Categories of algebraic contexts equivalent to idempotent semirings and domain semirings

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Abstract. A categorical equivalence between algebraic contexts with relational morphisms and join-semilattices with homomorphisms is presented and extended to idempotent semirings and domain semirings. These contexts are the Kripke structures for idempotent semirings and allow more efficient computations on finite models because they can be logarithmically smaller than the original semiring. Some examples and constructions such as matrix semirings are also considered.

Keywords: Algebraic contexts, idempotent semirings, domain semirings

1 Introduction

The characterization of complete and atomic Boolean algebras as powerset algebras, essentially due to Tarski, is the basis of the categorical duality between the category of complete and atomic Boolean algebras with complete homomorphisms and the category of sets with arbitrary functions. This duality has been extended in many ways to other dualities such as between modal algebras and Kripke frames, between Boolean algebras with operators (which include relation algebras) and atom structures, between distributive lattices with operators and partially ordered frames, and between residuated lattices and residuated frames, to name some of the main examples. Here we present a categorical equivalence that is suitable for idempotent semirings with additional operations and which is based on notions from formal concept analysis. Recall that two categories $\mathbf{C}$, $\mathbf{D}$ are *equivalent* if they are "essentially the same", i.e., there are covariant functors $F : \mathbf{C} \to \mathbf{D}$, $G : \mathbf{D} \to \mathbf{C}$ such that GF and FG are naturally isomorphic to the identity functors $1_{\mathbf{C}}$ and $1_{\mathbf{D}}$ respectively. If the same condition holds for contravariant functors, then the categories are *dually equivalent*.

In formal concept analysis a complete lattice is represented by a *context*, i.e., a triple $\mathbb{X} = (X_-, X_+, X)$ where $X \subseteq X_- \times X_+$ is the *incidence relation*. Since we are interested in idempotent semirings, we consider (completely) join-preserving maps as morphisms between complete lattices, which places us in the category called **SUP** (since joins are also called suprema). In a recent development M. A. Moshier [12] defined morphisms for contexts to obtain a *relational* category **Cxt** that is dual to the category **INF** of complete meet semilattices with completely meet-preserving homomorphisms. Specifically, a morphism R from $\mathbb{X}$ to $\mathbb{Y} = (Y_-, Y_+, Y)$ is a binary relation $R \subseteq X_- \times Y_+$ that satisfies a natural compatibility condition, and composition of morphisms is defined in such a way that the incidence relation of a context is the identity morphism for that object. Since **INF** is both dual and equivalent to **SUP** (a symmetry that is made explicit in the category **Cxt**) we find this setting well suited for our purposes. For the categories of lattices and complete perfect lattices, similar dualities are contained in [9, 3, 7]

We extend this duality to an equivalence between semilattices and algebraic contexts (i.e., contexts where the incidence relation induces an algebraic closure operator) with morphisms that are directed-join-preserving relations between contexts. An alternative presentation of algebraic contexts with approximable maps as morphisms, and their relation to various other categories in domain theory, is given in [10]. For applications to domain semirings, we add a multiplication on the context side via a ternary relation; the identity element corresponds to a unary relation; and a domain operation is given by a binary relation. One of the advantages of the equivalent category of contexts with relations is that the objects can be "logarithmic" in size relative to their algebraic counterparts. In fact objects on the algebraic side correspond to contexts of many different sizes which are all isomorphic in the category **Cxt** (this is possible since the notion of isomorphism in **Cxt** is not based on bijections). This makes the construction of examples on the context side less restricted, so for example idempotent semirings can be obtained from Gentzen-style proof systems. Another interesting aspect is that the equivalence maps products of domain semirings to certain disjoint unions of contexts, and other constructions like ordinal products and poset products can also be obtained by combinatorial means on the context side.

Furthermore the relational morphisms in the category of contexts give this setting a flavor of the category **Rel** (where the objects are sets and morphisms are binary relations), and it is indeed the case that **Rel** is isomorphic to a full subcategory of **Cxt**. We also observe that the ideal completions of Kleene algebras are related to the equivalence with algebraic contexts. Since the notion of relational context morphism is relatively recent, this area is currently still developing and is likely to yield further insight into proof-theoretic and algebraic properties of idempotent semirings with additional operations.

2 Background

We first recall some standard definitions and fix the notation that is convenient for our approach. A *context* is a structure $\mathbb{X} = (X_-, X_+, X)$ such that X_-, X_+ are sets and $X \subseteq X_- \times X_+$. Thus a context is simply a typed relation, called the *incidence relation*, and we will usually not distinguish between the relation X and the context $\mathbb{X}$ that it defines. The relation X determines two functions $X^\uparrow : \mathcal{P}(X_-) \to \mathcal{P}(X_+)$ and $X^\downarrow : \mathcal{P}(X_+) \to \mathcal{P}(X_-)$ by $X^\uparrow A = \{b : \forall a \in A\ aXb\}$ and $X^\downarrow B = \{a : \forall b \in B\ aXb\}$. As usual, these maps form a Galois connection from $\mathcal{P}(X_-)$ to $\mathcal{P}(X_+)$, which means that $A \subseteq X^\downarrow B \Leftrightarrow B \subseteq X^\uparrow A$ for all $A \subseteq X_-$ and $B \subseteq X_+$. Moreover, $X^\downarrow X^\uparrow$ and $X^\uparrow X^\downarrow$ are *closure operators* on X_- and X_+ respectively. The sets $\mathrm{Cl}_-(X) = \{X^\downarrow X^\uparrow A : A \subseteq X_-\}$ and $\mathrm{Cl}_+(X) = \{X^\uparrow X^\downarrow B :$

