The categorical equivalence between complete (semi)lattices with operators and contexts with relations

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Outline

- Algebraic Logic Overview
- Boolean Algebras with operators $\iff$ Relational structures
- Equivalence of contexts with complete join (semi)lattices
- The category of Chu-relation morphisms
- Complete lattices with operators $\iff$ Relational contexts
- Morphisms for relational contexts
- Example: Complete residuated lattices and residuated frames
- Application: Poset products
Algebraic Logic

Logic

Algebra

Combinatorics
Classical Algebraic Logic

Classical Propositional Logic
(true false and or not)

Boolean Algebras  Sets
Classical Algebraic Logic

Classical First-order Logic
(true false and or not ∀ ∃)

Cylindric Algebras ——— Cylindric Atom Structures
Classical Algebraic Logic

Classical Modal Logic
(true false and or not □ ◊)

Modal Algebras (A, f) ———— Kripke Frames (X, F)
Classical Algebraic Logic

Arrow Logic
(true false and or not composition converse loops)

Relation Algebras ———— Relation Atom Structures
Classical Algebraic Logic

Classical Multi-modal Logic

(true false and or not $\Box_i^{(n_i)} \Diamond_i^{(n_i)}$)

BAs with Operators ——— Relational Structures
Positive Logic

(true false and or)

Distributive Lattices

Priestley Spaces
Intuitionistic Logic

(true false and or implication)

Heyting Algebras ─────────── Esakia Spaces

Frames (Point-free Top.) ─── Topological Spaces
Substructural Logic

(residuals, identity)

Residuated Lattices ——— Residuated Frames
Universal Algebraic Logic

Equational Logics ——— Propositional Logics

Univ. Algebra Varieties ——— Model Classes
Modern Algebraic Logic

Computer Programs

Algebraic Semantics ——— Denotational Semantics
Modern Algebraic Logic

Computation

CECAT

Algebra

Topology
The category caBA of complete and atomic Boolean algebras and complete homomorphisms is dual to the category Set of sets and functions:

On objects $B(X) = \mathcal{P}(X)$ and $G(\mathcal{P}(X)) = X$

On morphisms, for $g : X \to Y$ and $h : \mathcal{P}(Y) \to \mathcal{P}(X)$

$B(g)(S) = g^{-1}[S]$ and $G(h)(x)$ is the unique $y$ s. t. $x \in h(\{y\})$

This works because $g^{-1}$ preserves $\cap$ and $\cup$
Adding complete operators and relations

caBAO$_\tau$ is the category of caBAs with completely join-preserving operations of type $\tau : \mathcal{F} \to \omega$

Each $f \in \mathcal{F}$ has arity $\tau(f)$

The objects are $A = (A, \wedge, \vee, ' , \{f^A : f \in \mathcal{F}\})$

The morphisms are complete homomorphisms i.e., $h : A \to B$ preserves all joins, meets, complement, and $h(f^A(a_1, \ldots, a_n)) = f^B(h(a_1), \ldots, h(a_n))$

A central duality of algebraic logic is that $\text{caBAO}_\tau \equiv \text{RS}_\tau$

$\text{RS}_\tau =$ category of relational (Kripke) structures $X = (X, \{F^X : F \in \mathcal{F}\})$ where $F^X \subseteq X^{\tau(F)+1}$.

But what are the morphisms in $\text{RS}_\tau$?
Bounded morphisms

A relation $F^X \subseteq X^{n+1}$ defines an operation $f : \mathcal{P}(X)^n \to \mathcal{P}(X)$ by

$$f(S_1, \ldots, S_n) = F^X[S_1, \ldots, S_n] = \{z : (x_1, \ldots, x_n, z) \in F^X \text{ for some } x_i \in S_i\}$$

Then $f$ is a $\bigcup$-preserving operation on $B(X)$

From $f$ we can recover $F^X$ by $(x_1, \ldots, x_n, z) \in F^X$ iff $z \in f(\{x_1\}, \ldots, \{x_n\})$

For a function $g : X \to Y$ to be a morphism we want

$$g^{-1}[F^Y[S_1, \ldots, S_n]] = F^X[g^{-1}[S_1], \ldots, g^{-1}[S_n]]$$

Since $F[\cdot]$ and $g^{-1}[\cdot]$ are $\bigcup$-preserving it suffices to check for $S_i = \{y_i\}$
Want $g^{-1}[F^Y[\{y_1\}, \ldots, \{y_n\}]] = F^X[g^{-1}[\{y_1\}], \ldots, g^{-1}[\{y_n\}]]$

