An element a in a residuated lattice $\mathbf{L}$ is called *invertible* if $a(a\backslash 1) = 1 = (1/a)a$, and it is called *integral* if $1/a = a\backslash 1 = 1$. We denote the set of invertible elements of $\mathbf{L}$ by $G(\mathbf{L})$ and the set of integral elements by $I(\mathbf{L})$. Recall that a *positive* element a is an element that satisfies $a \geq 1$.

Note that a is invertible if and only if there exists an element a^{-1} such that $aa^{-1} = 1 = a^{-1}a$. In this case $a^{-1} = 1/a = a\backslash 1$. It is easy to see that multiplication by an invertible element is an order automorphism.

LEMMA 13 [GT05] *Let $\mathbf{L}$ be a GBL-algebra.*

(i) *Every positive element of $\mathbf{L}$ is invertible.*

(ii) *$\mathbf{L}$ satisfies the identities $x/x = x\backslash x = 1$.*

(iii) *$\mathbf{L}$ satisfies the identity $1/x = x\backslash 1$.*

(iv) *For all $x, y \in L$, if $x \vee y = 1$ then $xy = x \wedge y$.*

(v) *$\mathbf{L}$ satisfies the identity $x = (x \vee 1)(x \wedge 1)$.*

Proof. For (i), note that if a is a positive element in L, $a(a\backslash 1) = 1 = (1/a)a$, by definition; that is, a is invertible. For (ii), we argue as is the proof of Lemma 4.

By (ii) and residuation, we have $x(1/x) \leq x/x = 1$, hence $1/x \leq x\backslash 1$. Likewise, $x\backslash 1 \leq 1/x$.

For (iv), we have $x = x/1 = x/(x \vee y) = x/x \wedge x/y = 1 \wedge x/y = y/y \wedge x/y = (y \wedge x)/y$. So, $xy = ((x \wedge y)/y)y = x \wedge y$.

Finally, by Lemma 4, $(1/x \wedge 1)x = x \wedge 1$. Moreover, by (i) $x \vee 1$ is invertible and $(x \vee 1)^{-1} = 1/(x \vee 1) = 1/x \wedge 1$. Thus, $(x \vee 1)^{-1}x = x \wedge 1$, or $x = (x \vee 1)(x \wedge 1)$. ∎

The following theorem shows that if $\mathbf{L}$ is a GBL-algebra then the sets $G(\mathbf{L})$ and $I(\mathbf{L})$ are subuniverses of $\mathbf{L}$. We denote the corresponding subalgebras by $\mathbf{G}(\mathbf{L})$ and $\mathbf{I}(\mathbf{L})$.

THEOREM 14 [GT05] *Every GBL-algebra* $\mathbf{L}$ *is isomorphic to* $\mathbf{G}(\mathbf{L}) \times \mathbf{I}(\mathbf{L})$.

Proof. We begin with a series of claims.
Claim 1: $G(\mathbf{L})$ is a subuniverse of $\mathbf{L}$.

Let x, y be invertible elements. It is clear that xy is invertible. Additionally, for all $x, y \in G(\mathbf{L})$ and $z \in L$, $z \leq x^{-1}y \Leftrightarrow xz \leq y \Leftrightarrow z \leq x\backslash y$. It follows that $x\backslash y = x^{-1}y$, hence $x\backslash y$ is invertible. Likewise, $y/x = yx^{-1}$ is invertible.

Moreover, $x \vee y = (xy^{-1} \vee 1)y$. So, $x \vee y$ is invertible, since every positive element is invertible, by Lemma 13(i), and the product of two invertible elements is invertible. By properties of residuation, $x \wedge y = 1/(x^{-1} \vee y^{-1})$, which is invertible, since we have already shown that $G(\mathbf{L})$ is closed under joins and the division operations.
Claim 2: $I(\mathbf{L})$ is a subuniverse of $\mathbf{L}$.

