Duality for partial algebras, bunched implication algebras and GBL-algebras

Peter Jipsen

School of Computational Sciences and
Center of Excellence in Computation, Algebra and Topology (CECAT)
Chapman University, Orange, California

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In Memoriam Franco Montagna

Outline

- Partial algebras
- Effect algebras
- Generalized orthoalgebras
- Concrete generalized orthoalgebras
- Natural duality
- Bunched implication algebras
- Residuated Heyting algebras
- Generalized basic logic algebras

Introduction

A **groupoid** is a set A with a binary operation $\cdot : A \times A \to A$

If A is finite, say $= \{a_1, \ldots, a_n\}$, then a groupoid can be defined by its operation table

$$\begin{array}{c|cccc} \cdot & a_1 & a_2 & \cdots & a_n \\ \hline a_1 & & & & \\ a_2 & & & & \\ \vdots & & & & \\ a_n & & & & \end{array} \quad \leftarrow \text{fill out with (some of) } a_1, \ldots, a_n \text{ any way you like}$$

Introduction

- What do we know about 2-element groupoids?
- How many are there? 16 Up to isomorphism? 10
-

$\cdot$	0	1
0		
1		

C	0	1
0	0	0
1	0	0

$\cdot$	0	1
0	0	0
1	0	1

$\leftarrow$	0	1
0	0	0
1	1	0

π_1	0	1
0	0	0
1	1	1

$\rightarrow$	0
0	0
1	0

- $\boxed{xy = zw}$ $\boxed{\text{assoc. comm. idem.}}$ $\boxed{\text{implic. reduct of BA}}$ $\boxed{xy = x \text{ (Lz sg)}}$ $\boxed{\text{implic. reduct of BA}}$
-

π_2	0	1
0	0	1
1	0	1

$+$	0	1
0	0	1
1	1	0

$\mid$	0	1
0	1	0
1	0	0

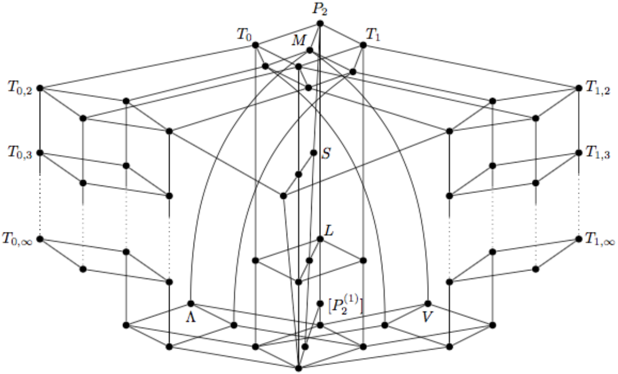
$\bar{\pi}_2$	0	1
0	1	0
1	1	0

$\bar{\pi}_1$	0	1
0	1	1
1	0	0

- $\boxed{xy = y \text{ (Rz sg)}}$ $\boxed{\text{assoc. } x+x=0,\ x+0=x}$ $\boxed{\text{BA axioms}}$ $\boxed{xy = yy,\ x^2x^2 = x}$ $\boxed{xy = xx,\ x^2x^2 = x}$
- Give axioms for the varieties they generate

Introduction

In fact, quite a bit is known about 2-element algebras



Post lattice (from Schölzel 2010)

Introduction

A **partial groupoid** is a set A with a **partial** binary operation $\cdot : A \times A \rightsquigarrow A$

If $A = \{a_1, \ldots, a_n\}$, then a partial groupoid can be defined by a **partially filled out** operation table

$\cdot$	a_1	a_2	$\cdots$	a_n
a_1				
a_2				
$\vdots$				
a_n				

Convention: every (total) algebra is a partial algebra

What do we know about 2-element partial groupoids?