$x \in g^{-1}[F^Y[\{y_1\}, \ldots, \{y_n\}]]$ iff $x \in F^X[g^{-1}[\{y_1\}], \ldots, g^{-1}[\{y_n\}]]$

$g(x) \in F^Y[\{y_1\}, \ldots, \{y_n\}]$ iff $\exists x_i \in g^{-1}[\{y_i\}]((x_1, \ldots, x_n, x) \in F^X)$

$(y_1, \ldots, y_n, g(x)) \in F^Y$ iff $\exists x_i(g(x_i) = y_i$ and $(x_1, \ldots, x_n, x) \in F^X)$

$g$ is a bounded morphism if it satisfies the above for all $F \in \mathcal{F}$

**Back:** $(x_1, \ldots, x_n, x) \in F^X \Rightarrow (g(x_1), \ldots, g(y_n), g(x)) \in F^Y$ and

**Forth:** $(y_1, \ldots, y_n, g(x)) \in F^Y \Rightarrow \exists x_i \in X$ such that $g(x_i) = y_i$ and $(x_1, \ldots, x_n, x) \in F^X$

**Aim:** extend this duality to (complete semi)lattices with operators
A context is a structure $X = (X_-, X_+, X)$ such that

$X_-, X_+$ are sets and $X \subseteq X_- \times X_+$.

The incidence relation $X$ determines two functions

$X^\uparrow : \mathcal{P}(X_-) \rightarrow \mathcal{P}(X_+)$ and $X^\downarrow : \mathcal{P}(X_+) \rightarrow \mathcal{P}(X_-)$ by

$X^\uparrow A = \{b : \forall a \in A \ aXb\}$ and $X^\downarrow B = \{a : \forall b \in B \ aXb\}$.

Gives a Galois connection from $\mathcal{P}(X_-)$ to $\mathcal{P}(X_+)$, i.e.,

$A \subseteq X^\downarrow B \iff B \subseteq X^\uparrow A$ for all $A \subseteq X_-$ and $B \subseteq X_+$.

$cl_-(X) = \{X^\downarrow X^\uparrow A : A \subseteq X_-\}$ and $cl_+(X) = \{X^\uparrow X^\downarrow B : B \subseteq X_+\}$

are dually isomorphic complete lattices with intersection as meet and Galois-closure of union as join.
Background

Contexts are due to Birkhoff; studied in Formal Concept Analysis

Let \( L \) be a (bounded) \( \lor \)-semilattice

The Dedekind-MacNeille context is \( C(L) = (L, L, \leq) \)

\( \text{cl}_-(C(L)) \) is the MacNeille completion \( \bar{L} \) of \( L \)

For a finite \( \lor \)-semilattice can take \( C(L) = (X_-, X_+, \leq) \) where \( X_- \) are the join-irreducibles and \( X_+ \) are the meet-irreducibles

For complete perfect lattices results similar to the ones below appear in [Dunn Gehrke Palmigiano 2005] and [Gehrke 2006]

For complete semilattices they are due to [Moshier 2011]

For complete residuated lattices this is joint work with N. Galatos
Complete lattices with complete homomorphisms form a category.

What are the appropriate morphisms for contexts?

For a context $\mathbf{X} = (X_-, X_+, X)$ the relation $X$ is an identity morphism that induces the identity map $X \downarrow X \uparrow$ on the closed sets.

A context morphism $R : \mathbf{X} \to \mathbf{Y} = (Y_-, Y_+, Y)$ is a relation $R \subseteq X_- \times Y_+$ such that $R \uparrow X \downarrow X \uparrow = R \uparrow = Y \uparrow Y \downarrow R \uparrow$ (R is compatible).

Lemma

If $R$ is compatible then $Y \downarrow R \uparrow : \text{cl}_-(X) \to \text{cl}_-(Y)$ preserves $\bigvee$. 

\[ \begin{array}{c}
X_+ & \quad & Y_+ \\
\downarrow & \quad & \downarrow \\
X & \quad & R \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
Theorem

(i) The collection $\text{Cxt}$ of all contexts with compatible relations as morphisms is a category

Composition

\[
\begin{array}{ccc}
X_+ & Y_+ & Z_+ \\
X & R & Y & S & Z \\
X_ & Y_ & Z_ \\
\end{array}
\]

so $xR;Sy$ iff $x \in R\downarrow J\uparrow S\downarrow \{y\}$

(ii) The category $\text{Cxt}$ is equivalent to the category $\text{SUP}$ of complete semilattices with completely join-preserving homomorphisms