Note that every integral element a is negative, since $1 = 1/a$ implies $1 \leq 1/a$ and $a \leq 1$. For $x, y \in I(\mathbf{L})$, we get:

$$1/xy = (1/y)/x = 1/x = 1, \text{ so } xy \in I(\mathbf{L}).$$

$$1/(x \vee y) = 1/x \wedge 1/y = 1, \text{ so } x \vee y \in I(\mathbf{L}).$$

$$1 \leq 1/x \leq 1/(x \wedge y) \leq 1/xy = 1, \text{ so } x \wedge y \in I(\mathbf{L}).$$

$$1 = 1/(1/y) \leq 1/(x/y) \leq 1/(x/1) = 1/x = 1, \text{ so } x/y \in I(\mathbf{L}).$$

Claim 3: For every $g \in (G(\mathbf{L}))^-$ and every $h \in I(\mathbf{L})$, $g \vee h = 1$.

Let $g \in (G(\mathbf{L}))^-$ and $h \in I(\mathbf{L})$. We have $1/(g \vee h) = 1/g \wedge 1/h = 1/g \wedge 1 = 1$, since $1 \leq 1/g$. Moreover, $g \leq g \vee h$, so $1 \leq g^{-1}(g \vee h)$. Thus, by the GBL-algebra identities and properties of residuation

$$\begin{aligned} 1 &= (1/[g^{-1}(g \vee h)])[g^{-1}(g \vee h)] \\ &= ([1/(g \vee h)]/g^{-1})g^{-1}(g \vee h) \\ &= (1/g^{-1})g^{-1}(g \vee h) \\ &= gg^{-1}(g \vee h) \\ &= g \vee h. \end{aligned}$$

Claim 4: For every $g \in (G(\mathbf{L}))^-$ and every $h \in I(\mathbf{L})$, $gh = g \wedge h$.

In light of Lemma 13(iv), $g^{-1}h = (g^{-1}h \vee 1)(g^{-1}h \wedge 1)$. Multiplication by g yields $h = (h \vee g)(g^{-1}h \wedge 1)$. Using Claim 3, we have $gh = g(g^{-1}h \wedge 1) = h \wedge g$, since multiplication by an invertible element is an order automorphism.

Claim 5: For every $g \in G(\mathbf{L})$ and every $h \in I(\mathbf{L})$, $gh = hg$.

The statement is true if $g \leq 1$, by Claim 4. If $g \geq 1$ then $g^{-1} \leq 1$, thus $g^{-1}h = hg^{-1}$, hence $hg = gh$. For arbitrary g, note that both $g \vee 1$ and $g \wedge 1$ commute with h. Using Lemma 13(iv), we get $gh = (g \vee 1)(g \wedge 1)h = (g \vee 1)h(g \wedge 1) = h(g \vee 1)(g \wedge 1) = hg$.

Claim 6: For every $x \in L$, there exist $g_x \in G(\mathbf{L})$ and $h_x \in I(\mathbf{L})$, such that $x = g_x h_x$.

By Lemma 13(iv), $x = (x \vee 1)(x \wedge 1)$. Since $1 \leq x \vee 1$ and $1 \leq 1/(x \wedge 1)$, by Lemma 13(i), these elements are invertible. Set $g_x = (x \vee 1)(1/(x \wedge 1))^{-1}$ and $h_x = (1/(x \wedge 1))(x \wedge 1)$. It is clear that $x = g_x h_x$, g_x is invertible and h_x is integral.

Claim 7: For every $g_1, g_2 \in G(\mathbf{L})$ and $h_1, h_2 \in I(\mathbf{L})$, $g_1 h_1 \leq g_2 h_2$ if and only if $g_1 \leq g_2$ and $h_1 \leq h_2$.