$B \subseteq X_+\}$ of *Galois-closed sets* are dually isomorphic complete lattices with intersection as meet and Galois-closure of union as join. Note that $\mathrm{Cl}_-(X)$ can also be defined as $\{X^{\downarrow}B : B \subseteq X_+\}$, so a set is Galois-closed if and only if it is in the image of $X^{\downarrow}$ (or similarly $X^{\uparrow}$ for $\mathrm{Cl}_+(X)$). The operations $X^{\uparrow}$ and $X^{\downarrow}$ both map unions to intersections, and sets of the form $X^{\downarrow}\{x\}$ are a *basis* from which all Galois-closed sets can be obtained by intersections.

For contexts X, Y a *context morphism* $R : X \to Y$ is a relation $R \subseteq X_- \times Y_+$ such that $X^{\downarrow}X^{\uparrow}R^{\downarrow} = R^{\downarrow} = R^{\downarrow}Y^{\uparrow}Y^{\downarrow}$ (or equivalently $X^{\downarrow}X^{\uparrow}A \subseteq R^{\downarrow}R^{\uparrow}A$ and $Y^{\uparrow}Y^{\downarrow}B \subseteq R^{\uparrow}R^{\downarrow}B$ for all $A \subseteq X_-$, $B \subseteq Y_+$), in which case the relation is said to be *compatible* with X, Y (see Fig. 1). Here the operations $R^{\uparrow} : \mathcal{P}(X_-) \to \mathcal{P}(Y_+)$ and $R^{\downarrow} : \mathcal{P}(Y_+) \to \mathcal{P}(X_-)$ are defined in the same way as for the binary relation X above, and we use juxtaposition for both function composition and function application (associating to the right). Note that the incidence relation X is itself

$$\begin{tikzcd}
X_+ & Y_+ \\
X_- \arrow[u, "X"] \arrow[ur, "R"] & Y_- \arrow[u, "Y"']
\end{tikzcd}$$

$$R \subseteq X_- \times X_+ \text{ and } X^{\downarrow}X^{\uparrow}R^{\downarrow} = R^{\downarrow} = R^{\downarrow}Y^{\uparrow}Y^{\downarrow}$$

Fig. 1. Context morphism

a morphism from $\mathbb{X}$ to $\mathbb{X}$ (since $X^{\downarrow}X^{\uparrow}X^{\downarrow} = X^{\downarrow}$), and with the composition defined below it is, in fact, the identity morphism on $\mathbb{X}$. Furthermore, compatibility implies that the map $R^{\downarrow}Y^{\uparrow}$ maps closed sets to closed sets, and it is easy to see that if B is a closed set then $A \subseteq R^{\downarrow}Y^{\uparrow}B$ if and only if $Y^{\uparrow}B \subseteq R^{\uparrow}A$ or equivalently $Y^{\downarrow}R^{\uparrow}A \subseteq B$. Hence the maps $Y^{\downarrow}R^{\uparrow}$ and $R^{\downarrow}Y^{\uparrow}$ are *residuals*, which implies that $Y^{\downarrow}R^{\uparrow} : \mathrm{Cl}_-(X) \to \mathrm{Cl}_-(Y)$ preserves arbitrary joins and $R^{\downarrow}Y^{\uparrow} : \mathrm{Cl}_-(Y) \to \mathrm{Cl}_-(X)$ preserves arbitrary intersections. Given a context Z and morphism $S : Y \to Z$, the composite morphism $R \fatsemi S : X \to Z$ is defined by $x \, R \fatsemi S \, y \Leftrightarrow x \in R^{\downarrow}Y^{\uparrow}S^{\downarrow}\{y\}$ (see Fig. 2).

$$\begin{tikzcd}
X_+ & Y_+ & Z_+ \\
X_- \arrow[u, "X"] \arrow[ur, "R"] & Y_- \arrow[u, "Y"] \arrow[ur, "S"] & Z_- \arrow[u, "Z"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X_+ & Z_+ \\
X_- \arrow[u, "X"] \arrow[ur, dotted, "R \fatsemi S"] & Z_- \arrow[u, "Z"']
\end{tikzcd}$$

Fig. 2. Composition of context morphisms

Finally, the *Dedekind-MacNeille context* of a poset L is $\mathrm{DM}(L) = (L, L, \leq)$.

