ON THE STRUCTURE OF GENERALIZED BL-ALGEBRAS

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Abstract. A *generalized BL-algebra* (or GBL-algebra for short) is a residuated lattice that satisfies the identities $x \wedge y = ((x \wedge y)/y)y = y(y\backslash(x \wedge y))$. It is shown that all finite GBL-algebras are commutative, hence they can be constructed by iterating ordinal sums and direct products of Wajsberg hoops. We also observe that the idempotents in a GBL-algebra form a subalgebra of elements that commute with all other elements.

Subsequently we construct subdirectly irreducible noncommutative integral GBL-algebras that are not ordinal sums of generalized MV-algebras. We also give equational bases for the varieties generated by such algebras. The construction provides a new way of order-embedding the lattice of ℓ-group varieties into the lattice of varieties of integral GBL-algebras.

The results of this paper also apply to pseudo-BL algebras.

1. Preliminaries

For some background on residuated lattices, we refer the reader to [11], but we recall here the definitions that are directly related to our topic. A *generalized BL-algebra* or *GBL-algebra* is a residuated lattice that satisfies *divisibility*: $x \leq y$ implies $x = uy = yv$ for some u, v. This condition is equivalent to the implication $x \leq y \implies x = (x/y)y = y(y\backslash x)$, which in turn is captured by the equations $x \wedge y = ((x \wedge y)/y)y = y(y\backslash(x \wedge y))$. Hence GBL-algebras form a variety. We also note that they have distributive lattice reducts and that the subvariety of integral GBL-algebras is defined by the simpler equations $x \wedge y = (x/y)y = y(y\backslash x)$ [11]. The variety of lattice-ordered groups (or ℓ-groups) is term-equivalent to the subvariety of residuated lattices determined by the equation $x(1/x) = 1$. The variety of pseudo-BL-algebras [6][8] is defined by expanding the signature with a constant

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0 denoting the least element of the algebra, together with the equations for residuated lattices, integral GBL-algebras, and *prelinearity*: $1 \leq (x/y \vee y/x) \wedge (x\backslash y \vee y\backslash x)$.

2. All finite GBL-algebras are commutative

By [9], any GBL-algebra is a direct product of an ℓ-group and an integral GBL-algebra, hence the structure of GBL-algebras can by understood by analyzing the structure of ℓ-groups and integral GBL-algebras.

In particular, any finite GBL-algebra is integral.

Lemma 1. *If a is an idempotent in an integral GBL-algebra A, then $ax = a \wedge x$ for all $x \in A$. Hence every idempotent is central, i.e. commutes with every element.*

Proof. Suppose $aa = a$. Then $ax \leq a \wedge x = a(a\backslash x) = aa(a\backslash x) = a(a \wedge x) \leq ax$. □

In an ℓ-group only the identity is an idempotent, hence it follows from the decomposition result mentioned above that idempotents are central in all GBL-algebras. Recall that a set $S \subseteq A$ is *upward closed* if for all $s \in S$ and all $a \in A$ we have $s \leq a$ implies $a \in S$. The next lemma shows that in a finite GBL-algebra the elements above a maximal non-unit idempotent form a chain.

Lemma 2. *Let a be a coatom in an integral GBL-algebra. Then $\{a^k : k = 0, 1, 2, \ldots\}$ is upward closed.*

Proof. We first show that either a^k is a cover of a^{k+1} or $a^k = a^{k+1}$ for $k = 0, 1, 2, \ldots\}$.

Consider any element b such that $a^{i+1} < b \leq a^i$ for some $i > 0$. By divisibility there exists $u \in A$ such that $b = a^i u$. Since $b \not\leq a^{i+1}$, we must have $u \not\leq a$, hence $a \vee u = 1$. It follows that $a^i = a^i(a \vee u) = a^{i+1} \vee a^i u = a^{i+1} \vee b = b$, hence $a^{i+1} \preceq a^i$.

