The lattice of varieties generated by small residuated lattices

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Outline

- Lattice of finitely generated CD varieties
- The HS order on finite subdirectly irreducibles
- Computing finite residuated lattices
- Using automated theorem provers
- Amalgamation in residuated lattices
A residuated lattice \((A, \vee, \wedge, \cdot, 1, \backslash, /)\) is an algebra where \((A, \vee, \wedge)\) is a lattice, \((A, \cdot, 1)\) is a monoid and for all \(x, y, z \in A\)

\[ x \cdot y \leq z \iff y \leq x \backslash z \iff x \leq z / y \]

Residuated lattices generalize many algebras related to logic, e.g. Boolean algebras, Heyting algebras, MV-algebras, Hajek’s basic logic algebras, linear logic algebras, ... 

\(\text{FL} = \text{Full Lambek calculus} = \text{the starting point for substructural logics}\)

corresponds to class FL of all residuated lattices with a new constant 0

Extensions of \(\text{FL}\) correspond to subvarieties of FL
Hiroakira Ono
(California, September 2006)
[1985] *Logics without the contraction rule*
   (with Y. Komori)
Provides a **framework** for studying many substructural logics, relating sequent calculi with semantics
The name **substructural logics** was suggested by K. Dozen, October 1990
[2007] *Residuated Lattices: An algebraic glimpse at substructural logics* (with Galatos, J., Kowalski)
Some propositional logics extending FL
A class $\mathcal{V}$ of algebras is a **variety** if it is defined by **identities**

$$\iff \mathcal{V} = \text{HSP}(\mathcal{K})$$ for some class $\mathcal{K}$ of algebras

$\mathcal{V}$ is **finitely generated** if $\mathcal{K}$ can be a **finite** class of **finite algebras**

An algebra is **congruence distributive** (CD) if its lattice of congruences is distributive

A class $\mathcal{V}$ of algebras is CD if every member is CD
Who is this?

Bjarni Jónsson

(AMS-MAA meeting in Madison, WI 1968)

Algebras whose congruence lattices are distributive [1967]

* Jónsson’s Lemma implies that the lattice of subvarieties of a CD variety is distributive
* The completely join-irreducibles in this lattice are generated by a single s. i. algebra
* for finite algebras $A, B$

$$\text{HSP}\{A\} \subseteq \text{HSP}\{B\} \iff A \in \text{HS}\{B\}$$
The relation $A \in HS\{B\}$ is a preorder on algebras (since $SH \leq HS$)

For finite s. i. algebras in a CD variety it is a partial order

Called the HS-poset of the variety

The lattice of finitely generated subvarieties is given by downsets in this poset
The HS-poset of MV-algebras

Komori [1981] *Super-Lukasiewicz propositional logics*
Computing finite residuated lattices

First compute all lattices with \( n \) elements

[J. and N. Lawless 2013]: there are \( 1\,901\,910\,625\,578 \) for \( n = 19 \)

Then compute all lattice-ordered \( z \)-monoids over each lattice

For residuated lattices there are \( 295\,292 \) for \( n = 8 \)

[Belohlavek and Vychodil 2010]: \( 30\,653\,419 \) CIRL of size \( n = 12 \)

Remove non-s. i. algebras from list (very few)

Compute maximal proper subalgebras of each algebra

Compute maximal homomorphic images (\( = \)minimal congruences)
A small sample

<table>
<thead>
<tr>
<th>n</th>
<th>RL</th>
<th>Chn</th>
<th>DE</th>
<th>F</th>
<th>M</th>
<th>N</th>
<th>FL</th>
<th>Chn</th>
<th>DE</th>
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<th>N</th>
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<td>15</td>
<td>20</td>
<td>11</td>
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</table>
### Residuated lattices of size $\leq 4$