Theorem 1. *[12]*

1. *The collection* **Cxt** *of all contexts with compatible relations as morphism is a category with the incidence relation of each context as the identity morphism.*
2. *The category* **Cxt** *is dually equivalent to the category* **INF** *of complete semilattices with completely meet-preserving homomorphisms. The adjoint functors are* Cl_{-} : **Cxt** $\to$ **INF** *and DM*: **INF** $\to$ **Cxt**. *On morphisms,* $\mathrm{Cl}_{-}(R) = R^{\downarrow}Y^{\uparrow} : \mathrm{Cl}_{-}(Y) \to \mathrm{Cl}_{-}(X)$ *and for an* **INF** *morphism* $h : L \to M$, $DM(h) = \{(x, y) \in M \times L : x \le h(y)\}$.

Lemma 2. *(i)* $R : X \to Y$ *is a monomorphism in* **Cxt** *if and only if* $R^{\downarrow}R^{\uparrow} = X^{\downarrow}X^{\uparrow}$.

(ii) $R : X \to Y$ *is an epimorphism in* **Cxt** *if and only if* $R^{\uparrow}R^{\downarrow} = Y^{\uparrow}Y^{\downarrow}$.

(iii) $R : X \to Y$ *is an isomorphism if and only if it is both mono and epi, or equivalently if* $R^{\downarrow}R^{\uparrow}X^{\downarrow} = X^{\downarrow}$ and $R^{\uparrow}R^{\downarrow}Y^{\uparrow} = Y^{\uparrow}$.

To illustrate the duality in Theorem 1(2) we discuss a few examples. On objects, the duality is simply Birkhoff's [1] notion of polarity that represents any complete lattice as the lattice of Galois-closed sets of some binary relation. This has been studied extensively in formal concept analysis [6] and many tools have been developed to compute with finite contexts. However the notion of morphism for contexts has not gotten as much attention, and several competing definitions have appeared [4, 7, 10]. The category **Cxt** of [12], defined above, has one of the most natural notions of morphism and fits best with the applications we are interested in.

1. Let S be any set, and consider the context $\mathbb{S} = (S, S, \ne)$. Then for any subset A of S we have $\ne^{\uparrow} A = S\backslash A$, hence the Galois closure of A is $S\backslash(S\backslash A) = A$. It follows that $\mathrm{Cl}_{-}(\mathbb{S}) = (\mathcal{P}(S), \bigcap)$, so the complete semilattice corresponding to the context $\mathbb{S}$ is the complete and atomic Boolean algebra of all subsets of S. Of course the duality between the category of complete and atomic Boolean algebras and the category of sets and functions is well known, but here the duality between **INF** and **Cxt** restricts to a duality of complete and atomic Boolean algebras with $\bigwedge$-preserving functions and the category **Rel** of sets and binary relations (since all relations are compatible in this case, hence **Cxt** morphisms. For example if $S = \{0, 1\}$ and $T = \{0, 1, 2\}$ then there are $2^{2\cdot 3} = 64$ binary relations from S to T, hence there are 64 morphisms from context $\mathbb{S}$ to $\mathbb{T} = (T, T, \ne)$, corresponding to 64 $\bigwedge$-preserving maps from an 8-element Boolean algebra to a 4-element Boolean algebra. One of these morphisms is illustrated in Figure 3.
2. Let $(P, \le)$ be a partially ordered set, and consider the context $\mathbb{P} = (P, P, \not\le)$. Then for any subset A of P we have $\not\le^{\uparrow} A = \{x \in P : a \not\le x \text{ for all } a \in A\}$, which is the largest downset of P that does not intersect A. Hence the Galois closure of A is $P \setminus (\not\le^{\uparrow} A)$ = the smallest upset containing A, and therefore $\mathrm{Cl}_{-}(\mathbb{P})$ is the complete $\bigcap$-semilattice of upsets of the poset P. This is in fact a complete and perfect distributive lattice (a lattice is *perfect* if every element is a join of completely join irreducible elements and a meet

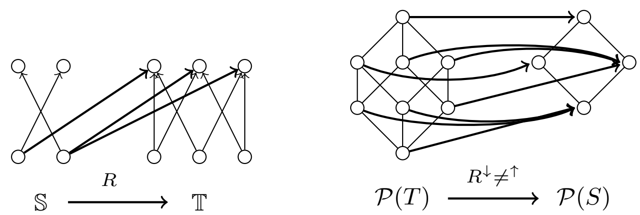

Fig. 3. One of 64 morphisms from $\mathbb{S}$ to $\mathbb{T}$.

of completely meet irreducible elements). The duality between finite posets and finite distributive lattices is due to Birkhoff, but as in 1., the morphisms are compatible relations for the contexts and $\bigwedge$-preserving functions for the complete $\bigwedge$-semilattices.