Now suppose the powers of a do not form an upward closed set, i.e. there exists an element $c \geq a^n$ for some n, and $c \neq a^i$ for all i. Let $m < n$ be minimal with respect to $a^{m+1} \not\geq c$, whence $a^m \leq a^{m+1} \vee c$. So $a^m \leq a^j \vee c$ for some j with $m < j$, and therefore $a^{j-1} \leq a^m \leq a^j \vee c$. Now we compute

$$a^m \leq a^j \vee c = aa^{j-1} \vee c \leq a(a^j \vee c) \vee c = a^{j+1} \vee ac \vee c = a^{j+1} \vee c.$$

By induction, it follows that $a^m \leq a^k \vee c$ for all $k > m$. Since we assumed $c \geq a^n$, we get $a^m \leq c$, a contradiction. □

In a finite residuated lattice, the central idempotent elements form a sublattice that is dually isomorphic to the congruence lattice of the algebra [11]. Hence a subdirectly irreducible finite residuated lattice has a unique largest central idempotent $c < 1$.

Recall that a *Wajsberg hoop* is an integral commutative residuated lattice that satisfies the identity $x \vee y = (x\backslash y)\backslash y$. It is well-known that for each positive integer n there is a unique subdirectly irreducible Wajsberg hoop with n elements: $1 > a > a^2 > \cdots > a^{n-1} = 0$.

We also make frequent use of ordinal sums. For posets A, B the *ordinal sum* $A \oplus B$ is defined on $A \cup B$ by extending the union of the two partial orders so that all elements of $A \setminus B$ are less than all elements of B. Note that we don't require A and B to be disjoint, but the orders should agree on the intersection, and $A \setminus B$ should be a downset of A to ensure that the resulting relation is again a partial order. If A and B are GBL-algebras that are either disjoint or $A \cap B = \{1^A\}$ and 1^A is the least element of B, then the resulting structure is again a GBL-algebra where for $a \in A$ and $b \in B$ one defines $a \cdot b = b \cdot a = a$.

We now show that any finite subdirectly irreducible GBL-algebra decomposes as the ordinal sum of a Wajsberg hoop on top of a smaller GBL-algebra.

Lemma 3. *Let A be a finite subdirectly irreducible GBL-algebra, and let c be its unique largest idempotent below 1. Then A is the ordinal sum of $\downarrow c$ and $\uparrow c$, where c is the identity of $\downarrow c$, and the residuals in the lower component are defined by $x\backslash^{\downarrow} y = x\backslash y \wedge c$ and $x/^{\downarrow} y = x/y \wedge c$. Furthermore $\uparrow c$ is a Wajsberg hoop.*

Proof. By the preceding lemmas, it suffices to show that there are no elements b incomparable to c. We may assume that $c = a^n$ for a suitable coatom a. Suppose by way of contradiction, that b is an element incomparable to c, and choose it to be maximal. We argue that b must also be a coatom.

Let i be maximal with respect to $a^i \geq b$. If $i > 0$, then $b \leq a$, so by divisibility $b = av$ for some $v \in A$. We cannot have $v = a^k$, since $b \not\geq c$. But now the preceding lemma and $b \leq v$ imply $b = v$ by maximality of b. Thus $b = ab$, which leads to $a^{i+1} \geq ab = b$, contradicting the maximality of i. Therefore $i = 0$, and so b is an atom distinct from a.

Using the preceding lemma once more, we see the $\{b^k : k = 0, 1, 2, \ldots\}$ is upward closed, with an idempotent, say d, as least element. Thus c and d are both central idempotents by Lemma 1, and they join to 1. This is impossible since we assumed that the algebra is subdirectly irreducible. □

Since all Wajsberg hoops are commutative, the main result follows by induction on the size of the algebra. More precisely, given a finite subdirectly irreducible GBL-algebra A, we decompose it into the ordinal sum of a smaller GBL-algebra and a Wajsberg hoop. The smaller GBL-algebra is a subdirect product of subdirectly irreducible homomorphic images, each smaller than A, hence by the inductive hypothesis, they are commutative. Since ordinal sums preserve commutativity, the result follows.

Note that the theorem also holds if we expand the signature with a constant 0 to denote the least element of the algebra.

Theorem 4. *Every finite GBL-algebra and every finite pseudo-BL-algebra is commutative.*

Corollary 5. *The varieties of all GBL-algebras and of all pseudo-BL-algebras do not have the finite model property, i.e. they are not generated by their finite members.*

With the help of Lemma 1, it is easy to see that the set of idempotents in a GBL-algebra is a sublattice that is closed under multiplication. We show that it is also closed under the residuals. It follows from our noncommutative examples below that the set of all central elements in a GBL-algebra is, in general, not a subalgebra.