- How many are there? $81 - 16 = 65$ Up to isomorphism? $45 - 10 = 35$
-

$\cdot_1$	0	1
0	0	0
1	0	

$\cdot_2$	0	1
0	0	0
1	1	

$\cdot_3$	0	1
0	0	0
1		0

$\cdot_4$	0	1
0	0	0
1		1

$\cdot_5$	0	1
0	0	0
1		

$\cdot_6$	0	1
0	0	1
1	0	

$\cdot_7$	0	1
0	0	1
1	1	

$\cdot_8$	0	1
0	0	1
1		0

$\cdot_9$	0	1
0	0	1
1		1

$\cdot_{10}$	0	1
0	0	1
1		

$\cdot_{11}$	0	1
0	0	
1	0	0

$\cdot_{12}$	0	1
0	0	
1	0	

$\cdot_{13}$	0	1
0	0	
1	1	0

$\cdot_{14}$	0	1
0	0	
1	1	

$\cdot_{15}$	0	1
0	0	
1		0

$\cdot_{16}$	0	1
0	0	
1		1

$\cdot_{17}$	0	1
0	0	
1		

$\cdot_{18}$	0	1
0	1	0
1	0	

$\cdot_{19}$	0	1
0	1	0
1	1	

$\cdot_{20}$	0	1
0	1	0
1		0

$\cdot_{21}$	0	1
0	1	0
1		

$\cdot_{22}$	0	1
0	1	1
1	0	

$\cdot_{23}$	0	1
0	1	1
1	1	

$\cdot_{24}$	0	1
0	1	1
1		0

$\cdot_{25}$	0	1
0	1	1
1		

$\cdot_{26}$	0	1
0	1	
1	0	

$\cdot_{27}$	0	1
0	1	
1	1	

$\cdot_{28}$	0	1
0	1	
1		0

$\cdot_{29}$	0	1
0	1	
1		

$\cdot_{30}$	0	1
0		0
1	0	

$\cdot_{31}$	0	1
0		0
1	1	

$\cdot_{32}$	0	1
0		0
1		

$\cdot_{33}$	0	1
0		1
1	0	

$\cdot_{34}$	0	1
0		1
1		

$\cdot_{35}$	0	1
0		
1		

Remove all partial projection groupoids

- Leaves 21 non-relational partial groupoids
-

$\cdot_1$	0	1
0	0	0
1	0	

$\cdot_3$	0	1
0	0	0
1		0

$\cdot_7$	0	1
0	0	1
1	1	

$\cdot_8$	0	1
0	0	1
1		0

$\cdot_{11}$	0	1
0	0	
1	0	0

$\cdot_{13}$	0	1
0	0	
1	1	0

$\cdot_{15}$	0	1
0	0	
1		0

$\cdot_{18}$	0	1
0	1	0
1	0	

$\cdot_{19}$	0	1
0	1	0
1	1	

$\cdot_{20}$	0	1
0	1	0
1		0

$\cdot_{21}$	0	1
0	1	0
1		

$\cdot_{22}$	0	1
0	1	1
1	0	

$\cdot_{23}$	0	1
0	1	1
1	1	

$\cdot_{24}$	0	1
0	1	1
1		0

$\cdot_{25}$	0	1
0	1	1
1		

$\cdot_{26}$	0	1
0	1	
1	0	

$\cdot_{27}$	0	1
0	1	
1	1	

$\cdot_{28}$	0	1
0	1	
1		0

$\cdot_{29}$	0	1
0	1	
1		

$\cdot_{30}$	0	1
0		0
1	0	

$\cdot_{31}$	0	1
0		0
1	1	

Remove partial groupoids that are isom. to opposite

- Leaves 16 non-dually-isomorphic partial groupoids
-

$\cdot_1$	0	1
0	0	0
1	0	

$\cdot_3$	0	1
0	0	0
1		0

$\cdot_7$	0	1
0	0	1
1	1	

$\cdot_8$	0	1
0	0	1
1		0

$\cdot_{15}$	0	1
0	0	
1		0

$\cdot_{18}$	0	1
0	1	0
1	0	

$\cdot_{19}$	0	1
0	1	0
1	1	

$\cdot_{20}$	0	1
0	1	0
1		0

$\cdot_{21}$	0	1
0	1	0
1		

$\cdot_{23}$	0	1
0	1	1
1	1	

$\cdot_{25}$	0	1
0	1	1
1		

$\cdot_{27}$	0	1
0	1	
1	1	

$\cdot_{28}$	0	1
0	1	
1		0

$\cdot_{29}$	0	1
0	1	
1		

$\cdot_{30}$	0	1
0		0
1	0	

$\cdot_{31}$	0	1
0		0
1	1	

Axioms for the ISP classes they generate?