3. Both 1. and 2. have dualities that can be described without the use of contexts (sets for 1. and posets for 2.). However for semilattices in general, contexts are required. Note that complete $\bigwedge$-semilattices are complete lattices since the $\bigvee A = \bigwedge\{b : a \leq b \text{ for all } a \in A\}$. If a lattice L is also perfect, then we can obtain a smallest context by taking $(J_\infty(L), M_\infty(L), \leq)$, where $J_\infty(L)$ is the set of completely join irreducible elements and $M_\infty(L)$ is the set of completely meet irreducible elements of L. Figure 4 shows some 6-element (semi)lattices with their contexts.

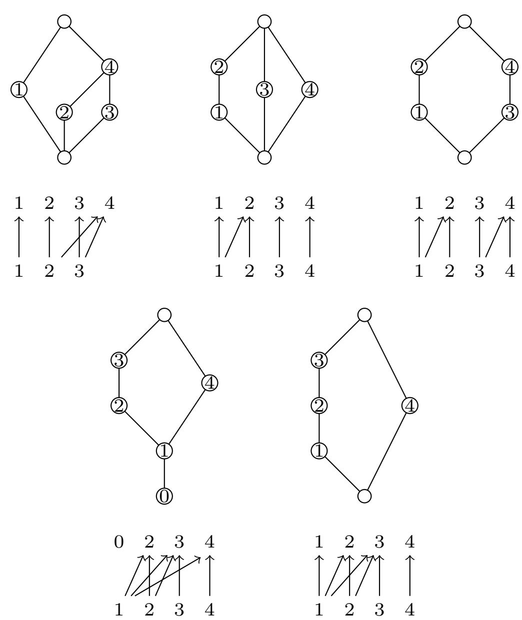

Fig. 4. Examples of 6-element (semi)lattices and corresponding contexts.

3 Semilattices and algebraic contexts

Since we are interested in idempotent semirings, we want to obtain a similar categorical connection between the category of (not necessarily complete) join-semilattices with bottom element and (finite) join-preserving maps, and the category of so-called *algebraic* contexts and certain context morphisms.

Recall [2] that a family $\{A_i : i \in I\}$ of sets is *directed* if for all $i, j \in I$ there exists $k \in I$ such that $A_i \cup A_j \subseteq A_k$. For a context X, the closure operator $X^{\downarrow}X^{\uparrow}$ is *algebraic* if the closure of any subset is the union of the closures of its finite subsets, or equivalently ([2], Theorem 7.14), if for any directed family of sets $\{A_i \subseteq X_- : i \in I\}$ we have $X^{\downarrow}X^{\uparrow}\bigcup_i A_i = \bigcup_i X^{\downarrow}X^{\uparrow}A_i$. The *compact sets* of a context X are given by $K(X) = \{X^{\downarrow}X^{\uparrow}A : A \text{ is a finite subset of } X_-\}$. For algebraic contexts X, Y, an *algebraic* morphism $R : X \to Y$ is a context morphism such that $R^{\downarrow}Y^{\uparrow}$ preserves directed unions, i.e., for any directed family $\{A_i \subseteq X_- : i \in I\}$ we have $R^{\downarrow}Y^{\uparrow}\bigcup_i A_i = \bigcup_i R^{\downarrow}Y^{\uparrow}A_i$. This property holds for identity morphisms and is preserved by composition, hence algebraic contexts form a subcategory of **Cxt** denoted by **ACxt**. Observe that all finite contexts are algebraic, and in this case the compact sets $K(X)$ coincide with the closed sets $\mathrm{Cl}_-(X)$.

The category of join-semilattices with bottom element 0 and join-preserving homomorphisms that preserve 0 is denoted by $\mathbf{JSLat}_0$. We will use $+$ to denote the join operation, since this is in agreement with idempotent semirings. For a join-semilattice L, an *ideal* is a subset J of L such that for all $x, y \in J$, $x+y \in J$ and for all $z \in L$ if $z \leq x$ then $z \in J$. The set of ideals of L is denoted by $I(L)$. Given a join-semilattice L, the ideal context of L is $C(L) = (L, I(L), \in)$, i.e., the incidence relation is the element-of relation. Note that the closure operator $\in^{\downarrow}\in^{\uparrow}$ generates ideals from subsets of L, and since every ideal is the union of its finitely generated subideals, this is an algebraic context.

Theorem 3. *The category* $\mathbf{JSLat}_0$ *is equivalent to* **ACxt**. *The adjoint functors are* $K : \mathbf{ACxt} \to \mathbf{JSLat}_0$ *and* $C : \mathbf{JSLat}_0 \to \mathbf{ACxt}$. *On morphisms,* $K(R) = Y^{\downarrow}R^{\uparrow} : K(X) \to K(Y)$ *and for a* $\mathbf{JSLat}_0$ *morphism* $h : L \to M$, $C(h) = \{(a, D) \in L \times I(M) : h(a) \in D\}$.