Theorem 6. *The idempotents in a GBL-algebra form a subalgebra.*

Proof. By the decomposition result of [9], it suffices to prove the result for integral GBL-algebras. Let $aa = a$ and $bb = b$ be two idempotents. Then

$$a\backslash b \le (a \vee a\backslash b)\backslash(a\backslash b) \le a\backslash(a\backslash b) = aa\backslash b = a\backslash b, \text{ and}$$
$$a(a\backslash b) = a \wedge b = (a \wedge b)^2 = (a(a\backslash b))^2.$$

By divisibility, we have

$$\begin{aligned} a\backslash b &= (a \vee a\backslash b)((a \vee a\backslash b)\backslash(a\backslash b)) \\ &= (a \vee a\backslash b)(a\backslash b) \\ &= a(a\backslash b) \vee (a\backslash b)^2 \\ &= (a(a\backslash b))^2 \vee (a\backslash b)^2 = (a\backslash b)^2. \end{aligned}$$

The last equality holds in integral algebras because $a(a\backslash b) \le a\backslash b$. By symmetry, it follows that $(a/b) = (a/b)^2$, hence the residuals of any two idempotents are again idempotents. ◻

A *Brouwerian algebra* is a residuated lattice that satisfies $xy = x \wedge y$. The previous result shows that any GBL algebra contains a largest Brouwerian subalgebra, given by the subalgebra of idempotents.

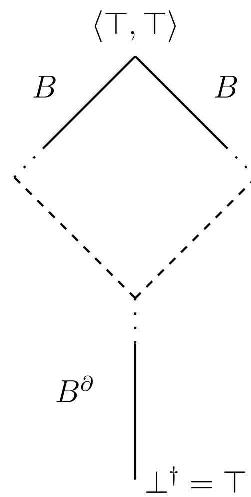

FIGURE 1

3. A construction of non-commutative GBL-algebras

We first note that any noncommutative ℓ-group G, the negative cone G^- of G, and any large enough principal lattice filter $\uparrow b$ of G^- is an example of a noncommutative GBL-algebra (in the latter case one defines $xy = (x \cdot^G y) \vee b$, $x\backslash y = (x^{-1} \cdot^G y) \wedge 1$, and "large enough" means the filter should include a failure of commutativity, hence it must be infinite by Theorem 4). However all these examples satisfy the identities $x \vee y = x/((x \vee y)\backslash x) = (x/(x \vee y))\backslash x$ that define generalized MV-algebras ([11]), called GMV-algebras for short (see also [7] for the bounded version). The variety of integral GBL-algebras is closed under the construction of ordinal sums (even infinite ones), hence one may construct further GBL-algebras by stacking the given examples on top of each other (only the top algebra is allowed to be nonintegral). In the case of commutative GBL-algebras, this observation produces a collection of examples that generates the variety of all commutative GBL-algebras, as can be deduced from results about hoops in [4]. So it is natural to ask whether the same structure theory might hold without commutativity. Our next examples show that this is not the case. Readers with a background in ℓ-groups may recognize these examples as certain modified intervals in the Scrimger 2-group. Similar modifications applied to Scrimger n-groups or other wreath products can be used to produce further nonisomorphic examples.

Let B be a residuated lattice with top element $\top$, and denote by B^∂ the dual poset of the lattice reduct of B. Let $B^\dagger$ be the ordinal sum of B^∂ and $B \times B$, i.e., every element of B^∂ is below every element of

$B \times B$ (see Figure 1). Note that $B^\dagger$ is a lattice under this partial order, with bottom element $\top$. To avoid confusion, we denote this element by $\perp^\dagger$. We define a binary operation $\cdot$ on $B^\dagger$ as follows:

$$\begin{aligned}
\langle a, b\rangle \cdot \langle c, d\rangle &= \langle ac, bd\rangle \\
\langle a, b\rangle \cdot u &= u/a \\
u \cdot \langle a, b\rangle &= b\backslash u \\
u \cdot v &= \top = \perp^\dagger
\end{aligned}$$

To avoid some ambiguity, we use juxtaposition for the monoid operation of B, but will continue to write $\cdot$ for the operation in $B^\dagger$, and $\backslash^\dagger, /^\dagger$ for the residuals. Note that even if B is a commutative residuated lattice, $\cdot$ is in general noncommutative. Our first result shows that, somewhat surprisingly, $\cdot$ is associative and residuated. The proof makes use of the identities $x/(yz) = (x/z)/y$ and $(x\backslash y)/z = x\backslash(y/z)$ that are equivalent to associativity for residuated binary operations.

