A Survey of Partially Ordered Algebras

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Outline

- Partially ordered algebras
- Partially ordered varieties
- Scope of the survey
- Current version
- Some results and open problems

Partially ordered algebras

Unisorted universal algebra has been developed over the last century

Central concepts: signature, algebras, homomorphisms, congruences, subalgebras, products, HSP, varieties, quasivarieties, free algebras, ...

Every algebra $\mathbf{A}$ has an underlying set A as its universe

A **partially ordered algebra A** has an underlying poset $(A, \leq^{\mathbf{A}})$, denoted simply by A, as its universe; the **dual poset** A^∂ is $(A, \geq^{\mathbf{A}})$

Each **fundamental operation symbol** $f \in \mathcal{F}$ corresponds to an operation $f^{\mathbf{A}}$ of $\mathbf{A}$ that is order-preserving **or order-reversing** in each argument

This information is part of the **signature** $\sigma : \mathcal{F} \to \{1, \partial\}^*$

E.g. $\sigma(f) = \partial 1$ means f is binary and $f^{\mathbf{A}}{:}A^\partial \times A^1 \to A$ is order-preserving

Partially ordered algebras

In standard universal algebra the base category is the category of **sets**

For **partially ordered algebras (po-algebras)** the base category is **Pos** = the category of posets with order-preserving maps as morphisms

However, term operations on **A** are not necessarily morphisms in **Pos**

If $\sigma(f) = \partial 1$, $f^{\mathbf{A}}(x, x)$ may **not** be order-preserving or order-reversing

Varieties of algebras with order-preserving operations have been studied by [Bloom 1976], [Bloom and Wright 1983], [Kurz and Velebil 2017], ...

However for algebraic logic, **negation** and **residuation** are important operations, and they are **not** order-preserving

The study of (nonorder-preserving) po-algebras is due to [Pigozzi 2004]

Motivation for studying po-algebras

Algebraic logic “is” the study of po-algebras

$\varphi \leq \psi$ means φ has ψ as a consequence

Algebraic proof theory “is” the study of po-algebras where all fundamental operations are **residuated, dually residuated, Galois connections or dual Galois connections** in each argument

$$f(x_1,\dots,x_{i-1},y,x_{i+1}\dots,x_n) \leq z \Leftrightarrow y \leq g(x_1,\dots,x_{i-1},z,x_{i+1},\dots,x_n)$$

$$y \leq f(x_1,\dots,x_{i-1},z,x_{i+1}\dots,x_n) \Leftrightarrow g(x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_n) \leq z$$

$$y \leq f(x_1,\dots,x_{i-1},z,x_{i+1}\dots,x_n) \Leftrightarrow z \leq g(x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_n)$$

$$f(x_1,\dots,x_{i-1},z,x_{i+1}\dots,x_n) \leq y \Leftrightarrow g(x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_n) \leq z$$

Every set is a poset ordered as an antichain, hence the study of po-algebras **includes** the study of algebras

[Pigozzi 2004]: many standard notions naturally generalize to po-algebras

Subalgebras and products

A po-algebra $\mathbf{B}$ is a **subalgebra** of $\mathbf{A}$ if $\leq^{\mathbf{B}} = \leq^{\mathbf{A}} \cap B^2$ and $f_i^{\mathbf{B}} = f_i^{\mathbf{A}}|_B$ all i

i.e., $(B, \leq^{\mathbf{B}})$ is a subposet of $(A, \leq^{\mathbf{A}})$ **with the induced partial order** and B is closed under all operations of $\mathbf{A}$.