For the non-trivial direction we have

$$g_1 h_1 \leq g_2 h_2 \Rightarrow g_2^{-1} g_1 h_1 \leq h_2 \Rightarrow g_2^{-1} g_1 \leq h_2/h_1 \leq e \Rightarrow g_1 \leq g_2.$$

Moreover,

$$\begin{aligned} g_2^{-1} g_1 \leq h_2/h_1 \quad & \Rightarrow e \leq g_1^{-1} g_2 (h_2/h_1) \\ & \Rightarrow 1 = [1/g_1^{-1} g_2 (h_2/h_1)] g_1^{-1} g_2 (h_2/h_1) \\ & \Rightarrow 1 = [(1/(h_2/h_1))/g_1^{-1} g_2] g_1^{-1} g_2 (h_2/h_1) \\ & \Rightarrow 1 = g_2^{-1} g_1 g_1^{-1} g_2 (h_2/h_1) \\ & \Rightarrow 1 = h_2/h_1 \\ & \Rightarrow h_1 \leq h_2. \end{aligned}$$

By Claims 1 and 2, $\mathbf{G}(\mathbf{L})$ and $\mathbf{I}(\mathbf{L})$ are subalgebras of $\mathbf{L}$. Define $f : \mathbf{G}(\mathbf{L}) \times \mathbf{I}(\mathbf{L}) \to \mathbf{L}$ by $f(g, h) = gh$. We will show that f is an isomorphism. It is onto by Claim 6 and an order isomorphism by Claim 7. So, it is a lattice isomorphism, as well. To verify that f preserves the other operations note that $gg'hh' = ghg'h'$, for all $g, g' \in G(\mathbf{L})$ and $h, h' \in I(\mathbf{L})$, by Claim 5. Moreover, for all $g, g', \bar{g} \in G(\mathbf{L})$ and $h, h', \bar{h} \in I(\mathbf{L})$, $\bar{g}\bar{h} \leq gh/g'h'$ if and only if $\bar{g}\bar{h}g'h' \leq gh$. By Claim 5, this is equivalent to $\bar{g}g'\bar{h}h' \leq gh$, and, by Claim 7, to $\bar{g}g' \leq g$ and $\bar{h}h' \leq h$. This is in turn equivalent to $\bar{g} \leq g/g'$ and $\bar{h} \leq h/h'$, which is equivalent to $\bar{g}\bar{h} \leq (g/g')(h/h')$ by Claim 7. Thus, for all $g, g' \in G(\mathbf{L})$ and $h, h' \in I(\mathbf{L})$, $gh/g'h' = (g/g')(h/h')$ and, likewise, $g'h' \backslash gh = (g' \backslash g)(h' \backslash h)$. ■

COROLLARY 15 *Every GBL-algebra is the direct product of an ℓ-group and an integral GBL-algebra.*

Combining this with Theorem 7 immediately gives the following result.

COROLLARY 16 *Every cancellative GBL-algebra is the direct product of an ℓ-group and the negative cone of an ℓ-group.*

6 Further results on the structure of GBL-algebras

In this section we briefly summarize results from a series of papers [JM06] [JM09] [JM10]. The collaboration that lead to these results started when the second author and Franco Montagna met at a wonderful ERCIM workshop on Soft Computing, organized by Petr Hájek in Brno, Czech Republic, in 2003. This is yet another example how Petr Hájek's dedication to the field of fuzzy logic has had impact far beyond his long list of influential research publications.

6.1 Finite GBL-algebras are commutative

LEMMA 17 *If a is an idempotent in an integral GBL-algebra A, then $ax = a \wedge x$ for all $x \in A$. Hence every idempotent is central, i.e. commutes with every element.*

Proof. Suppose $aa = a$. Then $ax \leq a \wedge x = a(a\backslash x) = aa(a\backslash x) = a(a \wedge x) \leq ax$. ■

In an ℓ-group only the identity is an idempotent, hence it follows from the decomposition result mentioned above that idempotents are central in all GBL-algebras.