We also recall that a subset N of a residuated lattice A is *normal* if it is closed under left and right conjungates, i.e. for all $a \in A$ and $x \in N$, $a\backslash xa \wedge 1, ax/a \wedge 1 \in N$. A *congruence filter* is a lattice filter that is also a normal submonoid of A. Hence the smallest congruence filter is $\uparrow 1$, and a congruence filter that properly includes this one is called *nontrivial*. Congruence filters are in one-to-one correspondence with congruence relations on A [5][11], and A is subdirectly irreducible if and only if it has a unique minimal nontrivial congruence filter.

Lemma 7. *For any residuated lattice B with top element, the algebra $B^\dagger$ defined above is a bounded residuated lattice. If B is nontrivial, then $B^\dagger$ is not a GMV-algebra, and if B is subdirectly irreducible, so is $B^\dagger$.*

Proof. Since $\cdot$ is defined pointwise on $B \times B$, it is clearly associative there. The remaining cases (omitting mirror images) are checked as follows:

$$\begin{aligned}
&(\langle a, b\rangle\cdot\langle c, d\rangle)\cdot u = \langle ac, bd\rangle\cdot u = u/(ac) = (u/c)/a \\
&\qquad = \langle a, b\rangle\cdot(u/c) = \langle a, b\rangle\cdot(\langle c, d\rangle\cdot u) \\
&(\langle a, b\rangle\cdot u)\cdot\langle c, d\rangle = (u/a)\cdot\langle c, d\rangle = c\backslash(u/a) = (c\backslash u)/a \\
&\qquad = \langle a, b\rangle\cdot(c\backslash u) = \langle a, b\rangle\cdot(u\cdot\langle c, d\rangle) \\
&(\langle a, b\rangle\cdot u)\cdot v = (u/a)\cdot v = \perp^\dagger = \langle a, b\rangle\cdot\perp^\dagger = \langle a, b\rangle\cdot(u\cdot v) \\
&(u\cdot\langle a, b\rangle)\cdot v = (b\backslash u)\cdot v = \perp^\dagger = u\cdot(v/a) = u\cdot(\langle a, b\rangle\cdot v) \\
&(u\cdot v)\cdot w = \perp^\dagger\cdot w = \perp^\dagger = u\cdot\perp^\dagger = u\cdot(v\cdot w)
\end{aligned}$$

To show that $B^\dagger$ is residuated, we define the residuals below, and leave it to the reader to check that this is indeed the correct definition. Note that the order for elements in the bottom ordinal sum component is *reversed.*

$$
\begin{array}{llll}
\langle a,b\rangle\backslash^\dagger\langle c,d\rangle & = \langle a\backslash c, b\backslash d\rangle & \langle a,b\rangle/^\dagger\langle c,d\rangle & = \langle a/c, b/d\rangle \\
\langle a,b\rangle\backslash^\dagger u & = ua & \langle a,b\rangle/^\dagger u & = \langle\top,\top\rangle \\
u\backslash^\dagger\langle a,b\rangle & = \langle\top,\top\rangle & u/^\dagger\langle a,b\rangle & = bu \\
u\backslash^\dagger v & = \langle\top, u/v\rangle & u/^\dagger v & = \langle u\backslash v,\top\rangle
\end{array}
$$

The GMV identity $x\vee y = x/((x\vee y)\backslash x)$ fails if we take $x = 1\in B^\partial$ and $y = \langle\top, b\rangle \in B^2$ for some $b\neq\top$, since $x\vee y = y$ but the right hand side evaluates to $\langle\top,\top\rangle$.