The **direct product** $\prod_{i \in I} \mathbf{A}_i$ of a family $\{\mathbf{A}_i \mid i \in I\}$ of po-algebras is defined as for ordinary algebras (pointwise)

The partial order on the product is the pointwise order:

$$a \leq b \iff a(i) \leq^{\mathbf{A}_i} b(i) \text{ for all } i \in I$$

Homomorphisms and isomorphisms

A **homomorphism** $h : \mathbf{A} \to \mathbf{B}$ is an **order-preserving** function $h : A \to B$ (i.e., $h[\leq^{\mathbf{A}}] \subseteq \leq^{\mathbf{B}}$) and for all i

$$h(f_i^{\mathbf{A}}(a_1, \ldots, a_{n_i})) = f_i^{\mathbf{B}}(h(a_1), \ldots, h(a_{n_i}))$$

As usual, h is **surjective** or **onto** if $h[A] = \{h(a) \mid a \in A\} = B$.

In this case $\mathbf{B} = h[\mathbf{A}]$ is called a **homomorphic image** of $\mathbf{A}$.

A homomorphism $h : \mathbf{A} \to \mathbf{B}$ is an **embedding** if it is one-to-one and **order-reflecting**

i.e., $h^{-1}[\leq^{\mathbf{B}}] \subseteq \leq^{\mathbf{A}}$, or equivalently $h(x) \leq^{\mathbf{B}} h(y) \implies x \leq y$

A homomorphism h is an **isomorphism** if h is a surjective embedding

In this case $\mathbf{A}$ is said to be **isomorphic** to $\mathbf{B}$, written $\mathbf{A} \cong \mathbf{B}$

It is easy to check that h^{-1} is an isomorphism as well

Precongruences and quotient algebras

Recall that a **preorder** is a reflexive and transitive binary relation

A **precongruence** on a po-algebra $\mathbf{A}$ is a preorder α on A with $\leq^{\mathbf{A}} \subseteq \alpha$

$$x\alpha y \implies f^{\mathbf{A}}(z_1,\dots,x,\dots,z_n)\alpha f^{\mathbf{A}}(z_1,\dots,y,\dots,z_n) \text{ if } \sigma(f)_i = 1$$
$$x\alpha y \implies f^{\mathbf{A}}(z_1,\dots,y,\dots,z_n)\alpha f^{\mathbf{A}}(z_1,\dots,x,\dots,z_n) \text{ if } \sigma(f)_i = \partial$$

for all $i \in \{1,\dots,n\}$ and all fundamental operations f of $\mathbf{A}$.

The set of all precongruences of $\mathbf{A}$ is denoted by $\mathrm{Pco}(\mathbf{A})$

Every precongruence α contains a largest **congruence** $\hat\alpha = \alpha \cap \alpha^{-1}$

However, $\hat\alpha$ may not contain $\leq^{\mathbf{A}}$, so in general $\hat\alpha$ is not in $\mathrm{Pco}(\mathbf{A})$.

The **quotient algebra** $\mathbf{A}/\alpha$ of a po-algebra $\mathbf{A}$ modulo a precongruence α is given by $(A/\hat\alpha, \leq^{\mathbf{A}/\alpha}, f_1^{\mathbf{A}/\alpha}, f_2^{\mathbf{A}/\alpha}, \dots)$, where $[x]_{\hat\alpha} \leq^{\mathbf{A}/\alpha} [y]_{\hat\alpha} \iff x\alpha y$

[Pigozzi 2004] proves the isomorphism theorems and correspondence theorem for po-algebras

Varieties of po-algebras

Let $\mathcal{K}$ be a class of po-algebras of the same signature

$\mathsf{H}_P\mathcal{K} =$ the class of **po-homomorphic images** of members of $\mathcal{K}$

$\mathsf{S}\mathcal{K} =$ the class of **subalgebras** of members of $\mathcal{K}$

$\mathsf{P}\mathcal{K} =$ the class of **products** of members of $\mathcal{K}$

$\mathcal{K}$ is a **po-variety** if $\mathcal{K}$ is closed under H_P, S, P

[Pigozzi 2004] $\mathsf{H}_P\mathsf{SP}\mathcal{K} =$ the **po-variety generated** by $\mathcal{K}$

An **inequation** is of the form $s \le t$ for terms s, t

$\mathrm{Mod}_\sigma(\mathcal{I})$ is the class of po-algebras of signature σ that satisfy all inequations in $\mathcal{I}$