- Which of these generate a **finitely axiomatizable** ISP class?
- **Natural duality theory** applies to partial algebras (*Davey* [2006])
- Which of these partial groupoids are **dualizable**?
- For this talk: want the operation to have an **identity element** 0
- Two **total** groupoids: 2-element semilattice and 2-element group
- $\begin{array}{c|cc} \vee & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \end{array} \qquad \begin{array}{c|cc} +_2 & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}$
- How many **partial groupoids with an identity element** are there?
- $P_1 = \begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & \end{array}$ Is $\mathsf{ISP}(P_1)$ **finitely axiomatizable**? **Dualizable**?

Subalgebras, products, homomorphisms

- A **partial algebra** $\mathbf{A} = \langle A, F^{\mathbf{A}} \rangle$ is a set A and a collection of partial operation $F^{\mathbf{A}}$ on A
- Every partial algebra can be extended to a total algebra $\tilde{\mathbf{A}}$ by adding one element $\infty \notin A$
- $\tilde{f}(x_1, \ldots, x_n) = \begin{cases} f(x_1, \ldots, x_n) & \text{if } f(x_1, \ldots, x_n) \text{is defined (i.e., exists)} \\ \infty & \text{otherwise} \end{cases}$
- $\mathrm{dom}(f) = \{(x_1, \ldots, x_n) \mid f(x_1, \ldots, x_n) \text{ exists}\}$
- $\mathbf{B} \subseteq \mathbf{A}$ is a **(partial) subalgebra** if $\mathbf{B}$ is closed under the partial operations of $\mathbf{A}$
- $\prod_{i \in I} \mathbf{A}_i$ is the **product**; operations are defined pointwise; exist **iff** they exist in all coords
- Note that $\tilde{\mathbf{A}} \times \tilde{\mathbf{B}} \neq \widetilde{\mathbf{A} \times \mathbf{B}}$
- $h : \mathbf{A} \to \mathbf{B}$ is a **homomorphism** if $h(f(x_1, \ldots, x_n)) = f(h(x_1), \ldots, h(x_n))$ for all $(x_1, \ldots, x_n) \in \mathrm{dom}(f)$
- **HSP** is defined using these operations

Identities and quasiidentities

The **signature** of a partial algebra is a set F of (partial) **function symbols**, each with an associated finite arity

The **interpretation** of f in a partial algebra $\mathbf{A}$ is denoted $f^{\mathbf{A}}$

Terms and **formulas** are defined as usual

A term $t(a_1, \ldots, a_n)$ is defined iff **all subterms are defined**

An identity $s(x_1, \ldots, x_n) = t(x_1, \ldots, x_n)$ holds in a partial algebra $\mathbf{A}$ if for all $x_1, \ldots, x_n \in A$ either **both sides are undefined**, or they are **defined and equal**

A **quasiidentity** $s_1 = t_1 \& \cdots \& s_n = t_n \implies s = t$ holds in $\mathbf{A}$ if for all assignments that make **the premises defined and equal**, s, t are **defined and equal**

Why bother with partial operations?

- **Boole** originally considered **union** undefined for **overlapping sets**
- In a **field**, the multiplicative **inverse** is a partial operation
- In quantum logic, **effect algebras** are partial algebras
- **Kripke frames** of **ordered** algebras are partial algebras
- **Products** of partial algebras are **cartesian** (not true with $\tilde{\mathbf{A}}$)
- **Natural duality** now allows **partial operations** and relations on both sides
- Partial algebras **include all relational structures** in an algebraic way
- $R \subseteq A^n$ corresponds to $f_R(x_1,\ldots,x_n) = \begin{cases} x_1 & \text{if } (x_1,\ldots,x_n) \in R \\ \text{undef} & \text{otherwise} \end{cases}$
- Note: f_R is the **restricted projection** $\pi_1{\restriction}_R$

Why bother with partial operations?