In fact, using this lemma, it is easy to see that the set of idempotents in a GBL-algebra is a sublattice that is closed under multiplication. In [JM06] it is proved that this set is also closed under the residuals.

THEOREM 18 *The idempotents in a GBL-algebra* $\mathbf{A}$ *form a subalgebra, which is the largest Brouwerian subalgebra of* $\mathbf{A}$.

By the results of the preceding section, the structure of any GBL-algebra is determined by the structure of its ℓ-group factor and its integral GBL-algebra factor. Since only the trivial ℓ-group is finite, it follows that any finite GBL-algebra is integral. A careful analysis of one-generated subalgebras in a GBL-algebra gives the following result.

THEOREM 19 [JM06] *Every finite GBL-algebra and every finite pseudo-BL-algebra is commutative.*

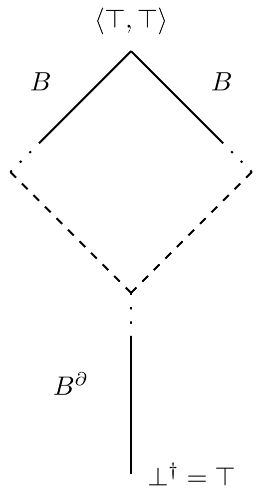

Figure 3.

Since there exist noncommutative GBL-algebras, such as any noncommutative ℓ-group, the next result is immediate.

COROLLARY 20 *The varieties* GBL *and* psBL *are not generated by their finite members, and hence do not have the finite model property.*

As mentioned earlier, BL-algebras are subdirect products of ordinal sums of commutative integral GMV-chains, and a similar result holds without commutativity for representable GBL-algebras. So it is natural to ask to what extent GBL-algebras are determined by ordinal sums of GMV-algebras and Heyting algebras. We briefly recall a construction of a GBL-algebra that does not arise from these building blocks.

For a residuated lattice $\mathbf{B}$ with top element $\top$, let $\mathbf{B}^{\partial}$ denote the dual poset of the lattice reduct of $\mathbf{B}$. Now we define $\mathbf{B}^{\dagger}$ to be the ordinal sum of $\mathbf{B}^{\partial}$ and $\mathbf{B}\times\mathbf{B}$, i.e., every element of $\mathbf{B}^{\partial}$ is below every element of $\mathbf{B}\times\mathbf{B}$ (see Figure 3). Note that $\mathbf{B}^{\dagger}$ has $\top$ as bottom element, so to avoid confusion, we denote this element by $\bot^{\dagger}$. A binary operation $\cdot$ on $\mathbf{B}^{\dagger}$ is defined as follows:

$$\begin{array}{ll} \langle a,b\rangle \cdot \langle c,d\rangle & = \langle ac, bd\rangle \\ \langle a,b\rangle \cdot u & = u/a \\ u \cdot \langle a,b\rangle & = b\backslash u \\ u \cdot v & = \top = \bot^{\dagger} \end{array}$$

Note that even if $\mathbf{B}$ is a commutative residuated lattice, $\cdot$ is in general noncommutative.

LEMMA 21

- *For any residuated lattice $\mathbf{B}$ with top element, the algebra $\mathbf{B}^\dagger$ defined above is a bounded residuated lattice.*
- *If $\mathbf{B}$ is nontrivial, then $\mathbf{B}^\dagger$ is not a GMV-algebra, and if $\mathbf{B}$ is subdirectly irreducible, so is $\mathbf{B}^\dagger$.*
- *$\mathbf{B}^\dagger$ is a GBL-algebra if and only if $\mathbf{B}$ is a cancellative GBL-algebra.*

6.2 The Blok-Ferreirim decomposition result for normal GBL-algebras

Building on work of Büchi and Owens [BO], Blok and Ferreirim [BF00] proved the following result.