Finally, to see that the construction preserves subdirect irreducibility, assume B has a unique minimal nontrivial congruence filter M. We claim that M^2 is the corresponding unique minimal nontrivial congruence filter for $B^\dagger$. Note that for $\langle a,b\rangle\in M^2$ and $u\in M$, the left conjugate $u\backslash^\dagger(\langle a,b\rangle\cdot u) = u\backslash^\dagger(u/a) = \langle\top, u/(u/a)\rangle \in M\times M$, since $a\in M$ implies that $u/(u/a)$ is congruent to $u/(u/1)\geq 1\in M$. The right conjugate is similar, and conjugation by elements of M^2 is computed pointwise, so it is closed under conjugation by such elements as well. Moreover, M^2 is obviously an up-closed subalgebra, hence it is a congruence filter.

To see that it is the smallest congruence filter, consider a congruence filter $F\subseteq M^2$ generated by a pair $\langle a,b\rangle\not\geq\langle 1,1\rangle = 1^\dagger$. By symmetry, we may assume that $a\not\geq 1$, and since $a/(a/a)\leq a$, it follows that the left conjugate $a\backslash^\dagger(\langle a,b\rangle\cdot a) = \langle\top, a/(a/a)\rangle\not\geq\langle 1,1\rangle$. Using the observation that conjugation by elements of M^2 is computed pointwise, it follows that $\{\langle\top,u\rangle : u\in M\}\subseteq F$. Similarly, letting $c = a/(a/a)$ and computing the right conjugate $(c\cdot\langle\top,c\rangle)/^\dagger c = \langle(c\backslash c)\backslash c,\top\rangle$, we see that $\{\langle u,\top\rangle : u\in M\}\subseteq F$. Since F is closed under pointwise meet, we conclude that $F = M^2$. □

Thus far we have obtained an interesting construction of noncommutative nonlinear subdirectly irreducible residuated lattices from possibly commutative and linear ones. However that is not particularly noteworthy since many such examples (even finite ones) are known. The strength of this construction comes from the next observation.

Lemma 8. *Let B be a residuated lattice with top element. Then $B^\dagger$ is a GBL-algebra if and only if B is a cancellative GBL-algebra.*

Proof. The GBL identities are equivalent to the quasiequation(s) $x\leq y$ implies $x = (x/y)\cdot y = y\cdot(y\backslash x)$. This condition holds in B if and only if it holds for the elements of B^2, since the operations act pointwise on this part of the algebra. The condition of cancellativity appears naturally when we consider the case $x = u\in B^\partial$ and $y = \langle a,b\rangle$. We compute $(u/\langle a,b\rangle)\cdot\langle a,b\rangle = (bu)\cdot\langle a,b\rangle = b\backslash(bu)$. Since $u,b\in B$ are arbitrary, divisibility holds if and only if $b\backslash(bu) = u$, and by symmetry $(ua)/a = u$, hold for all $u, a, b \in B$. It is well-known (and easy to see) that these two identities correspond to cancellativity.

Finally, if $u \leq^\dagger v \in B^\partial$ then $v \leq u \in B$, hence $(u/^\dagger v) \cdot v = \langle u\backslash v, \top\rangle \cdot v = v/(u\backslash v) = u \vee^B v = u$, where the second last equality follows from the GMV identities that are consequences of cancellativity and the GBL identities (see [3]). $\square$

Note that if a residuated lattice has a top element and is either cancellative or a GBL-algebra, then it is in fact integral.

By a result in [3], cancellative integral GBL-algebras are precisely the negative cones of ℓ-groups, so there are many choices for B. An easy example is obtained if one takes $B = \mathbb{Z}^-$.

Corollary 9. *There exists a GBL-algebra that is noncommutative, subdirectly irreducible, ordinal sum indecomposable, and is not a GMV-algebra.*

Recall that a GBL-algebra is prelinear if it satisfies $1 \leq (x/y \vee y/x) \wedge (x\backslash y \vee y\backslash x)$. It is simple to check that if B is integral and prelinear, then $B^\dagger$ is integral, bounded and prelinear, hence it can be expanded to a pseudo-BL-algebra that is not an ordinal sum of pseudo-MV-algebras and/or prelinear GMV-algebras. Alternatively, omitting the join operation, we obtain examples for generalized hoops with the analogous properties.