Proof. We first show that up to isomorphism C and K are inverses on objects of the category. For L in $\mathbf{JSLat}_0$ and $A \subseteq L$, the closure $\in^{\downarrow}\in^{\uparrow} A = \langle A\rangle$ is the ideal generated by A. For a finite subset A, this ideal is always principal, hence $K(C(L))$ is the set of principal ideals of L. Since any ideal is the union of the principal ideals it contains, and since they are the closure of a singleton, it follows that $C(L)$ is algebraic. It is easy to check that the map $a \mapsto {\downarrow} a = \{b \in L : b \leq a\}$ is an isomorphism from L to $K(C(L))$ ordered by inclusion.

Now let X be an algebraic context and consider $A, B \in K(X)$. Then $A+B = X^{\downarrow}X^{\uparrow}A_0 + X^{\downarrow}X^{\uparrow}B_0 = X^{\downarrow}X^{\uparrow}(A_0 \cup B_0)$ for some finite $A_0 \subseteq A$ and $B_0 \subseteq B$. Hence $K(X)$ is a semilattice, and the least element is $X^{\downarrow}X^{\uparrow}\emptyset$. We need to prove that $(K(X), I(K(X)), \in)$ is isomorphic to X, so we define a relation R from X_- to $I(K(X))$ by xRD if and only if $X^{\downarrow}X^{\uparrow}\{x\} \in D$. To see that R is an isomorphism, it suffices by Lemma 2(iii) to check that R is compatible and that

$R^{\downarrow}R^{\uparrow}X^{\downarrow} = X^{\downarrow}$ and $R^{\uparrow}R^{\downarrow}{\in}^{\uparrow} = {\in}^{\uparrow}$. Note also that each of these equations holds for all subsets if it is valid for singleton subsets.

$X^{\downarrow}X^{\uparrow}R^{\downarrow} = R^{\downarrow}$: Let D be an ideal of $K(X)$. Then $R^{\downarrow}\{D\} = \{x \in X_- : X^{\downarrow}X^{\uparrow}\{x\} \in D\} = \{x \in X_- : x \in A \text{ for some } A \in D\} = \bigcup D$. Since D is a directed set and X is an algebraic context, $X^{\downarrow}X^{\uparrow}\bigcup D = \bigcup_{A\in D} X^{\downarrow}X^{\uparrow}A = \bigcup D$.

${\in}^{\uparrow}{\in}^{\downarrow}R^{\uparrow} = R^{\uparrow}$: For $x \in X_-$ we have

$$R^{\uparrow}\{x\} = \{D \in I(K(X)) : X^{\downarrow}X^{\uparrow}\{x\} \in D\}$$

which is the collection of all ideals that include the principal ideal, say J, generated by $X^{\downarrow}X^{\uparrow}\{x\}$ in $K(X)$. Now ${\in}^{\downarrow}$ of this collection is the intersection of all these ideals, hence is equal to the ideal J. Since J is principal, it follows that ${\in}^{\uparrow}J = \{D : X^{\downarrow}X^{\uparrow}\{x\} \in D\} = R^{\uparrow}\{x\}$.

$R^{\downarrow}R^{\uparrow}X^{\downarrow} = X^{\downarrow}$: Note that for $A \subseteq X_-$ we have $R^{\uparrow}A = \{D : X^{\downarrow}X^{\uparrow}\{a\} \in D$ for all $a \in A\} = \{D : \{X^{\downarrow}X^{\uparrow}\{a\} : a \in A\} \subseteq D\}$. Hence $x \in R^{\downarrow}R^{\uparrow}A$ implies xRD_A, where D_A is the ideal generated by the set $\{X^{\downarrow}X^{\uparrow}\{a\} : a \in A\}$. Therefore $X^{\downarrow}X^{\uparrow}\{x\} \in D_A$, so $X^{\downarrow}X^{\uparrow}\{x\} \subseteq X^{\downarrow}X^{\uparrow}\{a_1, \ldots, a_n\}$ for some finite subset of A. It follows that $x \in X^{\downarrow}X^{\uparrow}A$, and replacing A with $X^{\downarrow}B$ proves the result.

$R^{\uparrow}R^{\downarrow}{\in}^{\uparrow} = {\in}^{\uparrow}$: We first observe that for $C \in K(X)$ we have

$$R^{\downarrow}{\in}^{\uparrow}\{C\} = R^{\downarrow}\{D : C \in D\} = \bigcap\{\bigcup D : C \in D\}$$

since $R^{\downarrow}\{D\} = \bigcup D$. But one of the ideals D is $\downarrow C$, and $\bigcup \downarrow C = C$, hence $R^{\downarrow}{\in}^{\uparrow}\{C\} = C$. It follows that $R^{\uparrow}R^{\downarrow}{\in}^{\uparrow}\{C\} = R^{\uparrow}C = \{D : X^{\downarrow}X^{\uparrow}\{x\} \in D$ for all $x \in C\}$. Therefore $R^{\uparrow}C \subseteq \{D : C \in D\} = {\in}^{\uparrow}\{C\}$ as required.