<table>
<thead>
<tr>
<th>RL var</th>
<th>FL var</th>
<th>Name, id, transformations</th>
<th>Sub</th>
<th>Hom</th>
</tr>
</thead>
<tbody>
<tr>
<td>GBA</td>
<td>BA</td>
<td>$\langle 2_1, 1 \rangle$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>WH</td>
<td>MV</td>
<td>$\langle 3_1, 2, 01 \rangle$</td>
<td>2_1</td>
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</tr>
<tr>
<td>RBr</td>
<td>GA</td>
<td>$\langle 3_2, 2, 11 \rangle$</td>
<td>2_1</td>
<td>2_1</td>
</tr>
<tr>
<td>CRRL</td>
<td>RInFL_e</td>
<td>$\langle 3_3, 1, 22 \rangle$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>WH</td>
<td>MV</td>
<td>$\langle 4_1, 3, 001, 012 \rangle$</td>
<td>2_1</td>
<td></td>
</tr>
<tr>
<td>BH</td>
<td>BL</td>
<td>$\langle 4_2, 3, 011, 122 \rangle$; $\langle 4_3, 3, 111, 112 \rangle$</td>
<td>3_1 3_2</td>
<td>3_1 2_1</td>
</tr>
<tr>
<td>RBr</td>
<td>GA</td>
<td>$\langle 4_4, 3, 111, 122 \rangle$</td>
<td>3_2</td>
<td>3_2</td>
</tr>
<tr>
<td>CIRRL</td>
<td>RInFL_{ew}</td>
<td>$\langle 4_5, 3, 001, 022 \rangle$</td>
<td>2_1</td>
<td>2_1</td>
</tr>
<tr>
<td>CIRRL</td>
<td>RFL_{ew}</td>
<td>$\langle 4_6, 3, 001, 002 \rangle$</td>
<td>3_1</td>
<td></td>
</tr>
<tr>
<td>IRRL</td>
<td>RFL_w</td>
<td>$\langle 4_7, 3, 001, 122 \rangle$; $\langle 4_8, 3, 011, 022 \rangle$</td>
<td>2_1</td>
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</tr>
<tr>
<td>CRRL</td>
<td>RInFL_e</td>
<td>$\langle 4_9, 1, 233, 333 \rangle$; $\langle 4_{10}, 2, 113, 333 \rangle$</td>
<td>3_3</td>
<td>2_1 3_3</td>
</tr>
<tr>
<td>CRRL</td>
<td>RFL_e</td>
<td>$\langle 4_{11}, 1, 223, 333 \rangle$; $\langle 4_{12}, 2, 011, 133 \rangle$; $\langle 4_{13}, 2, 111, 133 \rangle$</td>
<td>3_3</td>
<td></td>
</tr>
<tr>
<td>RRL</td>
<td>RFL</td>
<td>$\langle 4_{14}, 2, 111, 333 \rangle$; $\langle 4_{15}, 2, 113, 133 \rangle$</td>
<td></td>
<td>2_1</td>
</tr>
<tr>
<td>GBA</td>
<td>BA</td>
<td>$\langle D_{1}, 3, 101, 022 \rangle$</td>
<td>2_1</td>
<td>2_1</td>
</tr>
<tr>
<td>CDRL</td>
<td>DInFL_e</td>
<td>$\langle D_{2}, 1, 1, 202, 323 \rangle$; $\langle D_{3}, 1, 1, 213, 333 \rangle$; $\langle D_{4}, 2, 1, 233, 333 \rangle$</td>
<td>3_3</td>
<td></td>
</tr>
<tr>
<td>CDRL</td>
<td>DFL_e</td>
<td>$\langle D_{5}, 1, 222, 323 \rangle$</td>
<td>3_3</td>
<td></td>
</tr>
</tbody>
</table>
HS-poset of residuated lattices with $\leq 4$ elements
The Amalgamation Property

Let $\mathcal{K}$ be a class of mathematical structures

(e. g. sets, groups, residuated lattices, ...)

