The Blok-Ferreirim theorem for normal GBL-algebras and its application

Peter Jipsen* and Franco Montagna

Chapman University
Department of Mathematics and Computer Science
Orange, California

University of Siena
Department of Mathematics and Computer Science
Siena, Italy

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Outline

- Background
- Products, ordinal sums and poset sums
- The Blok-Ferreirim theorem for normal GBL-algebras
- Applications
  - n-potent GBL-algebras are commutative
  - FEP for commutative integral GBL-algebras
  - All finite GBL-algebras are poset sums of Wajsberg hoops

Definition

A *residuated lattice* is a system $(L, \wedge, \vee, \cdot, \backslash, /, e)$ where

- $(L, \wedge, \vee)$ is a lattice,
- $(L, \cdot, e)$ is a monoid,
- $\backslash$ and $/$ are binary operations such that the *residuation property* holds:

$$x \cdot y \le z \quad \text{iff} \quad y \le x\backslash z \quad \text{iff} \quad x \le z/y$$

Galatos, J., Kowalski, Ono, (2007) "Residuated Lattices: An algebraic glimpse at substructural logics", Studies in Logic, Elsevier, xxi+509 pp.

The symbol $\cdot$ is often omitted

Definition

A residuated lattice is:

- *commutative* if it satisfies $xy = yx$
- *integral* if it satisfies $x \leq e$
- **divisible if $\mathbf{x \leq y}$ implies $\mathbf{x = y(y\backslash x) = (x/y)y}$**
- *representable* if it is isomorphic to a subdirect product of totally ordered residuated lattices
- *bounded* if it has a mimimum element, and there is an additional constant 0 which denotes this minimum

In a commutative residuated lattice the operations $x\backslash y$ and y/x coincide and are denoted by $x \to y$

Definition

A GBL-algebra is a divisible residuated lattice

A *BL-algebra* is a bounded commutative integral representable GBL-algebra

A *GMV-algebra* is a GBL-algebra satisfying $x \le y$ implies $y = x/(y\backslash x) = (x/y)\backslash x$

An *MV-algebra* is a bounded commutative GMV-algebra

A *lattice ordered group* or *ℓ-group* is (term-equivalent to) a residuated lattice satisfying $x(x\backslash e) = e$

Commutative GMV-algebas (and MV-algebras) are always representable

BL-algebras were introduced by Hájek in 1998 as an algebraic semantics of Basic (fuzzy) Logic

Basic logic is a generalization of the three most important fuzzy logics:

Łukasiewicz logic, Gödel logic and product logics

Cignoli, Esteva, Godo, Torrens (2000) showed that the variety of BL-algebras is generated by the class of residuated lattices arising from continuous t-norms on $[0, 1]$ and their residuals

Mundici's categorical equivalence between MV-algebras (the algebraic semantics for Łukasiewicz logic) and abelian ℓ-groups has been extended to BL-algebras by Agliano and Montagna (2003):

Every totally ordered BL-algebra can be represented as an ordinal sum of an indexed family of negative cones of abelian ℓ-groups and of MV-algebras, which in turn arise from abelian ℓ-groups with a strong order unit via Mundici's functor Γ

This has recently been generalized to the non-commutative case, i.e. pseudo BL-algebras, by Dvurečenskij (2006)

Another generalization of BL-algebras is obtained by removing representability

For the $\cdot, \rightarrow, 1$ fragment, this generalization leads to the notion of hoop

In fact hoops were introduced by Bosbach (1966) before BL-algebras for reasons which are independent of fuzzy logic

Definition

A *hoop* is a commutative integral residuated partially ordered monoid $(M, \cdot, \rightarrow, e)$, with partial order $\leq$ defined by $x \leq y$ iff $x \rightarrow y = e$, satisfying the divisibility condition: $x \leq y$ iff $x = y \cdot (y \rightarrow x)$

A hoop is said to be a *Wajsberg hoop* iff it is a subreduct of an MV-algebra

Hoops are precisely the subreducts of commutative and integral GBL-algebras with respect to the signature $\{\cdot, \rightarrow, e\}$

In any hoop the meet is definable by $x \wedge y = x \cdot (x \to y)$

In a Wajsberg hoop, the join is also definable by $x \vee y = (x \to y) \to y$

Thus Wajsberg hoops are term-equivalent to commutative and integral GMV-algebras

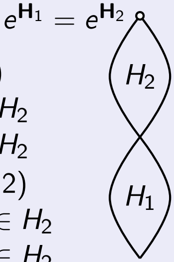
An interesting feature of hoops is that they include the $\wedge, \to, 1$ reducts of Heyting algebras