Theorem (Pigozzi 2004)

$\mathcal{K}$ is a po-variety if and only if $\mathcal{K} = Mod_\sigma(\mathcal{I})$ for a set of inequations $\mathcal{I}$

Examples of po-varieties

1. The class **Pos** of posets is a po-variety (no fundamental operations)

   For any set S, $\mathsf{H}_P(\{S\}) =$ all posets of cardinality $\le |S|$

   $\mathbf{Pos} = \mathsf{H}_P\mathsf{SP}(\mathbf{2})$, $\quad \mathrm{Mod}_\emptyset(x \le y) =$ only proper po-subvariety
2. Meet-semilattices: $\sigma(\wedge) = 11, \quad x \wedge y \le x, \quad x \wedge y \le y, \quad x \le x \wedge x$

   Hence $z \le x$ and $z \le y \implies z \le z \wedge z \le x \wedge z \le x \wedge y$
3. Lattices: add $\sigma(\vee) = 11, \quad x \le x \vee y, \quad y \le x \vee y, \quad x \vee x \le x$

   Hence $x \wedge (x \vee y) \le x \le x \wedge x \le x \wedge (x \vee y)$
4. Left residuated magmas: $\sigma(\backslash) = \partial 1, \sigma(\cdot) = 11, x \le y\backslash(yx), x(x\backslash y) \le y$
5. Residuated magmas: add $\sigma(/) = 1\partial, \quad x \le (xy)/y, \quad (x/y)y \le x$

$$xy \le z \iff x \le z/y \iff y \le x\backslash z$$

6. Partially ordered groups: $\sigma(\cdot) = 11, \sigma(e) = \lambda, \sigma(^{-1}) = \partial$, group axioms

Quasivarieties of po-algebras

A **quasi-inequality** is of the form $s_0 \le t_0 \ \& \ \dots \ \& \ s_{n-1} \le t_{n-1} \Rightarrow s_n \le t_n$

$\mathcal{K}$ is a **po-quasivariety** if $\mathcal{K} = \mathrm{Mod}_\sigma(\mathcal{Q})$ for a set $\mathcal{Q}$ of quasi-inequalities

Theorem (Pigozzi 2004)

$\mathcal{K}$ is a po-quasivariety if and only if $\mathcal{K} = SP_UP\mathcal{K}$

The class of sets is not a po-variety, but it is a po-quasivariety

A poset is an **antichain** if it satisfies $x \le y \implies x = y$

$\mathbf{Set} = \mathrm{Mod}_\emptyset(\{x \le y \implies x = y\}) = \mathrm{SP}_U\mathrm{P}(\{0,1\}, =)$

Inequational logic

Birkhoff's rules for equational logic give an elegant and complete system for deriving all equational consequences from a given set of identities

Similarly, let $\mathcal{I}$ be a set of inequalities and define $\mathcal{D}$ to be the smallest set containing $\mathcal{I}$ such that for all terms r, s, t

$t \leq t \in \mathcal{D}$

$s \leq t, t \leq s \in \mathcal{D} \implies s = t \in \mathcal{D}$

$r \leq s, s \leq t \in \mathcal{D} \implies r \leq t \in \mathcal{D}$

$\mathcal{D}$ is closed under uniform substitution

$r = s \in \mathcal{D} \implies t(r) = t(s) \in \mathcal{D}$ for any term $t(x)$

$r \leq s \in \mathcal{D} \implies t(r) \leq t(s) \in \mathcal{D}$ if $t(x)$ is order-preserving

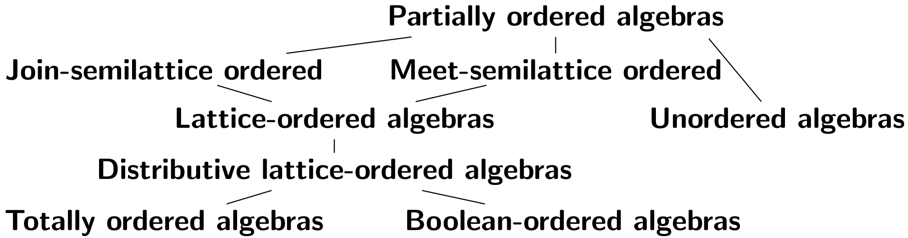
$r \leq s \in \mathcal{D} \implies t(s) \leq t(r) \in \mathcal{D}$ if $t(x)$ is order-reversing