- The main reason: **Computer Science**
- Consider the memory of a computer: a list of **cells** with **values** in them
- $(m_0, m_1, \ldots, m_i, \ldots)$ or more generally:
- a function $m : L \to V$ from a set L of **locations** to V of **values**
- As a program runs, it is allocated some of these cells
- The part of memory used is called a **heap** h, where $h : L \rightsquigarrow V$ is a **partial** function
- If several programs run **concurrently**, they use **separate** heaps

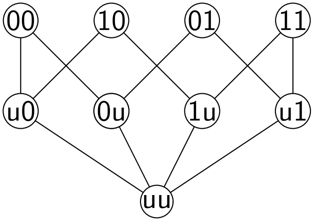
Cancellative commutative partial monoids

- A **cancellative commutative partial monoid** is of the form $(A, +, 0)$ where $+ : A \times A \rightsquigarrow A$ is
- $x + y = y + x$ (commutative)
- $(x + y) + z = x + (y + z)$ (associative)
- $x + 0 = x$ (0 is the identity)
- $x + z = y + z \implies x = y$ (cancellative)
- *Calcagno, O'Hearn, Yang* [2007] call them **separation algebras**
- Typical model: $P_{L,V} = \{$ all heaps ($=$ partial functions L to V) $\}$ and
$$h + k = \begin{cases} h \cup k & \text{if } \mathrm{dom}(h) \cap \mathrm{dom}(k) = \emptyset \\ \text{undef.} & \text{otherwise} \end{cases}$$
- E.g., if $L = \{0, 1\}$ and $V = \{0, 1\}$ we have $A = \{uu, u0, u1, 0u, 1u, 00, 01, 10, 11\}$ where $h = ab$ is the heap that satisfies $h(0) = a$, $h(1) = b$; $h(x) = u$ means undefined
- Define **heap algebras** $= \mathsf{S}(\{P_{L,V} \mid L, V \text{ are sets}\})$. Note that $0 = uu$

Heap algebra examples

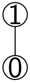
- $P_{2,2} =$

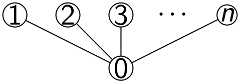
$+$	uu	$u0$	$u1$	$0u$	$1u$	00	01	10	11
uu	uu	$u0$	$u1$	$0u$	$1u$	00	01	10	11
$u0$	$u0$			00	10				
$u1$	$u1$			01	11				
$0u$	$0u$	00	01						
$1u$	$1u$	10	11						
00	00								
01	01								
10	10								
11	11								



- Define $x \leq y$ if $x + z = y$ for some z (the **natural order**)
- Can you find another (smaller) example? Guess what! $P_1 = P_{1,1}$
- $P_1 =$

$+$	0	1
0	0	1
1	1	

$=$ 

$P_{1,n} =$ 

Heap algebras = ISP(P_1)

Products of P_1 are Boolean lattice reducts with $x + y = x \vee y$ if $x \wedge y = 0$

What do the (partial) subalgebras of products of P_1 look like?

Theorem

The class of heap algebras is ISP(P_1)

Proof.

$P_{L,V} \cong (V \cup \{u\})^L$ where $u \notin V$

1. $P_{1,V}$ is a subalgebra of $(P_1)^V$
2. Observe that $P_{L,V} = (P_{1,V})^L$

□

Subquasivarieties of canc. comm. partial monoids

Let CCpM $=$ quasivariety of **cancellative commutative partial monoids**

CCpM is larger than $\mathsf{ISP}(P_1)$ since $\mathbb{Z}_2$ is a CCpM

P_1 is **positive**, i.e., satisfies $x + y = 0 \implies x = 0$, which fails in $\mathbb{Z}_2$

A **generalized effect algebra** is a positive CCpM

Fact

*$x \leq y$ is a **partial order** in generalized effect algebras*

Effect algebras come from quantum logic, *Foulis and Bennett* [1994]