Unlike Heyting and BL-algebras, hoops need not be closed under join

A useful construction for hoops is the ordinal sum, which leads to the Blok-Ferreirim decomposition theorem for hoops

Definition

The *ordinal sum* $\mathbf{H}_1 \oplus \mathbf{H}_2$ of two hoops $\mathbf{H}_1$ and $\mathbf{H}_2$ is defined as follows: up to isomorphism, we may assume that $H_1 \cap H_2 = \{e^{\mathbf{H}_1}\} = \{e^{\mathbf{H}_2}\}$
The universe of $\mathbf{H}_1 \oplus \mathbf{H}_2$ is $H_1 \cup H_2$, and the top element is e $(= e^{\mathbf{H}_1} = e^{\mathbf{H}_2})$
The operations are defined as follows:



$$x \cdot y = \begin{cases} x \cdot_i y & \text{if} \quad x, y \in H_i \ (i = 1, 2) \\ x & \text{if} \quad x \in H_1 \setminus \{e\}, y \in H_2 \\ y & \text{if} \quad y \in H_1 \setminus \{e\}, x \in H_2 \end{cases}$$

$$x \to y = \begin{cases} x \to_i y & \text{if} \quad x, y \in H_i \ (i = 1, 2) \\ e & \text{if} \quad x \in H_1 \setminus \{e\}, y \in H_2 \\ y & \text{if} \quad y \in H_1 \setminus \{e\}, x \in H_2 \end{cases}$$

Theorem (Blok and Ferreirim, 2000)

Every subdirectly irreducible hoop is the ordinal sum of a proper subhoop $\mathbf{H}$ *and a subdirectly irreducible nontrivial Wajsberg hoop* $\mathbf{W}$.

GBL-algebras are a common generalization of ℓ-groups and of Heyting algebras (and BL-algebras)

They constitute a bridge between algebra and substructural logics

Contrary to BL-algebras and pseudo BL-algebras, at the moment only a few significant results are known about GBL-algebras:

Theorem (Galatos and Tsinakis, 2005)

Every GBL-algebra decomposes as a direct product of an ℓ-group and an integral GBL-algebra

Therefore we can mainly concentrate on integral GBL-algebras

Theorem (J. and Montagna, 2006)

Every finite GBL-algebra is commutative and integral.

Now extend ordinal sums to integral GBL-algebras and residuated lattices

If we just copy the definition of ordinal sum of hoops, we meet a difficulty:

if e is not join irreducible in $\mathbf{H}_1$, and $\mathbf{H}_2$ has no minimum, then the ordinal sum defined as for hoops is not closed under join.

Thus the ordinal sum construction splits into the following cases:

Ordinal sums of type (a): If e is join irreducible in $\mathbf{H}_1$

Then do the same as for hoops, defining both $\backslash$ and $/$

Type (b): If e is not join-irreducible in $\mathbf{H}_1$ and $\mathbf{H}_2$ has a minimum m

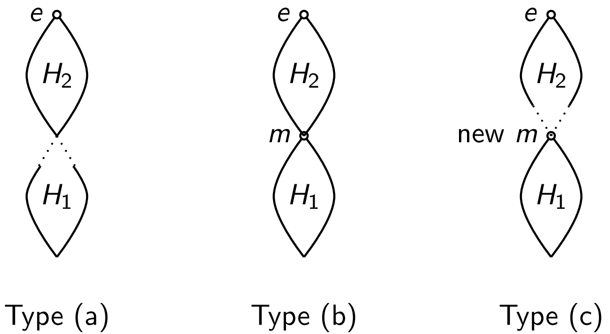
Then do the same as for hoops, except if $x, y \in H_1$ and $x \vee_1 y = e^{\mathbf{H}_1}$ then $x \vee y = m$

Type (c): If e is not join irreducible in $\mathbf{H}_1$, and $\mathbf{H}_2$ has no minimum

Then add a new minimum element m to $\mathbf{H}_2$, which forces $xm = mx = m$, $m\backslash x = x/m = e^{\mathbf{H}_2}$, and $x\backslash m = m = m/x$. Now proceed as for type (b)