The adjoint functors are $\text{cl}_- : \text{Cxt} \to \text{SUP}$ and $C : \text{SUP} \to \text{Cxt}$

On morphisms, $\text{cl}_-(R) = Y\downarrow R\uparrow : \text{cl}_-(X) \to \text{cl}_-(Y)$ and for a $\text{SUP}$ morphism $h : L \to M$, $C(h) = \{(x, y) \in L \times M : h(x) \leq y\}$
Lattice compatible morphisms

**Lemma.** \( Y\downarrow R\uparrow \) preserves \( \land \) iff there exists a compatible relation \( R_* : Y_- \to X_+ \) such that \( Y\downarrow R\uparrow = R_* X\uparrow \) (call \( R \) lattice compatible)

**Theorem.** The category \( \text{LCxt} \) of all contexts with lattice compatible relations as morphisms is equivalent to the category \( \text{CLat} \) of complete lattices with complete lattice morphisms.

**Lemma.** (i) \( R : X \to Y \) is a monomorphism in \( \text{Cxt} \) iff \( R\downarrow R\uparrow = X\downarrow X\uparrow \)

(ii) \( R : X \to Y \) is a epimorphism in \( \text{Cxt} \) iff \( R\uparrow R\downarrow = Y\uparrow Y\downarrow \)

Every morphism has itself as epi-mono factorization

\[
\begin{array}{ccc}
X_+ & \to & Y_+ \\
\downarrow & R & \downarrow \\
X & \to & Y \\
X_- & \to & X_-
\end{array}
\]
**LCxt** is “much larger” than **CLat** since many different contexts represent the same lattice.

It is easier to construct examples in **LCxt** than in **CLat**.

Contexts can be logarithmic in size compared to their lattices of Galois closed sets.

So checking compatibility of $R$ is much more efficient than checking the homomorphism property between large lattices.

Now want to extend to complete lattices with operators, relational contexts.
Chu-relation morphisms

A Chu-relation pair \((S, T) : X \to Y\) is a pair of relations \(S \subseteq X_- \times Y_-\) and \(T \subseteq Y_+ \times X_+\) such that

\[ y \in Y^\uparrow S[\{x\}] \iff x \in X^\downarrow T[\{y\}] \]
Chu-relation morphisms

Lemma

For a relation $S \subseteq X_- \times Y_-$ the following are equivalent:

1. for all $A \subseteq X_-$ we have $S[X_\downarrow X_\uparrow A] \subseteq Y_\downarrow Y_\uparrow S[A]$
2. for all $y \in Y^+$, $\{ x \in X_- : S[\{x\}] \subseteq Y_\downarrow \{y\}\}$ is in $\text{cl}-(X)$
3. there exists a relation $T \subseteq Y_+ \times X_+$ such that $(S, T)$ are a Chu-relation pair
4. the relation $R = \{(x, y) \in X_- \times Y_+ : y \in Y_\uparrow S[\{x\}]\}$ is a context morphism

and they imply that the map $Y_\downarrow Y_\uparrow S[\ ] : \text{cl}-(X) \rightarrow \text{cl}-(Y)$ is $\bigvee$-preserving.

1. is the nuclear condition from [Galatos-J. Residuated Frames]
Contexts with Chu-relation morphisms

Lemma

If \((S, T) : X \to Y\) and \((S', T') : Y \to Z\) are Chu-relation pairs, then \((S; S', T'; T)\) is a Chu-relation pair from \(X\) to \(Z\), and \((\text{id}_X, \text{id}_X)\) is a Chu-relation pair from \(X\) to \(X\), hence contexts with Chu-relation pairs form a category, called \(\text{RelCxt}\).

This category is not equivalent to \(\text{Cxt}\) since many Chu-relation pairs can correspond to the same context morphism \(R\).

However there are functors \(F : \text{Cxt} \to \text{RelCxt}\) and \(G : \text{RelCxt} \to \text{Rel}\) given by \(F(R) = (R_0, R_1)\) where

\[
x R_0 y \iff y \in Y_{\downarrow} R_{\uparrow}\{x\} \quad \text{and} \quad y R_1 x \iff x \in X^{\uparrow} R_{\downarrow}\{y\}
\]

\[
x G(S, T) y \iff y \in Y_{\uparrow} S[\{x\}]
\]

\(GF(R) = R\) and \(FG(S, T)\) is naturally isomorphic to \((S, T)\)
Complete lattices with complete operators

Let $\text{SUP}_\tau$ be the class of $\lor$-semilattices with $\lor$-preserving operations of type $\tau: \mathcal{F} \to \omega$, where each $f \in \mathcal{F}$ has arity $\tau(f)$.