Effect algebras are generalized EA that have unary $'$ such that $x + x' = 0'$

Generalized Orthogalgebras

Addition in P_1 is **orthogonal**, i.e., $x + x = x + x \implies x = 0$

Unit interval with truncated $+$ is a **non-orthogonal** pos. CCpM: $\frac{1}{2} + \frac{1}{2} = 1$

Lemma: Orthogonal CCpMs are positive.
Proof: If $x + y = 0$ then $x + (x + y)$ is defined, so $(x + x) + y$ is defined, hence $x + x = x + x$ so we get $x = 0$

Orthogonal CCpMs are also known as **generalized orthoalgebras**

P_1 is **coherent**, i.e., if $x + y$, $x + z$ and $y + z$ are defined, so is $(x + y) + z$

Example of a **non-coherent** generalized orthoalgebra: Take an 8-element BA and remove the top element

Coherent orthoalgebras are first-order equivalent to **orthomodular posets**

Concrete generalized orthoalgebras

Let U be any set and define $+$ on $\mathscr{P}(U)$ by

$$X + Y = \begin{cases} X \cup Y & \text{if } X \cap Y = \emptyset \\ \text{undef.} & \text{otherwise} \end{cases}$$

Then $(\mathscr{P}(U), +, \emptyset)$ is a **coherent generalized orthoalgebra** $\cong {P_1}^{|U|}$

A **concrete generalized orthoalgebra** is any partial algebra embedded in this powerset algebra

Hence the class of **concrete generalized orthoalgebras** is $\mathrm{ISP}(P_1)$

But this is **smaller** than the class of coherent generalized orthoalgebras

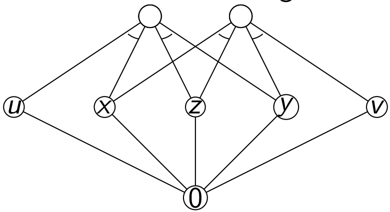
Is it **finitely axiomatizable**?

More quasiidentities

Lemma: The following quasiequations hold in P_1 and are not consequences of previous axioms:

1. $x \leq y + z \;\&\; y \leq x + z \implies x = y$
2. $w + x = y + z \;\&\; w + y = u \;\&\; x + z = v \implies x = y$

   Proof that 1. holds in P_1: Suppose $x \leq y + z \;\&\; y \leq x + z$ but $x \neq y$.
   By symmetry can assume $x = 0$, $y = 1$.
   Then $x \leq y + z$ implies $z = 0$ (since $y + z$ must be defined).
   But now $1 = y \leq x + z = 0 + 0 = 0$ is a contradiction.
   1. fails in this coherent generalized orthoalgebra:



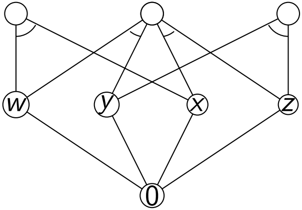
where only $u + x, y + z, x + z, y + v$ are defined

Similarly 2. $w+x=y+z$ & $w+y=u$ & $x+z=v \implies x=y$ holds in P_1: Suppose $x=0, y=1$.

$w+x=y+z$ implies $z=0$ and $w=1$.

But now $w+y=1+1$ is undefined, contradicting $w+y=u$.

Below is a coherent generalized orthoalgebra that satisfies 1. but fails 2.



Concrete GOAs are not finitely axiomatized

Theorem

$\mathbf{ISP}(P_1)$ *is not finitely axiomatizable.*

Proof. Consider the following quasiidentities q_n:

$$\&_{i=0}^{n-1}(x_{2i} + x_{2i+1} = x_{2i+2} + x_{2i+3}) \;\&\; \&_{i=0}^{n-1}(x_{2i+1} + x_{2i+2} = y_i) \implies x_0 = x_2$$

where index addition is modulo $2n$.

We also define an algebra $Q_n = \{0, a_0, a_1, \ldots, a_{2n-1}, b_0, b_1, \ldots, b_n\}$

by $0 + x = 0 = x + 0$, $a_{2i} + a_{2i+1} = b_n$, and $a_{2i+1} + a_{2i+2} = b_i$ (index addition mod $2n$)

It is not difficult to check that this algebra is a coherent orthogonal CCpM

Claim 1. For all $n > 1$ the formula q_n holds in $\mathbf{ISP}(P_1)$ but fails in Q_n.