PROPOSITION 22 *Every subdirectly irreducible hoop is the ordinal sum of a proper subhoop $\mathbf{H}$ and a subdirectly irreducible nontrivial Wajsberg hoop $\mathbf{W}$.*

This result was adapted to BL-algebras in [AM03]. To discuss further extensions to GBL-algebras we first recall some definitions about filters and congruences in residuated lattices.

A *filter* of a residuated lattice $\mathbf{A}$ is an upward closed subset F of $\mathbf{A}$ which contains 1 and is closed under the meet and the monoid operation. A filter F is said to be *normal* if $aF = Fa$ for all $a \in A$, or equivalently if $a\backslash(xa) \in F$ and $(ax)/a \in F$ whenever $x \in F$ and $a \in A$. A residuated lattice is said to be *normal* if every filter of it is a normal filter. A residuated lattice is said to be *n-potent* if it satisfies $x^{n+1} = x^n$, where $x^n = x \cdot \ldots \cdot x$ (n times). Note that n-potent GBL-algebras are normal ([JM09]).

In every residuated lattice, the lattice of normal filters is isomorphic to the congruence lattice: to any congruence θ one associates the normal filter $F_\theta = \uparrow\{x : (x,1) \in \theta\}$. Conversely, given a normal filter F, the set θ_F of all pairs (x,y) such that $x\backslash y \in F$ and $y\backslash x \in F$ is a congruence such that the upward closure of the congruence class of 1 is F. Hence a residuated lattice is subdirectly irreducible if and only if it has a smallest nontrivial normal filter.

In [JM09] the following result is proved.

THEOREM 23 (i) *Every subdirectly irreducible normal integral GBL-algebra is the ordinal sum of a proper subalgebra and a non-trivial integral subdirectly irreducible GMV-algebra.*

(ii) *Every n-potent GBL-algebra is commutative and integral.*

A variety $\mathcal{V}$ has the *finite embeddability property (FEP)* iff every finite partial subalgebra of an algebra in $\mathcal{V}$ partially embeds into a finite algebra of $\mathcal{V}$. The FEP is stronger than the finite model property: for a finitely axiomatized variety $\mathcal{V}$, the finite model property implies the decidability of the equational theory of $\mathcal{V}$, while the FEP implies the decidability of the universal theory of $\mathcal{V}$. With the help of the ordinal sum decomposition, the following result is proved in [JM09].

THEOREM 24 *The variety of commutative and integral GBL-algebras has the FEP.*

However, by interpreting the quasiequational theory of ℓ-groups into that of GBL-algebras, it is shown that without the assumption of commutativity the quasiequational theory of GBL-algebras is undecidable. The decidability of the *equational* theory of GBL-algebras is currently an open problem.

6.3 Poset products

The success of ordinal sum decompositions for subclasses of GBL-algebras has prompted the use of some generalizations to obtain representation and embedding theorems for larger subclasses. The *poset product* uses a partial order on the index set to define a subset of the direct product. Specifically, let $\mathbf{I} = (I, \leq)$ be a poset, assume $\{\mathbf{A}_i : i \in I\}$ is a family of residuated lattices, and that for nonmaximal $i \in I$ each $\mathbf{A}_i$ is integral, and for nonminimal $i \in I$ each $\mathbf{A}_i$ has a least element denoted by 0. The poset product of $\{A_i : i \in I\}$ is

$$\prod_{\mathbf{I}} A_i = \{f \in \prod_{i \in I} A_i : f(i) = 0 \text{ or } f(j) = 1 \text{ for all } i < j \text{ in } I\}.$$

The monoid operation and the lattice operations are defined pointwise. The residuals are defined by

$(f\backslash g)(i) = f(i)\backslash_i g(i)$ if $f(j) \leq_j g(j)$ for all $j > i$, and 0 otherwise,

$(f/g)(i) = f(i)/_i g(i)$ if $f(j) \leq_j g(j)$ for all $j > i$, and 0 otherwise.