Ordinal sums I

In all cases $e^{\mathbf{H}_1} = e^{\mathbf{H}_2} = e$



In all cases if $\mathbf{H}_1$ and $\mathbf{H}_2$ are integral GBL-algebras, so is $\mathbf{H}_1 \oplus \mathbf{H}_2$

Ordinal sums are also applicable to integral residuated lattices in general

BUT the Blok-Ferreirim Theorem does not apply to integral GBL-algebras

J. and Montagna (2006) contains an example of a subdirectly irreducible integral GBL-algebra which *cannot* be decomposed as ordinal sum of two GBL-algebras and is not a GMV-algebra

We now consider some subclasses of integral GBL-algebras which satisfy the Blok-Ferreirim decomposition theorem

Definition

A *filter* of a residuated lattice $\mathbf{A}$ is a set $F \subseteq A$ such that $e \in F$, F is upwards closed and F is closed under product and meet

A filter is said to be *normal* iff $a \in F$ implies $b\backslash(ab) \in F$ and $(ba)/b \in F$ for every $b \in A$

Definition

An integral GBL-algebra is said to be *normal* if every filter of it is normal

A GBL-algebra is said to be *n-potent* if it satisfies $x^{n+1} = x^n$

Lemma

An n-potent GBL-algebra is integral and normal

Proof.

This follows since, for every x, x^n is an idempotent

In a GBL-algebra all idempotents are $\leq e$ and commute with all elements

So we get $x^n y = yx^n \leq yx$ and $yx^n = x^n y \leq xy$

Now, if F is any filter and $x \in F$, then $x^n \leq y\backslash(xy)$ and $x^n \leq (yx)/y$

Since $x^n \in F$, we have that $y\backslash(xy) \in F$ and $(yx)/x \in F$, therefore F is normal □

Clearly, commutative GBL-algebras are normal, but the converse does not hold (consider any weakly abelian and nonabelian ℓ-groups)

We can now state our main decomposition result for GBL-algebras

Theorem

Every normal subdirectly irreducible integral GBL-algebra decomposes as the ordinal sum, either of type (a) or of type (b), of an integral GBL-algebra and a subdirectly irreducible integral GMV-algebra.

Application: *n*-potent GBL-algebras are commutative

J. and Montagna (2006) show that every finite GBL-algebra is commutative

Now we extend this result to *n*-potent GBL-algebras

Clearly, every finite GBL-algebra is n-potent for some n, but the converse does not hold (consider e.g. any infinite Heyting algebra)

Lemma

In any n-potent subdirectly irreducible GBL-algebra, e is join irreducible.

Lemma

Every subdirectly irreducible n-potent GMV-algebra is a finite chain, hence it is commutative.

Theorem

Any n-potent GBL-algebra is commutative.

Neither GBL-algebras nor integral GBL-algebras have the finite model property (since all finite GBL-algebras are commutative and integral)

Application: Commutative and integral GBL-algebras have the finite embeddability property (FEP for short)

$\mathcal{V}$ *has the FEP* iff every finite partial subalgebra of any algebra of $\mathcal{V}$ embeds into a finite algebra of $\mathcal{V}$

FEP implies the decidability of the universal theory of $\mathcal{V}$.

Theorem

The variety of commutative and integral GBL-algebras has the FEP

Proof.

(Outline) Let **A** be a commutative and integral GBL-algebra, and let **C** be a finite partial subalgebra of **A** with $e \in C$.

We prove by induction on the cardinality of **C** that **C** embeds into a finite GBL-algebra.

Without loss of generality one can restrict to the case when **A** is subdirectly irreducible and generated by **C**.

Use the decomposition theorem to find a GBL-algebra **B** and a s.i. integral GMV-algebra **W** such that $\mathbf{A} \cong \mathbf{B} \oplus \mathbf{W}$

Considering several cases one can show that $\mathbf{B} \cap \mathbf{C}$ has cardinality $< n$, so by the induction hypothesis it embeds in a finite commutative GBL-algebra, say $\mathbf{B}'$

Blok and Ferreirim (2000) showed that Wajsberg hoops have FEP, so $\mathbf{W} \cap \mathbf{C}$ embeds in a finite commutative GMV-algebra, say $\mathbf{W}'$

Hence **C** embeds in the finite algebra $\mathbf{B}' \oplus \mathbf{W}'$ □

Corollary

The universal theory of commutative and integral GBL-algebras is decidable

The universal theory of commutative GBL-algebras is decidable

What about the quasiequational theory of all GBL-algebras?