The objects are $L = (L, \lor, \{ f^L : f \in \mathcal{F} \})$.

The morphisms are complete homomorphisms, i.e. $h: L \to M$ preserves all joins and $h(f^L(a_1, \ldots, a_n)) = f^M(h(a_1), \ldots, h(a_n))$.

$\text{Cxt}_\tau = \text{category of contexts with relations}$

$X = (X_-, X_+, X, \{ F^X : F \in \mathcal{F} \})$ where the $F^X \subseteq X^{\tau(F)+1}_-$ satisfy

$F^X[X^\downarrow X^\uparrow S_1, \ldots, X^\downarrow X^\uparrow S_n] \subseteq X^\downarrow X^\uparrow F^X[S_1, \ldots, S_n]$ for all $S_i \subseteq X_-$.
Bounded morphisms for contexts

A relation $F^X \subseteq X_{n+1}$ defines an operation $f : \text{cl}_-(X)_n \to \text{cl}_-(X)$ by $f(S_1, \ldots, S_n) = X \downarrow X \uparrow F^X[S_1, \ldots, S_n]$

$= X \downarrow X \uparrow \{z : (x_1, \ldots, x_n, z) \in F^X \text{ for some } x_i \in S_i, \text{ all } i = 1, \ldots, n\}$

Then $f$ is a $\lor$-preserving operation on $\text{cl}_-(X)$

For a relation $R : X \to Y$ to be a $\text{Cxt}_\tau$ morphism we want

$Y \downarrow R \uparrow f(S_1, \ldots, S_n) = f(Y \downarrow R \uparrow S_1, \ldots, Y \downarrow R \uparrow S_n)$

Since $f$ and $Y \downarrow R \uparrow$ are $\lor$-preserving, enough to check for $S_i = \{x_i\}$

$Y \downarrow R \uparrow X \downarrow X \uparrow F^X[\{x_1\}, \ldots, \{x_n\}] = Y \downarrow Y \uparrow F^Y[Y \downarrow R \uparrow \{x_1\}, \ldots, Y \downarrow R \uparrow \{x_n\}]$

$\implies \quad R \uparrow F^X[\{x_1\}, \ldots, \{x_n\}] = Y \uparrow F^Y[Y \downarrow R \uparrow \{x_1\}, \ldots, Y \downarrow R \uparrow \{x_n\}]$
This is the **bounded morphism** condition for contexts with relations

Note that the morphism condition is just for *points* of the context.

Can give a similar condition for relational context morphisms using Chu-relation pairs.

This has *practical* advantages since composition of Chu-relation pairs is *ordinary relation composition* that does not require intermediate closure calculations.

**Theorem** *The category* \( \text{Cxt}_\tau \) *of all contexts with bounded compatible relations as morphisms is equivalent to the category* \( \text{SUP}_\tau \) *with complete homomorphisms.*

*This equivalence restricts to the subcategories of* \( \text{LCxt}_\tau \) *with bounded compatible lattice relations and* \( \text{CLat}_\tau \), i.e., complete lattices with \( \lor \)-preserving operators and complete homomorphisms.*
Example: Complete residuated lattices

The general theory can also be applied to algebras with quasi-operators, defined in the distributive case by [Gehrke Priestley 2007], and for operations that satisfy Sahlqvist-type equations

\((L, \land, \lor, \cdot, \backslash, /, e)\) is a **complete residuated lattice** if

- \((L, \land, \lor)\) is a complete lattice,
- \((L, \cdot, e)\) is a monoid and
- \(x \cdot y \leq z \iff y \leq x \backslash z \iff x \leq z / y\) for all \(x, y, z \in L\)

It follows that \(\cdot\) is completely join-preserving in each argument

\(\mathbf{cRL}\) is the category of complete residuated lattices with complete homomorphisms
Residuated frames