The poset product is distinguished visually from the direct product since the index set is a poset $\mathbf{I}$ rather than just an index set I. If $\mathbf{I}$ is an antichain then the poset product reduces to the direct product, and if $\mathbf{I}$ is a finite chain, the poset product gives the ordinal sum over the reverse order of $\mathbf{I}$. In [JM09] the following is proved.

THEOREM 25 (i) *The poset product of a collection of residuated lattices is a residuated lattice, which is integral (divisible, bounded respectively) when all factors are integral (divisible, bounded respectively).*

(ii) *Every finite GBL-algebra can be represented as the poset product of a finite family of finite MV-chains.*

The next result, from [JM10], extend this to larger classes of GBL-algebras, but in this case one only gets an embedding theorem.

THEOREM 26 *Every n-potent GBL-algebra embeds into the poset product of a family of finite and n-potent MV-chains.*

Every normal GBL-algebra embeds into a poset product of linearly ordered integral bounded GMV-algebras and linearly ordered ℓ-groups.

For Heyting algebras the above theorem reduces to the well-known embedding theorem into the complete Heyting algebra of all upward closed subsets of some poset.

Various properties can be imposed on poset products to obtain embedding theorems for other subclasses of GBL-algebras. The follow result from [JM10] collects several of them.

THEOREM 27 *A GBL-algebra is*

- *a BL-algebra iff it is isomorphic to a subalgebra $\mathbf{A}$ of a poset product $\bigotimes_{i\in\mathbf{I}}\mathbf{A}_i$ such that*

  (a) *each $\mathbf{A}_i$ is a linearly ordered MV-algebra,*

  (b) *$\mathbf{I}=(I,\leq)$ is a* root system*, i.e. every principal filter of $\mathbf{I}$ is linearly ordered, and*

  (c) *the function on I which is constantly equal to 0 is in $\mathbf{A}$;*

- *an MV-algebra iff it is isomorphic to a subalgebra $\mathbf{A}$ of a poset product $\bigotimes_{i\in\mathbf{I}}\mathbf{A}_i$ such that conditions* (a) *and* (c) *above hold and*

  (d) *$\mathbf{I}=(I,\leq)$ is a poset such that $\leq$ is the identity on I;*

- *representable iff it is embeddable into a poset product $\bigotimes_{i\in\mathbf{I}}\mathbf{A}_i$ such that each $\mathbf{A}_i$ is a linearly ordered GMV-algebra and* (b) *holds;*

- *an abelian ℓ-group iff it is embeddable into a poset product $\bigotimes_{i\in\mathbf{I}}\mathbf{A}_i$ such that each $\mathbf{A}_i$ is a linearly ordered abelian ℓ-group and condition* (d) *above holds;*

- *n-potent iff it is embeddable into a poset product of linearly ordered n-potent MV-algebras;*

- *a Heyting algebra iff it is isomorphic to a subalgebra $\mathbf{A}$ of a poset product $\bigotimes_{i\in\mathbf{I}}\mathbf{A}_i$ where condition* (c) *holds and in addition*

  (e) *every $\mathbf{A}_i$ is the two-element MV-algebra;*

- *a Gödel algebra iff it is isomorphic to a subalgebra* $\mathbf{A}$ *of a poset product* $\bigotimes_{i\in\mathbf{I}}\mathbf{A}_i$ *where* (b), (c) *and* (e) *hold;*
- *a Boolean algebra iff it is isomorphic to a subalgebra* $\mathbf{A}$ *of a poset product* $\bigotimes_{i\in\mathbf{I}}\mathbf{A}_i$ *where* (c), (d) *and* (e) *hold.*

Of course this survey covers only some of the highlights of a few research papers concerned with GBL-algebras and related classes. Volumes have been written about BL-algebras, MV-algebras, ℓ-groups and Heyting algebras, as well as about many other algebras that satisfy divisibility, and the reader is encouraged to explore the literature further, starting with the references below.

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