Usually there is a natural notion of morphism for $\mathcal{K}$

(e. g. function, homomorphism, ...)

$\mathcal{K}$ has the **amalgamation property** if

for all $A, B, C \in \mathcal{K}$ and all **injective** $f : A \hookrightarrow B$, $g : A \hookrightarrow C$

there exists $D \in \mathcal{K}$ and **injective** $h : B \hookrightarrow D$, $k : C \hookrightarrow D$ such that

$$h \circ f = k \circ g$$
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Connections with logic

Bill Craig
(Berkeley, CA 1977)

Craig interpolation theorem [1957]

If $\phi \implies \psi$ is true in first order logic
then there exists $\theta$ containing only
the relation symbols in both $\phi, \psi$
such that $\phi \implies \theta$ and $\theta \implies \psi$

Also true for many other logics, including classical propositional
logic and intuitionistic propositional logic

Let $\mathcal{K}$ be a class of algebras of an algebraizable logic $\mathcal{L}$

Then $\mathcal{K}$ has the (strong/super) amalgamation property iff $\mathcal{L}$
satisfies the Craig interpolation property
What is known?

There are two versions: 1. the **amalgamation property** (AP)

for all $A, B, C \in \mathcal{K}$ and all injective $f : A \hookrightarrow B$, $g : A \hookrightarrow C$

there exists $D \in \mathcal{K}$ and injective $h : B \hookrightarrow D$, $k : C \hookrightarrow D$ such that

$$h \circ f = k \circ g$$

2. the **strong amalgamation property** (SAP): in addition to

$h \circ f = k \circ g$ also require $h[f[A]] = h[B] \cap k[C]$

**Equivalently:** If $A$ is a subalgebra of $B$, $C$ in $\mathcal{K}$ and $A = B \cap C$
then there exists $D \in \mathcal{K}$ such that $B, C$ are subalgebras of $D$
A sample of what is known

These categories have the strong amalgamation property:

Sets
Groups [Schreier 1927]
Sets with any binary operation [Jónsson 1956]
Variety of all algebras of a fixed signature
Partially ordered sets [Jónsson 1956]
Lattices [Jónsson 1956]

These categories only have the amalgamation property:

Distributive lattices [Pierce 1968]
Abelian lattice-ordered groups [Pierce 1972]

These categories fail to have the amalgamation property:

Semigroups [Kimura 1957]
Lattice-ordered groups [Pierce 1972]
Kiss, Márki, Pröhle and Tholen [1983] Categorical algebraic properties. A compendium on amalgamation, congruence extension, epimorphisms, residual smallness and injectivity

They summarize some general techniques for establishing these properties

They give a table with known results for 100 categories

Day and Jezek [1984] The only lattice varieties that satisfy AP are the trivial variety, the variety of distributive lattices and the variety of all lattices

Busianiche and Montagna [2011]: Amalgamation, interpolation and Beth’s property in BL (Section 6 in Handbook of Mathematical Fuzzy Logic)

Metcalfe, Montagna and Tsinakis [2014]: Amalgamation and interpolation in ordered algebras, Journal of Algebra
How to prove/disprove the AP

Look at three examples:

1. Why does SAP hold for class of all Boolean algebras?
2. Why does AP hold for distributive lattices?
3. Why does AP fail for distributive residuated lattices?

1. Boolean algebras (BA) can be embedded in complete and atomic Boolean algebras (caBA)

\[ A \xrightarrow{g} C \xleftarrow{g^\sigma} C^\sigma \xrightarrow{k} D \]

\[ A^\sigma \xrightarrow{f} B \xleftarrow{f^\sigma} B^\sigma \]

caBA is dually equivalent to Set
Amalgamation for BA

So we need to fill in the following dual diagram in $\text{Set}$

$\xymatrix{ & Uf(B) \ar[dl]_h \ar[dr]^{Uf(f)} & \\
Uf(B) \times Uf(C) \ar[dl]_k \ar[dr] & & Uf(A) \\
Uf(C) \ar[ul] & & \ar[ul]_\ \ Uf(g) }$

Can take $P$ to be the pullback, so

$P = \{(b, c) \in Uf(B) \times Uf(C) : Uf(f)(b) = Uf(g)(c)\}$

Then $h = \pi_1|_P$ and $k = \pi_2|_P$

$h$ is surjective since for all $b \in Uf(B)$, there exists $c \in Uf(C)$ s.t.