$(X_-, X_+, X, \circ, \|, //, E)$ is a unital residuated frame (or ru-frame) if

- $(X_-, X_+, X)$ is a context
- $\circ \subseteq X^3$, $\| \subseteq X_- \times X_+^2$, $// \subseteq X_+ \times X_- \times X_+$ and $E \subseteq X_-$
- $X^{\uparrow}(x \circ E) = X^{\uparrow}\{x\} = X^{\uparrow}(E \circ x)$
- $x \circ y \subseteq X^{\downarrow}\{z\} \iff y \in X^{\downarrow}(x \| z) \iff x \in X^{\downarrow}(z // y)$

The last two conditions use the notation $x \circ y = \circ[\{x\}, \{y\}]$

The last condition implies $A \cdot B := X^{\downarrow}X^{\uparrow}(A \circ B)$ is a completely join-preserving operation on the lattice of closed sets

An associative ru-frame satisfies $X^{\uparrow}((x \circ y) \circ z) = X^{\uparrow}(x \circ (y \circ z))$
Morphisms for ru-frames

Recall a relation $R : X_\bot \times Y^\top$ is a lattice context morphism if it is compatible $R^\uparrow X^\downarrow X^\uparrow = R^\uparrow = Y^\uparrow Y^\downarrow R^\uparrow$ and there exists a compatible relation $R_* : Y_\bot \times X_\bot$ such that $Y^\downarrow R^\uparrow = R_* X^\uparrow$

Also want $Y^\downarrow R^\uparrow (A \cdot B) = (Y^\downarrow R^\uparrow A) \cdot (Y^\downarrow R^\uparrow B)$,
$Y^\downarrow R^\uparrow (A \setminus B) = (Y^\downarrow R^\uparrow A) \setminus (Y^\downarrow R^\uparrow B)$ and
$Y^\downarrow R^\uparrow (A / B) = (Y^\downarrow R^\uparrow A) / (Y^\downarrow R^\uparrow B)$ for all $A, B \in \text{cl}_\bot (X)$

Since $\cdot$ is $\lor$-preserving, the first simplifies to
$Y^\downarrow R^\uparrow (\{x_1\} \cdot \{x_2\}) = (Y^\downarrow R^\uparrow \{x_1\}) \cdot (Y^\downarrow R^\uparrow \{x_2\})$,
$R^\uparrow (x_1 \circ x_2) = Y^\uparrow (Y^\downarrow R^\uparrow \{x_1\} \circ Y^\downarrow R^\uparrow \{x_2\})$ and the others reduce to

$Y^\downarrow R^\uparrow (\{x\} \setminus X^\downarrow \{x'\}) = (Y^\downarrow R^\uparrow \{x\}) \setminus R_* \{x'\}$ and
$Y^\downarrow R^\uparrow (X^\downarrow \{x'\} / \{x\}) = R_* \{x'\} / (Y^\downarrow R^\uparrow \{x\})$

**Theorem.** With this definition of morphisms, the category of associative unital residuated frames is equivalent to the category of complete residuated lattices.
In [Galatos, J.] residuated frames are also defined for the case of involutive FL-algebras.

Gentzen systems for (involutive) residuated lattices are used to construct (involutive) residuated frames to prove cut-elimination, finite model properties and finite embeddability results for a range of subvarieties.

In these applications the contexts are rarely separating or reduced, so it is important to have an equivalence with all contexts.

**Extension to all semilattices:** A context is *algebraic* if $X \downarrow X \uparrow$ preserves all directed unions.

From [Hofmann Mislove Stralka 1974] get an equivalence between all join-semilattices and algebraic contexts with compatible morphisms s.t. $Y \downarrow R \uparrow$ preserve all directed unions.

Can use Hartung’s topological contexts to get equivalence for all lattices with operators.
Application: Poset products

Products of lattices = disjoint union of contexts with full incidence relation between distinct parts

Ordinal sums = disjoint unions but with “half-full” incidence relation

*Poset products* of bounded (residuated) lattices are an intermediate concept

Let \((P, \leq)\) be a poset and consider the context \((P, P, \not\leq)\)

Given contexts \(X_p\) for \(p \in P\), let \(X_\neg = \bigcup_P X_{p\neg}\), \(X_\neg = \bigcup_P X_{p\neg}\) and for \(x \in X_p\) and \(y \in X_q\) define \(x X y\) iff \(p \not\leq q\)

The context \(X\) is the *poset sum* of \(\{X_p : p \in P\}\), and \(\text{cl}_\neg(X)\) is the *poset product* of the lattices \(\text{cl}_\neg(X_p)\)

Poset products of complete residuated lattices \(\iff\) Poset sums of contexts
References

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Thank You