Recall that an ℓ-group can be regarded as a residuated lattice, with $x\backslash y = x^{-1}y$ and $y/x = yx^{-1}$, and that the inverse operation can be written in the language of residuated lattices as $x^{-1} = x\backslash e$.

Theorem

To each quasiequation Φ of residuated lattices we can constructively associate a quasiequation Φ' such that Φ holds in all ℓ-groups iff Φ' holds in all GBL-algebras

Thus the quasiequational theory of GBL-algebras is undecidable

Application: Poset sum representability for finite GBL-algebras

Now we define *poset sums* as a generalization of ordinal sums

Using the decomposition theorem for normal GBL-algebras, we then show that every finite GBL-algebra is isomorphic to a poset sum of finite Wajsberg chains

Let $I(\mathbf{A})$ be the set of *idempotents* $\mathbf{A}$

J. and Montagna 2006 show that $I(\mathbf{A})$ is a subalgebra that satisfies $xy = x \wedge y$ and hence is a Brouwerian algebra (= 0-free subreduct of a Heyting algebra)

Let P be the poset of join-irreducibles of this Brouwerian algebra

Each $i \in P$ has a unique lower cover $i^* \in I(\mathbf{A})$

Using the decomposition theorem we show that the interval $\mathbf{A}_i = [i^*, i]$ is a chain, and has the multiplicative structure of a Wajsberg hoop

The GBL-algebra **A** can be reconstructed from a subset of the direct product of these Wajsberg chains

For BL-algebras this result was proved by Di Nola and Lettieri (2003)

In this case the representability of BL-algebras implies that the poset of join-irreducibles is a forrest (i.e. disjoint union of trees)

Our more general approach is somewhat simpler and shows that representability plays no role in this result

A generalized ordinal sum for residuated lattices is defined as follows:

Let P be a poset, and let $\mathbf{A}_i$ ($i \in P$) be a family of residuated lattices

In addition we require that for nonmaximal $i \in P$ each $\mathbf{A}_i$ is integral, and for nonminimal $i \in P$ each $\mathbf{A}_i$ has a least element denoted by 0_i

The *poset sum* is defined as

$$\bigoplus_{i\in P} A_i = \{a \in \prod_{i\in P} A_i : a_j < e \Longrightarrow a_k = 0_k \text{ for all } j < k\}$$

This subset of the product contains the constant function $\underline{e}$

Note that an element a is in the poset sum if and only if $\{i \in P : 0_i < a_i < e\}$ is an antichain and $\{i \in P : a_i = e\}$ is downward closed (hence $\{i \in P : a_i = 0_i\}$ is upward closed)

The operations $\wedge$, $\vee$ and $\cdot$ are defined pointwise (as in the product)

For the definition of the residuals, we have

$$(a\backslash b)_i = \begin{cases} a_i\backslash b_i & \text{if } a_j \le b_j \text{ for all } j < i \\ 0_i & \text{otherwise} \end{cases}$$ and similarly for $(a/b)_i$

Note that poset sums generalize both ordinal sums and direct products

Indeed, if the poset P is a chain, the poset sum produces an ordinal sum (of type (a) or (b) since $\mathbf{A}_i$ has a least element for all nonminimal $i \in P$)

If P is an antichain then it produces a direct product

If i is a maximal (minimal) element of P, we refer to $\mathbf{A}_i$ as a maximal (minimal) summand

Theorem

The poset sum of residuated lattices is again a residuated lattice

If all maximal summands are integral then the poset sum is integral, and if all minimal summands have a least element, then the poset sum has a least element

Note that since poset sums are generalizations of ordinal sums, we cannot expect the varieties of Boolean algebras, MV-algebras or involutive lattices to be closed under poset sums

However it does preserve the defining property of generalized basic logic

Theorem

The variety of integral GBL-algebras is closed under poset sums

In fact, for GBL-algebras, this construction describes **all** the finite members

For a residuated lattice $\mathbf{A}$ and an idempotent element i of A, we let $\mathbf{A}_{\downarrow i} = (\downarrow i, , \wedge, \vee, \cdot, \backslash_i, /_i, i)$, where $x\backslash_i y = (x\backslash y) \wedge i$ and $x/_i y = (x/y) \wedge i$

Lemma

For any integral GBL-algebra $\mathbf{A}$ *and any idempotent element* i *of* $\mathbf{A}$*, the map* $\hat{i} : A \to {\downarrow}i$ *given by* $\hat{i}(a) = a \wedge i$ *is a homomorphism from* $\mathbf{A}$ *to* $\mathbf{A}_{\downarrow i}$

Hence the algebra $\mathbf{A}_{\downarrow i}$ *is an integral GBL-algebra*

The next result follows from the observation that lattice operations and residuals are first-order definable from the partial order and the monoid operation of a residuated lattice.