Proof.

Suppose the premises hold in P_1 but $x_0 \neq x_2$.
If $x_0 = 0$ then $x_2 = 1$, and since $x_1 + x_2$ is defined, it follows that $x_1 = 0$.
However, this contradicts $x_0 + x_1 = x_2 + x_3$.
If $x_0 = 1$ then $x_1 = 0$ since $x_0 + x_1$ is defined, and $x_2 = 0$ since we are assuming $x_0 \neq x_2$.
Now $x_0 + x_1 = x_2 + x_3$ implies $x_3 = 1$, and since $x_3 + x_4$ is defined, we have $x_4 = 0$.
If $n = 2$ then $x_4 = x_0$ since indices are calculated modulo 4, but this contradicts $x_0 = 1$.
Assume we have shown $x_{2i-1} = 1$ and $x_{2i} = 0$.
Then $x_{2i-2} + x_{2i-1} = x_{2i} + x_{2i+1}$ implies $x_{2i+1} = 1$, hence $x_{2i+2} = 0$.
By induction we have $x_{2n} = 0$, which again contradicts $x_0 = 1$.
To see that q_n fails in Q_n, take a_i to be the value of the variable x_1. $\square$

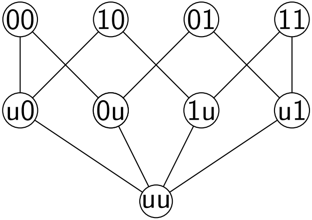
Claim 2. The ultraproduct $(\prod_{n\in\omega} Q_n)/\mathscr{U}$ is in $\mathbf{ISP}(P_1)$ for any nonprincipal ultrafilter $\mathscr{U}$ on ω, hence $\mathbf{ISP}(P_1)$ is not finitely axiomatizable.

Proof.

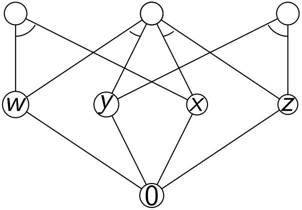
(outline) In each Q_n, the term $a_i + a_j$ is defined iff $j = i \pm 1 \pmod{2n}$, and the terms $a_{2i} + a_{2i+1}$ are all equal to b_n.
This same structure holds in the ultraproduct, except that the addition is now done in $\mathbb{Z}$.
To see that the ultraproduct is in $\mathbf{ISP}(P_1)$, it suffices to embed this algebra in the powerset algebra $\mathscr{P}(\omega)$ with disjoint union as partial operation and the empty set as identity.
Let $a_0 = 2\mathbb{Z}$ and $a_1 = \omega - a_0$. In general, let $a_k = 2k\mathbb{Z}$ and $a_{k+1} = \omega - a_k$, and check that this map is an embedding. □

ISP(P_1) is not closed under H

$(P_{1,2})^2 \cong$

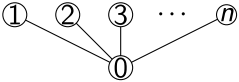


and has a homomorphic image



$\notin$ ISP(P_1)

Heap algebras satisfy no congruence equations

- Consider the heap algebra $P_{1,n} =$ 

- Can identify any two maximal elements without collapsing any others
- Can identify any maximal element with 0 without collapsing any others
- Therefore $\mathrm{Con}(P_{1,n}) = \mathrm{Eq}(n) =$ the lattice of equivalence relations on an n element set
- Any lattice equation fails in $\mathrm{Eq}(n)$ for some n

Natural duality (briefly)

- Duality theory aims to find **categorical (dual) equivalences** between two categories
- **Natural dualities** provide a framework using homomorphisms into a generating object
- E.g. **Stone duality** $D : \mathrm{BA} \to \mathrm{Stone}$, $E : \mathrm{Stone} \to \mathrm{BA}$ given by