This isomorphism also shows that R is an algebraic morphism. To check that K is a functor, recall that the identity morphism id_X of a context is the incidence relation X. Hence $K(id_X) = X^{\downarrow}X^{\uparrow}$ which is the identity map on the semilattice $K(X)$ since the elements of $K(X)$ are closed. Composition is preserved since if $R : X \to Y$ and $S : Y \to Z$ then

$$K(R;S) = Z^{\downarrow}(R;S)^{\uparrow} = Z^{\downarrow}S^{\uparrow}Y^{\downarrow}R^{\uparrow} = K(S)K(R).$$

For the map C we have $C(id_L) = \{(a,D) \in L \times I(L) : id_L(a) \in D\} = \{(a,D) : a \in D\} = {\in}$, and this is the incidence relation of the context $C(L)$. Let $h : L \to M$ and $g : M \to N$ be $\mathbf{JSLat}_0$ homomorphisms, then $a\, C(h); C(g)\, D$ if and only if

$$\begin{aligned} a \in C(h)^{\downarrow}{\in}^{\uparrow}C(g)^{\downarrow}\{D\} &= C(h)^{\downarrow}{\in}^{\uparrow}g^{-1}[D] = C(h)^{\downarrow}\{g^{-1}[D]\} \\ &= h^{-1}[g^{-1}[D]] = (gh)^{-1}[D] \end{aligned}$$

and this is equivalent to $a\, C(gh)\, D$. Moreover, it is not difficult to check that $C(h)$ is an algebraic morphism, that $K(R)$ is a $\mathbf{JSLat}_0$ homomorphism, and that $K(C(h))$ and $C(K(R))$ are naturally isomorphic to h and R respectively. □

The above equivalence is of course closely related to the Hofmann-Mislove-Stralka duality [11] between join-semilattices and algebraic lattices with maps that preserve all meets and directed joins (see also [8], p. 274). However, the category of algebraic contexts is much "bigger" than the category of algebraic lattices, since there are many contexts of different sizes that correspond to the same algebraic lattice. Consequently one has much more freedom constructing contexts, and for many semilattices one can obtain contexts that are logarithmically smaller. For example, if a semilattice is given by the compact elements of a complete and atomic Boolean algebra B, then one may take the algebraic context $(At(B), At(B), \neq)$ to represent the semilattice. As mentioned at the end of the previous section, in formal concept analysis many algorithms and visualization tools have been developed for contexts, and with the above equivalence they can be readily applied to arbitrary semilattices.

We have also noted that the category of algebraic contexts contains a subcategory that is isomorphic to **Rel**, the category of sets and binary relations. The objects of this subcategory are the contexts $(A, A, \neq)$ where A is any set, and the morphisms are any binary relation, since the compatibility conditions are automatically satisfied. Another interesting subcategory is obtained by considering posets $(P, \leq)$ and defining the contexts $(P, P, \not\geq)$. The algebraic lattice in this case is the lattice of all downsets of P, which is a complete perfect distributive lattice, and the semilattice of compact elements consists of the downsets of finite subsets of P.

4 Contexts for idempotent semirings and domain semirings

We now show how additional join-preserving operations on the semilattice are represented on the context side. We use the example of domain semirings, but it will be clear that the framework can handle semilattices with join-preserving operations of any arity. Thus the categorical equivalence with algebraic contexts is extended to a proper generalization of the duality for complete and atomic Boolean algebras with operators and relational structures with bounded morphisms.

Recall that an *idempotent semiring* is an algebra $(L, +, 0, \cdot, 1)$ such that $(L, +, 0)$ is in $\mathbf{JSLat}_0$, $(L, \cdot, 1)$ is a monoid, $\cdot$ is join-preserving in both arguments, and $0x = 0 = x0$. A *domain semiring* is of the form $\mathbf{L} = (L, +, 0, \cdot, 1, d)$ such that $(L, +, 0, \cdot, 1)$ is an idempotent semiring, d is join-preserving, $d(0) = 0$,

$$d(x) + 1 = 1$$

$$d(x)x = x \text{ and}$$

$$d(xd(y)) = d(xy).$$

When confusion is unlikely, we usually refer to a domain semiring $\mathbf{L}$ simply by the name of its underlying set L.

Let X be an algebraic context. To capture the operations of the domain semiring on the semilattice $K(X)$, we need a ternary relation $\circ \subseteq X^3_-$, a unary relation $E \subseteq X_-$ and a binary relation $D \subseteq X_-^2$. For $A, B \subseteq X_-$, we define the notation

$$A \circ B = \{c \in X_- : (a,b,c) \in \circ \text{ for some } a \in A, b \in B\} \text{ and}$$

$$D[A] = \{b \in X_- : aDb \text{ for some } a \in A\}.$$

For $x, y \in X_-$ we further abbreviate $x \circ y = \{x\} \circ \{y\}$ and $D(x) = D[\{x\}]$. The closure operation $X^{\downarrow}X^{\uparrow}$ is called a *nucleus with respect to* $\circ$ if for all $A, B \subseteq X_-$ we have

$$(X^{\downarrow}X^{\uparrow}A) \circ (X^{\downarrow}X^{\uparrow}B) \subseteq X^{\downarrow}X^{\uparrow}(A \circ B)$$