Lemma

Suppose $\mathbf{A}, \mathbf{B}$ *are residuated lattices and* $h : A \to B$ *is an order-preserving monoid isomorphism*

Then h *is a residuated lattice isomorphism*

Theorem

Let $\mathbf{A}$ *be a finite GBL-algebra and let* P *be the set of all join-irreducible idempotents of* $\mathbf{A}$

For $i \in P$, *let* i^* *be the unique maximal idempotent below* i, *and let* $A_i = \{x : i^* \leq x \leq i\} = [i^*, i]$

Then $\mathbf{A}_i = (A_i, \wedge, \vee, \cdot, \backslash^i, /^i, i)$ *is a Wajsberg chain and* $\mathbf{A} \cong \bigoplus_{i \in P} \mathbf{A}_i$

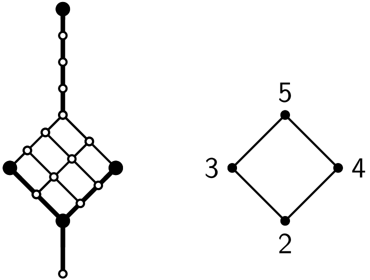
Moreover, there is a bijective correspondence between finite GBL-algebras and finite posets labelled with natural numbers > 1, denoting the size of the corresponding Wajsberg chain in the poset sum.

If the poset is a forrest, the GBL-algebra is representable hence can be expanded to a BL-algebra

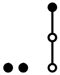
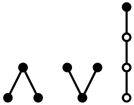
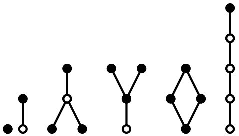
Thus the result proved here extends DiNola and Letieri's result.

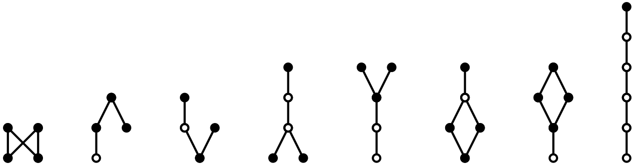
Our representation result is useful for constructing and counting finite GBL-algebras

For example, consider the following lattice structure of a GBL-algebra with 17 elements that is obtained from a poset sum of a 2, 3, 4, and 5-element Wajsberg chain over the poset $\mathbf{2} \times \mathbf{2}$ (the join irreducible idempotents are denoted by black dots)



By the previous theorem the same lattice supports $2^6 = 64$ nonisomorphic GBL-algebras since six other join irreducibles could be idempotents

Posets of join-irreducibles of distributive lattices of size n with number of nonisomorphic GBL-algebras below each poset					
1	2	3	$n = 4$	$n = 5$	$n = 6$
$\emptyset$					
1	1	2	1 4	1 1 8	2 2 2 1 16

$n = 7$

1 2 2 4 4 2 2 32

A finite GBL-algebra is subdirectly irreducible iff the poset of join-irreducibles has a top element

It is representable (and hence expands to a finite BL-algebra) iff the poset of join-irreducibles is a forest.

Since subdirectly irreducible BL-algebras are chains, it follows that for $n > 1$ there are precisely 2^{n-2} nonisomorphic subdirectly irreducible n-element BL-algebras.

Size $n =$	1	2	3	4	5	6	7
GBL-algebras	1	1	2	5	10	23	49
si GBL-algebras	0	1	2	4	9	19	42
BL-algebras	1	1	2	5	9	20	38
si BL-algebras	0	1	2	4	8	16	32

Note that the variety of idempotent GBL-algebras is (term-equivalent to) the variety of Brouwerian algebras

Now the finite members in this variety are just finite distributive lattices, expanded with the residual of the meet operation

Thus Birkhoff's duality between finite posets and finite distributive lattices shows that finite Brouwerian algebras are isomorphic to poset sums of the two-element generalized Boolean algebra (= two-element Wajsberg chain)

Using the preceding theorem, this result can be generalized to n-potent GBL-algebras

Corollary

Any finite n-potent GBL-algebra is isomorphic to a poset sum of Wajsberg chains with at most $n + 1$ *elements*

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