This is in contrast with the situation for BL-algebras and basic hoops, where every subdirectly irreducible member is an ordinal sum of MV-algebras or Wajsberg hoops [1][2]. Thus the examples indicate that a structure theorem for GBL-algebras will be more complicated than for BL-algebras.

We also observe that if B is integral and commutative, then the elements $\langle a, a\rangle \in B^\dagger$ are central, and of course $= \perp^\dagger$ is central. However $\langle a, a\rangle\backslash\perp^\dagger = a^\partial$ is not central, hence the center of a GBL-algebra need not be a subalgebra.

Finally, we consider the varieties generated by algebras of the form $B^\dagger$. For the remainder we assume that these algebras have constants $\perp^\dagger$ and $0^\dagger$ to denote the elements $\top$ and 1 in B^∂ (but B is any residuated lattice with $\top$).

Theorem 10. *If $A = (\mathbb{Z}^-)^\dagger$ then $Var(A)$ is a variety that covers the variety of Boolean algebras.*

Proof. We first note that $\{0^A, 1^A\}$ is the only proper subalgebra of A that is not isomorphic to A. In particular it follows that any element other than 0^A or 1^A generates a subalgebra isomorphic to A. Moreover, A has only one proper nontrivial congruence and this congruence has exactly two congruence classes. We show that any subdirectly irreducible member of $\mathrm{Var}(A)$ is either the 2-element Boolean algebra or has a subalgebra isomorphic to A.

Let C be a subdirectly irreducible member of $\mathrm{Var}(A)$ and suppose C has more than two elements. By Jónsson's Lemma, C is a homomorphic image of a subalgebra D of an ultrapower of A. Let $c \in C$ be an element other than 0 or 1, and let $d \in D$ be a preimage of c under the homomorphism. It suffices to show that $a = \neg d$ generates a subalgebra of D that is isomorphic to A, since the homomorphism cannot collapse any elements of this subalgebra. Now $a = \langle a_i \rangle/\mathcal{U}$, where $\mathcal{U}$ is an ultrafilter on the index set I. Consider the set $J = \{i \in I : a_i^2 = 0^A\}$. If $J \in \mathcal{U}$ then a generates an isomorphic copy of A in precisely the way $-1 \in (\mathbb{Z}^-)^\partial$ generates A, and if $J \notin \mathcal{U}$ then the generation proceeds as with $\langle 0, -1 \rangle \in \mathbb{Z}^- \times \mathbb{Z}^-$. In either case we get the desired result. $\square$

We now show how one may construct an equational basis $E^\dagger$ for $B^\dagger$, given an equational basis E for B. Let $0^\dagger = 1 \in B^\partial$ and define

$$\begin{aligned}
&\neg x = x\backslash 0^\dagger, \quad {\sim}x = 0^\dagger/x, \quad x \leftrightarrow y = x\backslash y \wedge y\backslash x \\
&d(x) = \neg(\neg x^2)^2, \quad \text{where } \neg x^2 \text{ is read as } \neg(x^2) \\
&\pi_1(x) = \neg\neg x \wedge {\sim}\neg x, \qquad \pi_2(x) = \neg{\sim}x \wedge {\sim}{\sim}x \\
&p(x, y) = {\sim}\neg x \wedge \neg{\sim}y \wedge d(x) \wedge d(y) \\
&\delta(x) = \pi_1(x) \vee \neg d(x) \qquad \delta'(x) = ({\sim}x \wedge \neg x) \vee d(x).
\end{aligned}$$

The next lemma summarizes the properties of the terms defined here, as is easily checked by direct calculation in $B^\dagger$.

Lemma 11. *For $x, y \in B^\partial$,*

$$\begin{aligned}
&\neg x = \langle \top, x \rangle, \quad {\sim}x = \langle x, \top \rangle, \quad \neg\langle x, y \rangle = x, \quad {\sim}\langle x, y \rangle = y \\
&d(x) = \bot^\dagger, \qquad d(\langle x, y \rangle) = \langle \top, \top \rangle = \delta(x), \qquad \delta'(x) = \langle x, x \rangle \\
&p(\langle x, x \rangle, \langle y, y \rangle) = \langle x, y \rangle, \quad \pi_1(\langle x, y \rangle) = \langle x, x \rangle, \quad \pi_2(\langle x, y \rangle) = \langle y, y \rangle.
\end{aligned}$$

Thus p, π_1, π_2 satisfy the properties for a pairing function and projections, when restricted to (the diagonal of) $B \times B$.