and a *nucleus with respect to* D if for all $A \subseteq X_-$ we have

$$D[X^{\downarrow}X^{\uparrow}A] \subseteq X^{\downarrow}X^{\uparrow}D[A].$$

The nucleus property ensures that the operations $X^{\downarrow}X^{\uparrow}(A \circ B)$ and $X^{\downarrow}X^{\uparrow}D[A]$ are join-preserving in each argument. For example, the following calculation shows that the first operation is join-preserving in the second argument (recall that join $\sum$ is the closure of union):

$$\begin{aligned} X^{\downarrow}X^{\uparrow}(A \circ \sum_i B_i) &= X^{\downarrow}X^{\uparrow}(A \circ X^{\downarrow}X^{\uparrow}\bigcup_i B_i) \subseteq X^{\downarrow}X^{\uparrow}(A \circ \bigcup_i B_i) \\ &= X^{\downarrow}X^{\uparrow}\bigcup_i (A \circ B_i) = \sum_i (A \circ B_i) \subseteq \sum_i X^{\downarrow}X^{\uparrow}(A \circ B_i) \end{aligned}$$

where the first $\subseteq$ follows from the nucleus property, and the reverse inclusion always holds.

The relations $\circ$ and D are called *algebraic* if for all $A, B \in K(X)$ the operations $X^{\downarrow}X^{\uparrow}(A \circ B)$ and $X^{\downarrow}X^{\uparrow}D[A]$ are also in $K(X)$.

An *idempotent semiring context* is of the form $(X_-, X_+, X, \circ, E)$ such that (X_-, X_+, X) is an algebraic context, $\circ, E$ are an algebraic ternary and unary relation on X_-, the closure operator is a nucleus with respect to $\circ$, and for all $x, y, z \in X_-$ we have

$$X^{\uparrow}((x \circ y) \circ z) = X^{\uparrow}(x \circ (y \circ z)) \text{ and}$$

$$X^{\uparrow}(x \circ E) = X^{\uparrow}\{x\} = X^{\uparrow}(E \circ x).$$

A *domain context* is a structure $\mathbb{X} = (X_-, X_+, X, \circ, E, D)$ such that $(X_-, X_+, X, \circ, E)$ is an idempotent semiring context, the closure operator is also a nucleus with respect to D, and for all $x, y \in X_-$ we have

$$D(x) \subseteq X^{\downarrow}X^{\uparrow}E,$$

$$X^{\uparrow}(D(x) \circ x) = X^{\uparrow}\{x\} \text{ and}$$

$$X^{\uparrow}D[x \circ D(y)] = X^{\uparrow}D[x \circ y]$$

corresponding to the axioms for the domain operation d. Note that the last 3 conditions need only hold for all elements of X_-, whereas the domain axioms would have to be checked for all elements of the potentially much bigger semilattice of compact sets.

Let $\mathbb{X}, \mathbb{Y}$ be two domain contexts. A relation $R \subseteq X_- \times Y_+$ is a *domain context morphism* if it is compatible, algebraic, $R^\uparrow(E^\mathbb{X}) = Y^\uparrow(E^\mathbb{Y})$, and for all $A, B \in \mathrm{Cl}_-(\mathbb{X})$ we have

$$R^\uparrow(x \circ y) = Y^\uparrow(Y^\downarrow R^\uparrow\{x\} \circ Y^\downarrow R^\uparrow\{y\}) \text{ and}$$

$$R^\uparrow D(x) = Y^\uparrow D[Y R^\uparrow\{x\}].$$

An *idempotent semiring context morphism* is defined likewise, but without the last equation.

As with bounded morphisms (also called p-morphisms) in modal logic the notion of domain context morphism can be written as a first-order formula with variables ranging only over elements of the context. We have not done this here since it is less compact and is no more efficient in implementations than the given formulation.

The functor K from contexts to join-semilattices is extended to domain contexts by defining $K(\mathbb{X}) = (K(X), +, 0, \cdot, 1, d)$ where $A + B = X^\downarrow X^\uparrow(A \cup B)$, $0 = X^\downarrow X^\uparrow \emptyset$, $A \cdot B = X^\downarrow X^\uparrow(A \circ B)$, $1 = X^\downarrow X^\uparrow E$ and $d(A) = X^\downarrow X^\uparrow D[A]$. Likewise the functor C is extended to domain semirings by $C(\mathbf{L}) = (L, I(L), \in, \circ, \{1\}, D)$ where $\circ = \{(x, y, z) \in L^3 : x \cdot y = z\}$ and $D = \{(x, y) \in L^2 : d(x) = y\}$. With these definitions one can check that $K(\mathbb{X})$ is a domain semiring and $C(\mathbf{L})$ is a domain context. For example to check that $X^\downarrow X^\uparrow$ is a nucleus with respect to the relation D, recall that the closure operator generates an ideal from a subset of L. So $D[X^\downarrow X^\uparrow A] = D[\langle A \rangle] = \{y : d(x) = y \text{ for some } x \in \langle A \rangle\} = \{d(a_1 + \cdots + a_n) : a_i \in A, n \in \mathbb{N}\} = \{d(a_1) + \cdots + d(a_n) : a_i \in A, n \in \mathbb{N}\} \subseteq \langle D[A] \rangle = X^\downarrow X^\uparrow D[A]$. We are now ready to state the extended versions of the previous result.