Then $\mathcal{D}$ contains **all** $s \leq t$ that are **true in all models** of $\mathcal{I}$

Scope of the survey

350 page PDF file covering ∼500 classes of partially ordered algebras

The survey contains an introduction and 8 chapters covering classes of a particular order type:



Each section contains definition(s) of the class of algebras, a list of properties, fine spectrum, a list of subclasses, superclasses and (some finite) algebras not in any of its subclasses.

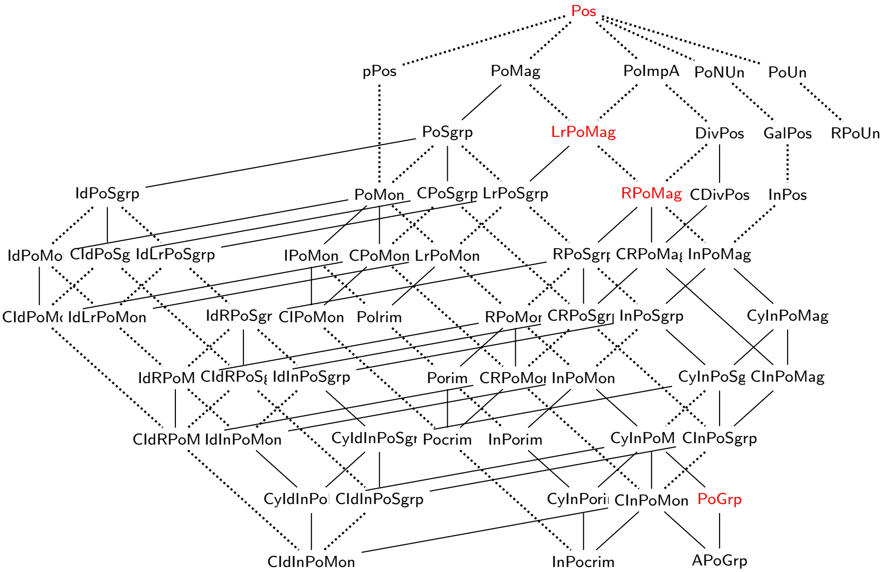


Figure: Classes of partially ordered algebras in Chapter 2

Naming Conventions

Some of the abbreviations used in the survey:

- C = commutative $\quad x \cdot y \leq y \cdot x$
- D = distributive $\quad x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$
- I = integral $\quad x \leq 1$
- Id = idempotent $\quad x \cdot x = x$
- Lr = left-residuated $\quad xy \leq z \Leftrightarrow y \leq x \backslash z$
- Po = partially ordered
- R = residuated $\quad xy \leq z \Leftrightarrow y \leq x \backslash z \Leftrightarrow x \leq z/y$
- To = totally ordered $\quad x \leq y$ or $y \leq x$