$D(\mathbf{A}) = \mathrm{Hom}(\mathbf{A}, \mathbf{2})$ with product topology from $\underline{\mathbf{2}}^A$, $D(h)(x) = x \circ h$
$E(\mathbf{X}) = \mathrm{Hom}(\mathbf{X}, \underline{\mathbf{2}})$ with operations inherited from $\mathbf{2}^X$, $E(k)(a) = a \circ k$

- Or **Priestley duality** $D : \mathrm{BDL} \to \mathrm{Pri}$, $E : \mathrm{Pri} \to \mathrm{BDL}$ given by

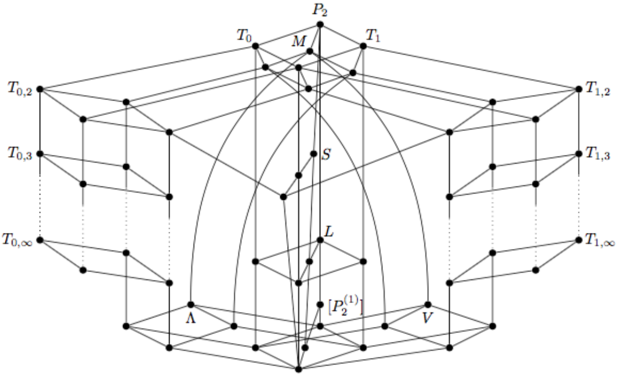
$D(\mathbf{A}) = \mathrm{Hom}(\mathbf{A}, \mathbf{C}_2)$ with product topology from $\underline{\mathbf{C}}_2^A$, $D(h)(x) = x \circ h$
$E(\mathbf{X}) = \mathrm{Hom}(\mathbf{X}, \underline{\mathbf{C}}_2)$ with operations inherited from $\mathbf{2}^X$, $E(k)(a) = a \circ k$

- Then $E(D(\mathbf{A})) \cong \mathbf{A}$ and $D(E(\mathbf{X})) \cong \mathbf{X}$ via a **natural** isomorphism

Dualizability of 2-element algebras

Theorem

[Clark, Davey 1998] All 2-element (total) algebras are dualizable, except for the 8 that are limits of descending chains



Natural duality for partial algebras

- Davey [2006] extends natural dualities to **categories of partial algebras** and **relational structures**
- Davey, Pitkethly and Willard [2012] give a **symmetric** formulation:
- Let f be m-ary, g be n-ary partial functions on a set A
- f, g are **compatible** ($f \sim g$) if for all $(a_{ij}) \in A^{m\times n}$ with $f(a_{*j}), g(a_{i*})$ defined we have $f(g(a_{1*}),\dots,g(a_{m*})) = g(f(a_{*1}),\dots,f(a_{*n}))$
- $F^{\Diamond} = \{g : f \sim g \text{ for all } f \in F\}$
- **Lemma** [DPW'12]. (i) $SP(\langle A, F\rangle)$ and $SP(\langle A, F'\rangle)$ are **isomorphic categories** if $F^{\Diamond\Diamond} = F'^{\Diamond\Diamond}$
- (ii) $g \in F^{\Diamond\Diamond}$ iff g has an extension in $\mathrm{Clo}_p(F)$ and the domain of g is **conjunct-atomic** definable from F
- Here $\mathrm{Clo}_p(F)$ is the clone of partial functions generated by F
- A k-ary relation R is **conjunct-atomic definable** from F if $R = \{\mathbf{a} \in A^k : \psi(\mathbf{a}) \text{ is true in } \langle A, F\rangle\}$ for some formula ψ that is a conjunction of atomic formulas

An algorithm for computing $F^{\Diamond\Diamond}$

Let $\mathbf{A} = \langle A, F\rangle$ be a finite partial algebra with F finite

To compute all k-ary partial functions in $F^{\Diamond\Diamond}$, compute the k-**ary partial clone** G_k = subalgebra of $\mathbf{A}^{A^k}$ generated by the k-ary projections $\pi_1, \ldots, \pi_k$
Next, close G_k under **equalizers of partial functions**, i.e., $f, g \in G_k$ implies $E(f,g) \in G_k$ where $E(f,g)(\mathbf{a}) = \begin{cases} f(\mathbf{a}) & \text{if } f(\mathbf{a}) = g(\mathbf{a}) \\ \text{undef.} & \text{otherwise} \end{cases}$
Finally, close G_k under the **restriction of** f **to the domain of** g, for all $f, g \in G_k$