Theorem 4. *The category* **IS** *of idempotent semirings is equivalent to the category* **ISCxt** *of idempotent semiring contexts. The adjoint functors are* $K : \mathbf{ISCxt} \to \mathbf{IS}$ *and* $C : \mathbf{IS} \to \mathbf{ISCxt}$. *On morphisms,* $K(R) = Y^\downarrow R^\uparrow : K(X) \to K(Y)$ *and* $C(h) = \{(a, D) \in L \times I(M) : h(a) \in D\}$.

Similarly the category **DS** *of domain semirings is equivalent to the category* **DSCxt** *of domain semiring contexts. The adjoint functors are* $K : \mathbf{DSCxt} \to \mathbf{DS}$ *and* $C : \mathbf{DS} \to \mathbf{DSCxt}$ *with the operation on morphisms as for idempotent semirings.*

With this result one can specify any domain semiring by a context, a subset, a binary relation and a ternary relation on the first component of the context. For example the first (semi)lattice in Figure 4 can be expanded into 5 nonisomorphic domain semirings where 1 is the identity element. Using the context from the figure with $X_- = \{1, 2, 3\}$, in each case $E = \{1\}$, the binary relation $D = \{(1, 1), (2, 1), (3, 1)\}$, and the 5 ternary relations are

$\circ_1$	2	3
2	$\{2\}$	$\{2\}$
3	$\{2\}$	$\{2\}$

$\circ_2$	2	3
2	$\{2\}$	$\{2\}$
3	$\{2\}$	$\{2,3\}$

$\circ_3$	2	3
2	$\{2,3\}$	$\{2,3\}$
3	$\{2,3\}$	$\{2\}$

$\circ_4$	2	3
2	$\{2,3\}$	$\{2,3\}$
3	$\{2,3\}$	$\{2,3\}$

$\circ_5$	2	3
2	X_-	X_-
3	X_-	X_-

where $1 \circ x = x = x \circ 1$ for all $x \in X_-$. Clearly this is more economical than giving the multiplication tables for five 6-element monoids.

Given a semiring L, one can construct the semiring $M_n(L)$ of all $n \times n$ matrices with entries from L in the usual way. This object has $|L|^{n^2}$ many elements, but for idempotent semirings the context $\mathbb{Y}$ of $M_n(L)$ is much smaller since it can be constructed from n^2 disjoint copies of the idempotent semiring context $\mathbb{X} = C(L)$ as follows. Let $Y_- = \{(i,j,a) : a \in X_-, i,j = 1,\ldots,n\}$ and define $(i,j,a)Y(i',j',a')$ iff $i \neq i'$ or $j \neq j'$ or aXa', $E = \{(i,i,a) : a \in E, i = 1,\ldots,n\}$, and $(i,j,a) \circ (k,l,b) = \{(i,l,c) : j = k \text{ and } c \in a \circ b\}$.

Kripke-style semantics, as provided for semilattice-expansions by algebraic contexts, are also nicely related to completions. Instead of the functor K to the category of semilattices, one can use the functor Cl_- to the category of complete semilattices. Indeed, the functor C followed by Cl_- is simply the ideal completion of semilattices, which also applies to idempotent semirings, domain semirings and (domain) Kleene algebras. For example there is a Kleene * induced on the completion of a domain semiring by defining $x^* = \sum_{i=0}^{\omega} x^i$, where $x^0 = 1$ and $x^{i+1} = x^i x$ for $i \geq 1$. It is currently ongoing research to adapt the equivalence with algebraic contexts so that it can represent a given Kleene * directly on contexts.

The equivalences with contexts (plus relations) can also be used to get insight into constructions on the algebraic side. In particular, products and coproducts in the algebraic categories are mapped to significantly different types of constructions on the context side. It remains to be seen whether this produces new results about, for example, the structure of free objects on the algebraic side. In the related area of residuated lattices the notion of residuated frame (= residuated context) has already produced significant results about decidability and finite embeddability [5]. Exploring connections with proof theory and coalgebras are other promising directions.

5 Conclusion

Based on a duality by Moshier [12] between complete semilattices and contexts, we have defined algebraic contexts and algebraic morphisms and proved an equivalence between the category of semilattice and the category of algebraic contexts. We then extended this equivalence to idempotent semirings and domain semirings, thereby obtaining Kripke-style semantics for these two categories. The same approach can be used to define categories of algebraic contexts with additional relations that are equivalent to categories of idempotent semilattices with operations that are join-preserving in each argument, thus generalizing the duality between complete and atomic Boolean algebras with operators and relational Kripke structures.

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