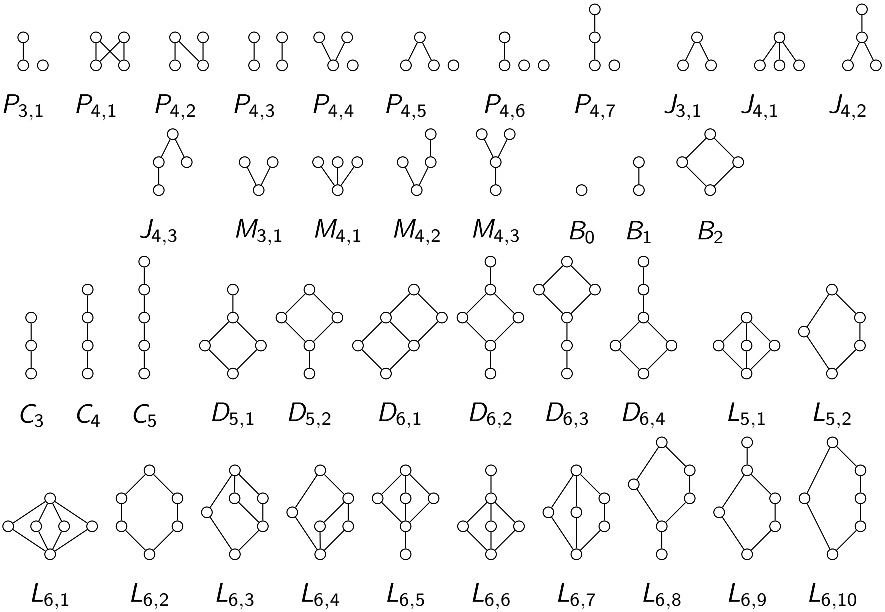
Abbreviations for the most common algebras:

- Grp = groups
- Lat = lattices
- Mag = magmas
- Mon = monoids
- Pos = posets
- Sgrp = semigroups

Properties recorded for the po-variety of lattices

Classtype	Variety
Equational theory	Decidable in PTIME
Quasiequational theory	Decidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes [FN1942]
Congruence modular	Yes
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Congruence extension property	No
Definable principal congruences	No
Equationally def. pr. cong.	No
Amalgamation property	Yes
Strong amalgamation property	Yes [Jón1956]
Epimorphisms are surjective	Yes

Small posets, semilattices, lattices, chains, Bool. algebras

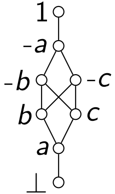


Small members in InPocrim

InPocrim
= involutive partially ordered commutative residuated integral monoids

Subclasses: CIInFL = commutative integral involutive FL-algebras

List of smallest InPocrims that are not in CIInFL



Pos$_{8,1}$

$\cdot$	$-a$	$-c$	$-b$	c	b	a
$-a$	a	a	a	a	a	$\perp$
$-c$	a	a	a	$\perp$	a	$\perp$
$-b$	a	a	a	a	$\perp$	$\perp$
c	a	$\perp$	a	$\perp$	$\perp$	$\perp$
b	a	a	$\perp$	$\perp$	$\perp$	$\perp$
a	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$

InPocrim$_{8,1}$

$\cdot$	$-a$	$-c$	$-b$	c	b	a
$-a$	b	b	a	a	a	$\perp$
$-c$	b	b	a	$\perp$	a	$\perp$
$-b$	a	a	a	a	$\perp$	$\perp$
c	a	$\perp$	a	$\perp$	$\perp$	$\perp$
b	a	a	$\perp$	$\perp$	$\perp$	$\perp$
a	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$

InPocrim$_{8,2}$

$\cdot$	$-a$	$-c$	$-b$	c	b	a
$-a$	$-c$	$-c$	b	a	b	$\perp$
$-c$	$-c$	$-\mathbf{c}$	b	$\perp$	b	$\perp$
$-b$	b	b	a	a	$\perp$	$\perp$
c	a	$\perp$	a	$\perp$	$\perp$	$\perp$
b	b	b	$\perp$	$\perp$	$\perp$	$\perp$
a	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$

InPocrim$_{8,3}$

Fine spectra of classes of algebras

The **fine spectrum** of a class of models is the number of models (up to isomorphism) of each cardinality $n = 1, 2, 3, 4, \ldots$

It is an invariant for each class, preserved by term equivalence. E.g.,

Abelian groups: $f_n = 1, 1, 1, 2, 1, 1, 1, 3, 2, 1, 1, 2, \ldots =$ number of factorizations of n into prime powers.

MV-algebras: $f_n = 1, 1, 1, 2, 1, 2, 1, 3, 2, 1, 1, 4, \ldots =$ number of ways of factoring n into a product with nontrivial factors.