$Uf(f)(b) = Uf(g)(c)$ because $Uf(g)$ is surjective

Similarly $k$ is surjective
2. Amalgamation for distributive lattices

**Theorem** [J. and Rose 1989]: Let $\mathcal{V}$ be a congruence distributive variety whose members have one-element subalgebras, and assume that $\mathcal{V}$ is generated by a finite simple algebra that has no proper nontrivial subalgebras. Then $\mathcal{V}$ has the amalgamation property.

The variety of distributive lattices is generated by the two-element lattice, which is simple and has only trivial proper subalgebras, hence **AP holds**.

**Corollary:** The AP holds for the variety of Sugihara algebras ($= V(3_3)$), and for $V(4_{12})$, $V(4_{14})$, $V(4_{15})$ as well as for any variety generated by an atom of the HS-poset.
Finally we get to mention some **computational tools**

To **disprove** AP or SAP, we essentially want to search for 3 **small** models $A, B, C$ in $\mathcal{K}$ such that $A$ is a **submodel** of both $B$ and $C$

We use the **Mace4 model finder** from Bill McCune [2009] to enumerate nonisomorphic models $A_1, A_2, \ldots$ in a **finitely axiomatized** first-order theory $\Sigma$

For each $A_i$ we construct the **diagram** $\Delta_i$ and use Mace4 again to find all **nonisomorphic** models $B_1, B_2, \ldots$ of $\Delta_i \cup \Sigma \cup \{\neg(c_a = c_b) : a \neq b \in A_i\}$ with slightly more elements than $A_i$

Note that by **construction**, each $B_j$ has $A_i$ as submodel
Checking failure of AP

Iterate over distinct pairs of models $B_j, B_k$ and construct the theory $\Gamma$ that extends $\Sigma$ with the diagrams of these two models, using only one set of constants for the overlapping submodel $A_i$.

Add formulas to $\Gamma$ that ensure all constants of $B_j$ are distinct, and same for $B_k$.

Use Mace4 to check for a limited time whether $\Gamma$ is satisfiable in some small model.

If not, use the Prover9 automated theorem prover (McCune [2009]) to search for a proof that $\Gamma$ is inconsistent. If yes, then a failure of AP has been found.

To check if SAP fails, add formulas that ensure constants of each pair of models cannot be identified, and also iterate over pairs $B_j, B_j$. 

Open problem: Does AP hold for all residuated lattices?

Commutative residuated lattices satisfy $x \cdot y = y \cdot x$

Kowalski, Takamura ['04] AP holds for commutative resid. lattices

Distributive residuated lattices satisfy $x \land (y \land z) = (x \land y) \lor (x \land z)$

Theorem [J. 2014]: AP fails for any variety of distributive residuated lattices that includes two specific 5-element commutative distributive integral residuated lattices

In particular, AP fails for the varieties DRL, CDRL, IDRL, CDIRL and any varieties between these
Conclusion

Many other **minimal** failures of **AP** and **SAP** can be found automatically.

By studying the **amalgamations of small algebras** one can get **hints** of how **AP** may be proved in general.

The method of enumerating small models and using **diagrams** of structures in **automated theorem provers** is applicable to many other problems, e.g., in coalgebra, combinatorics, finite model theory, ... 

**Computational research tools** like Sage, Prover9, UACalc, Isabelle, Coq, ... are becoming **very useful** for research in algebra, logic and combinatorics.

Thanks!