Let $E^\dagger$ be the set of identities determined as follows. We may assume that the identities in E are all of the form $1 \leq t$ for some term t. For each such identity, we add to $E^\dagger$ the identity $d(x_1) \wedge \cdots \wedge d(x_n) \leq t$, where $x_1, \ldots, x_n$ are the variables appearing in t. In addition we add the following identities:

(1) $d(x)d(y) = d(x) \wedge d(y), \quad \neg\neg d(x) = d(x)$
(2) $d(x \diamond y) = d(x) \diamond d(y)$, where $\diamond \in \{\vee, \wedge, \cdot, \backslash, /\}$
(3) $d(1^\dagger) = \top, \quad \neg\top \leq x, \quad d(d(x)) = d(x)$
(4) $(d(x) \wedge 1) \vee (d(y) \wedge 1) = (d(x) \vee d(y)) \wedge 1$
(5) $(d(x) \wedge 1)y = y(d(x) \wedge 1), \quad (d(x) \wedge 1)^2 = d(x) \wedge 1$
(6) $\pi_i(x \diamond y) = \pi_i(x) \diamond \pi_i(y)$, where $\diamond \in \{\vee, \wedge, \cdot, \backslash, /\}$, $i = 1, 2$
(7) $\delta(\delta(x)) = \delta(x)$
(8) $d(x) \wedge d(y) \wedge 1 \leq (\pi_1 p(\delta(x), y) \leftrightarrow \delta(x)) \wedge (\pi_2 p(x, \delta(y)) \leftrightarrow \delta(y))$
(9) $d(x) \wedge 1 \leq (p(\pi_1(x), \pi_2(x)) \leftrightarrow x)$
(10) $\neg d(x) \wedge 1 \leq x \leftrightarrow \neg\delta'(x)$, $d(x) \wedge 1 \leq \delta(x) \leftrightarrow \delta'(\neg x)$
(11) $d(x) \wedge \neg d(y) \wedge 1 \leq x/y$
(12) $\neg d(x) \wedge \neg d(y) \wedge 1 \leq \delta'(x \wedge y) \leftrightarrow \delta'(x) \vee \delta'(y)$
(13) $\neg d(x) \wedge \neg d(y) \wedge 1 \leq \delta'(x \vee y) \leftrightarrow \delta'(x) \wedge \delta'(y)$
(14) $d(x) \wedge \neg d(y) \wedge 1 \leq \delta'(x \cdot y) \leftrightarrow \delta'(y)/\pi_1(x)$
(15) $\neg d(x) \wedge d(y) \wedge 1 \leq \delta'(x \cdot y) \leftrightarrow \pi_2(y)\backslash\delta'(x)$
(16) $\neg d(x) \wedge \neg d(y) \wedge 1 \leq (x \cdot y) \leftrightarrow \bot$
(17) $d(x) \wedge \neg d(y) \wedge 1 \leq \delta'(x\backslash y) \leftrightarrow \delta'(y) \cdot \pi_1(x)$
(18) $\neg d(x) \wedge d(y) \wedge 1 \leq \delta'(x\backslash y) \leftrightarrow \top$
(19) $\neg d(x) \wedge \neg d(y) \wedge 1 \leq \delta'(x\backslash y) \leftrightarrow p(\top, \delta'(x)/\delta'(y))$
(20) $d(x) \wedge \neg d(y) \wedge 1 \leq \delta'(x/y) \leftrightarrow \top$
(21) $\neg d(x) \wedge d(y) \wedge 1 \leq \delta'(x/y) \leftrightarrow \pi_2(y) \cdot \delta'(x)$
(22) $\neg d(x) \wedge \neg d(y) \wedge 1 \leq \delta'(x/y) \leftrightarrow p(\delta'(x)\backslash\delta'(y), \top)$

The first two equations state that the image of d is a Boolean algebra, the next two lines express that d is a homomorphism onto this algebra, followed by identities useful to prove results about principal congruence filters (4)(5) and subalgebras (6)(7). The properties for the projections and pairing function are given by (8)(9), and the remaining identities express how the operations $\cdot, \backslash, /$ in $B^\dagger$ are defined from the operations of B, as given before and in the proof of Lemma 7. It is straightforward to check that $B^\dagger$ satisfies all the identities in $E^\dagger$, since the term $d(x)$ maps all the elements of $B \times B$ to $\langle \top, \top \rangle$, and those of B^∂ to $\bot^\dagger$.