Monoids: $f_n = 1, 2, 7, 35, 228, 2237, 31559, 1668997, \ldots$

Lexicographic list of fine spectra (in Appendix)

Name	Fine spectrum	OEIS
PoMag	1, 16, 4051	No
PoImpA	1, 16, 3981	No
PoSgrp	1, 11, 173, 4753, 198838,...	No
Mag	1, 10, 3330, 178981952,...	A001329
Srng	1, 10, 132, 2341	No
CPoSgrp	1, 7, 83, 1468, 37248,...	No
MedMag	1, 7, 75, 3969	No
IdPoSgrp	1, 7, 69, 1035	No
MMag	1, 6, 280	
JImpA	1, 6, 245	
MImpA	1, 6, 220	
JMag	1, 6, 220	
ToMag	1, 6, 175	
ToImpA	1, 6, 175	
MultLat	1, 6, 175	
DLMag	1, 6, 175	
DLImpA	1, 6, 175	
LMag	1, 6, 175	
LImpA	1, 6, 175	
DivPos	1, 6, 123	
LrPoMag	1, 6, 110	
MSgrp	1, 6, 70, 1437	No
JSgrp	1, 6, 61, 866	No
CDivPos	1, 6, 55, 1434	No
DLSgrp	1, 6, 44, 479	No
LSgrp	1, 6, 44, 479	No
ToSgrp	1, 6, 44, 386	A084965
PoUn	1, 6, 43, 452	No
PoNUn	1, 6, 39, 386, 5203	No
BMag	1, 6, 0, 1176, 0, 0, 0	No
BImpA	1, 6, 0, 1176, 0, 0, 0	No
BSgrp	1, 6, 0, 93, 0, 0, 0	No

LrPoSgrp	1, 5, 28, 273, 3788	No
Sgrp	1, 5, 24, 188, 1915, 28634,...	A027851
DivJslat	1, 4, 281	No
DivMslat	1, 4, 216	
DivLat	1, 4, 216	
ToDivLat	1, 4, 216	
DDivLat	1, 4, 216	
CnjMag	1, 4, 215	
CMag	1, 4, 129, 43968, 254429900,...	A001425
CDivJslat	1, 4, 79, 7545	No
CDivMslat	1, 4, 64, 6208	No
CDivLat	1, 4, 64, 6208	No
PoMon	1, 4, 37, 549	No
CMSgrp	1, 4, 32, 432	??
CJSgrp	1, 4, 29, 289	No
IdMSgrp	1, 4, 28, 308, 4694	No
CPoMon	1, 4, 27, 301, 4887	No
IdJSgrp	1, 4, 23, 166, 1379	No
Srng_0	1, 4, 22, 283	No
Srng_1	1, 4, 22, 169, 1819	No
CDLSgrp	1, 4, 20, 149, 1106	No
CLSgrp	1, 4, 20, 149, 1427	No
CToSgrp	1, 4, 20, 114, 710, 4726,...	A346414
IdLSgrp	1, 4, 17, 100, 674	No
DIdLSgrp	1, 4, 17, 100, 576	No
IdToSgrp	1, 4, 17, 82, 422	??
RPoUn	1, 4, 16, 87, 562	No
GalPos	1, 4, 15, 83, 539	No
InPoMag	1, 4, 12, 77, 498	No
CyInPoMag	1, 4, 12, 76, 481	No
CInPoMag	1, 4, 12, 69, 354, 3632	No
InPoSgrp	1, 4, 10, 50, 210, 1721	No
CyInPoSgrp	1, 4, 10, 50, 196, 1397	No

Original structures database vs. current survey

"*An online database of classes of algebraic structures*", June 2003, Annual Meeting of the Assoc. for Symbolic Logic, Univ. of Illinois at Chicago

This list of mathematical structures is still at
http://math.chapman.edu/~jipsen/structures

An alphabetical list of links that point to (sometimes incomplete) axiomatic descriptions of about 300 categories of universal algebras

Current version from a summer project with Bianca Newell to recreate this list of (partially-ordered) structures as a **computable** LaTeX document

Can be checked for consistency and updated more reliably

The po-algebra background is from Don Pigozzi, *Partially ordered varieties and quasivarieties*, 2004, https://orion.math.iastate.edu/dpigozzi/notes/santiago_notes.pdf

The current DRAFT survey

Would not exist without the tireless efforts of Bianca Newell

The survey is not finished – it's a continuously updated document

Single PDF file with many navigation links to browse the pages

Introduction (Chapter 1) is a very incomplete DRAFT

Download the latest DRAFT version of the PDF file at

`http://math.chapman.edu/~jipsen/Survey-of-po-algebras-DRAFT.pdf`

Software for reading theories and models into Python/Prover9/Mace4 is at

`https://github.com/jipsen/Survey-of-po-algebras`

Results about po-algebras

If $\leq$ is equationally definable then $\mathrm{Pco}(\mathbf{A}) \cong \mathrm{Con}(\mathbf{A})$

This coincides with the notion of **algebraizable** in algebraic logic

Theorem (Gil-Ferez, J.)

If a po-variety is Pco-distributive then Jónsson's Lemma holds

E.g., if all po-variety members $\mathbf{A}$ have lattice reducts $\mathbf{L_A}$ and $\leq^{\mathbf{A}}$ is a subrelation of $\leq^{\mathbf{L_A}}$ then Jónsson's Lemma applies

Theorem

For any po-algebra $\mathbf{A}$, the connected components of $\leq^{\mathbf{A}}$ are the kernel of a homomorphism to an unordered algebra

Partially ordered clones

For a poset $(P, \leq)$, a **po-clone** $(\mathcal{C}, \leq)$ is a clone $\mathcal{C}$ on P that is generated by operations that are order-preserving or order-reversing in each argument

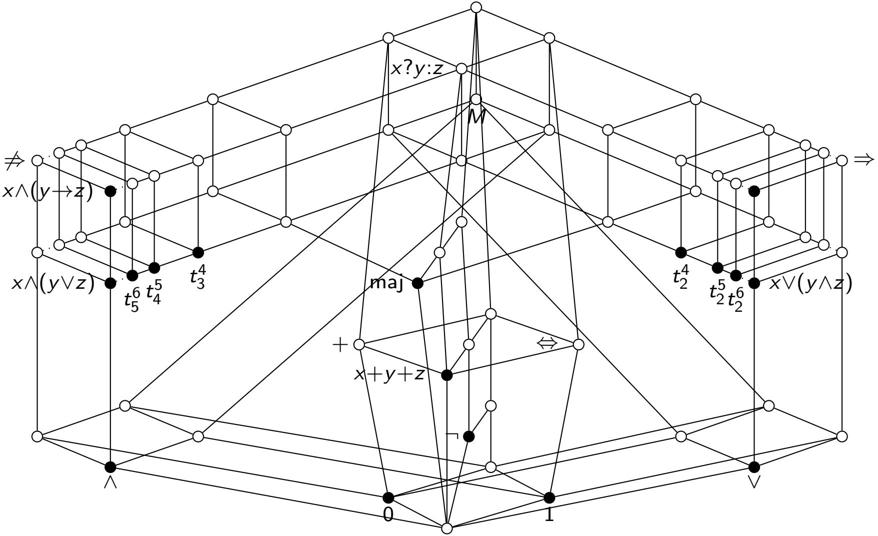
In the Post lattice the clones $\langle x + y + z \rangle$, $\langle + \rangle$, $\langle \Leftrightarrow \rangle$ are **not** po-clones

What about $\langle x + y + z, \neg \rangle$, $\langle x + y + z, \mathrm{maj} \rangle$ or $\langle +, 1 \rangle$?