Lemma 12. *Let A be a subdirectly irreducible bounded residuated lattice with 0 that satisfies all the identities in $E^\dagger$, and define $B = \{\delta(a) : a \in A\}$. Then B is a subalgebra of the $\{\bot, 0\}$-free reduct of A that satisfies all the identities in E, and $B^\dagger \cong A$.*

Proof. Let M be the unique minimal nontrivial congruence filter of A, and let $F = \{a \in A : d(a) = \top\}$. Since d is a homomorphism, F is a congruence filter. If $F = \uparrow 1$, the trivial congruence filter, then d is an injective homomorphism onto a Boolean algebra, hence A is the two-element residuated lattice. In this case B is the one-element algebra and the result follows.

If F is not the trivial congruence filter, then $M \subseteq F$, so $d(a) = \top$ for all $a \in M$. We claim that the image of d is the two element algebra $\{\bot, \top\}$. Suppose $d(a) \vee d(b) = \top$ and $d(a) \wedge d(b) = \bot$ for some $a, b \in A$. Using identities (4) and (5) in $E^\dagger$, we see that $\{x \in A : x \geq d(a) \wedge 1\}$ and $\{x \in A : x \geq d(b) \wedge 1\}$ are congruence filters of A that intersect to give the trivial congruence filter, and their union generates the largest congruence filter A, hence they are factor congruences. Since A is subdirectly irreducible, it follows that $d(a) = \top$ and $d(b) = \bot$ (or vice versa).

Now consider the set B as defined in the statement of the lemma. It is a subalgebra since $E^\dagger$ includes the identities (6)(7), and it satisfies all the identities in E since we added translations of them to $E^\dagger$. Let $f : A \to B^\dagger$ be given by

$$f(x) = \begin{cases} \langle \pi_1(x), \pi_2(x) \rangle & \text{if } d(x) = \top \\ \delta'(x) & \text{if } d(x) = \bot. \end{cases}$$

Then the identities in $E^\dagger$ may be used to show that f is an isomorphism. In particular, f is a bijection because of the identities for the pairing function and the projections (8)(9)(10) and it is a homomorphism since $E^\dagger$ includes (6) and (11)-(22). For example, to see that $f(x \cdot y) = f(x) \cdot f(y)$, we first consider the case $d(x \cdot y) = \top$. From (1) and (2) it follows that $d(x) = \top = d(y)$, hence

$$\begin{aligned} f(x) \cdot f(y) = \langle \pi_1(x), \pi_2(x) \rangle \cdot \langle \pi_1(y), \pi_2(y) \rangle &= \langle \pi_1(x) \cdot \pi_1(y), \pi_2(x) \cdot \pi_2(y) \rangle \\ &= \langle \pi_1(x \cdot y), \pi_2(x \cdot y) \rangle = f(x \cdot y). \end{aligned}$$

The remaining cases, with $d(x) = \bot$ and/or $d(y) = \bot$ are similar, but use the identities (14)(15)(16). $\square$

For a variety $\mathcal{V}$, we let $\mathcal{V}^\dagger$ denote the variety generated by the class $\{B^\dagger : B$ is a subdirectly irreducible member of $\mathcal{V})\}$. From the lemma above, we deduce the following result.

Theorem 13. *Let $\mathcal{V}$ be a variety of residuated lattices with $\top$, and let E be a set of identities. If $\mathcal{V} = Mod(E)$ then $\mathcal{V}^\dagger = Mod(E^\dagger)$ and the map $\mathcal{V} \mapsto \mathcal{V}^\dagger$ is injective and preserves inclusions of varieties.*

By a result in [3] the lattice of ℓ-group varieties is isomorphic to the lattice of varieties of negative cones of ℓ-groups. The latter varieties map via † to pseudo-BL varieties that are not pseudo-MV varieties.

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