The remaining clones on 2 elements are po-clones

For a bounded poset $(P, \leq)$, the clone of all operations on P is a po-clone

The Post lattice



$$t_k^n(x_1, \dots, x_n) = \begin{cases} 1 & \text{if } |\{i \mid x_i = 1\}| \geq k \\ 0 & \text{otherwise.} \end{cases}$$

Top clone over bounded poset

(Kurz 2017)

For a bounded poset $(P, \leq)$, the clone of all operations on P is a po-clone

Proof: Let $d : P^n \to (P^n)^\partial$ be given by $d(x) = x$.

Given any operation $g : P^n \to P$, define $f : (P^n)^\partial \times P^n \to P$ by

$$f(x,y) = \begin{cases} g(x) & \text{if } x = y \\ \top & \text{if } x < y \\ \bot & \text{otherwise} \end{cases}$$

Then f is order-preserving: $(x, y) \leq (x', y') \iff x' \leq x$ and $y \leq y'$

Case $x = y$: $f(x, y) = g(x) \leq \top$; Case $x < y$: Then $x' < y'$ so both $f(x, y) = f(x', y') = \top$. Otherwise $f(x, y) = \bot \leq f(x', y')$.

Now $f(d(x), x) = f(x, x) = g(x)$.

Residuation

Definition (Residuated po-magma)

$\mathbf{A} = (A, \leq, \cdot, \backslash, /)$ is a **residuated po-magma** or in **RPoMag** if

- $(A, \leq)$ is a poset and
- $\backslash, /$ are residuals: for all $x, y, z \in A$

$$x \leq z/y \iff x \cdot y \leq z \iff y \leq x\backslash z$$

Note: The residuation formula can be expressed by inequalities and implies $\sigma(\cdot) = 11$, $\sigma(\backslash) = \partial 1$, $\sigma(/) = 1\partial$

E.g., $yz \leq yz \Rightarrow y \leq yz/z$, hence $x \leq y \Rightarrow x \leq yz/z \Rightarrow xz \leq yz$

Theorem

In **RPoMag** *every connected component of* $\leq$ *is up- and down-directed*
In a finite **RPoMag** *every connected component is bounded*

Proof

Two elements x, y in a poset are connected if there exists a zigzag

$$x < z_1 > z_2 < z_3 > \cdots < z_n > y$$

If $a, b \leq c$ then $(a/(z\backslash c))((c/(z\backslash c))\backslash b) \leq a, b$

If $c \leq a, b$ then $a, b \leq c/[((a\backslash c)/(z\backslash(c\backslash c)))((c\backslash c)/(z\backslash(c\backslash c)))\backslash(b\backslash c))]$

Now apply these two results repeatedly to each V and Λ in the zigzag to get an upper and a lower bound of x, y

Residuated po-semigroups as generalized groups

Definition (Residuated po-semigroups or Lambek algebras)

$\mathbf{A} = (A, \leq, \cdot, \backslash, /)$ is a **residuated po-semigroup** or **Lambek algebra** is a residuated po-magma where $\cdot$ is associative

If we add the quasi-inequation $x \leq y \implies x = y$ then we get the po-quasivariety of groups

$\mathsf{H}_P(\mathbb{Z}, =, +, -)$ includes the po-group $(\mathbb{Z}, \leq, +, -)$

[Pigozzi 2004, Cor. 5.7] shows that for left-residuated unital po-magmas $p(x, y, z) = x(y \backslash z)$ is a po-Mal'cev term: $p(x, x, y) \leq y \leq p(y, z, z)$ and $\sigma(p) = 1\partial 1$, hence precongruences are permutable

Questions and open problems

Is there a characterization of po-clones other than finding a generating set operations that are order-preserving or -reversing in each argument?

Which po-clones are generated by operations that are residuated, dually residuated, Galois connections or dual Galois connections?

Describe the free 1-generated residuated po-magma

Find more examples of po-varieties that are Pco-distributive

[Pigozzi 2004] has a Mal'cev condition for permutability of precongruences, but this does not imply modularity.
Can it be strengthened to give modularity or distributivity?

Describe the structure of finite totally ordered bands

. . . (e.g., see ?? in table of fine spectra)

References

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Thanks!