Sets of the form $F^{\Diamond\Diamond}$ are called **structural clones**

There are 17 unary and 1693 binary structural clones on $\{0, 1\}$ compared to 6 unary and 26 binary (total) clones

Computing the structural clone of $\mathbf{P}_1$

x	y	0	$x+y$	$2x$	$2y$	$2x+y$	$x+2y$	$2x+2y$	$E(x,y)$	$x\restriction_{x+y}$	$y\restriction_{x+y}$	$0\restriction_{x+y}$	$0\restriction_{E(x,y)}$
0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	1	0	–	1	–	–	–	0	1	0	–
1	0	0	1	–	0	–	1	–	–	1	0	0	–
1	1	0	–	–	–	–	–	–	1	–	–	–	0

P_1 is dualizable at the finite level

Theorem

(Joint with M. A. Moshier) $\mathbf{P}_1 = \begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & \end{array}$ *is dualizable at the finite level*

Let $\underline{\mathbf{P_1}} = \langle\{0,1\}, G\rangle$ where $G = \{+,0\}^{\Diamond}$, so $g \in G$ if $g(0,\ldots,0) = 0$ and if $g(x_1,\ldots,x_k), g(y_1,\ldots,y_k)$, and $x_i + y_i$ **defined** for $i = 1,\ldots,k$ then $g(x_1+y_1,\ldots,x_k+y_k) = g(x_1,\ldots,x_k) + g(y_1,\ldots,y_k)$ (both defined)

Show for all finite $\mathbf{A} \in SP(\mathbf{P}_1)$ we have $E(D(\mathbf{A})) \cong \mathbf{A}$

Bunched implication algebras

A **bunched implication algebra** (BI-algebra) is of the form

$(A, \vee, \wedge, \to, \top, \bot, *, \backslash, /, 1)$ where $(A, \vee, \wedge, \to, \top, \bot)$ is a **Heyting algebra**

(i.e. a bounded distributive lattice with $x \wedge y \leq z$ iff $y \leq x \to z$) and

$(A, \vee, \wedge, *, \backslash, /, 1)$ is a **commutative residuated lattice**

(i.e. a commutative monoid with $x * y \leq z$ iff $y \leq x \backslash z$ iff $x \leq z/y$)

If $(x \to \bot) \to \bot = x$ we get **classical** BI-algebras

CBI-algebras = commutative residuated Boolean monoids
= *crm*-algebras of Jónsson-Tsinakis [1993]

BI-algebras come from **Separation Logic**, a **Hoare programming logic** for reasoning about pointers and concurrent resources

BI-algebras from generalized effect algebras

Let $(P, \oplus, 0)$ be a **generalized effect algebra** (GEA)
Recall the **natural order** $x \leq y$ iff $\exists z\, x \oplus z = y$

$Up(P)$ is the set of upward closed subsets of P
$=$ a completely distributive complete lattice under intersection and union

Hence $(Up(P), \cup, \cap, \to, P, \emptyset)$ is a **Heyting algebra**

Define $X * Y = \{x \oplus y \mid x \in X, y \in Y\}$,

$X \backslash Y = \{z \mid x \oplus z \in Y \text{ for all } x \in X\}$, $\quad X/Y = Y \backslash X$ and $1 = \{0\}$

Then $(Up(P), \cup, \cap, \to, P, \emptyset, *, \backslash, /, 1)$ is a bunched implication algebra

Let $Up(\mathbf{GEA}) = \{Up(P) \mid P \in \mathbf{GEA}\}$

Problem: Axiomatize the class $\mathrm{HSP}(Up(\mathbf{GEA}))$ or $\mathrm{HSP}(Up(SP(\mathbf{P}_1)))$

Residuated Heyting algebras

BI-algebras are residuated lattices with a Heyting algebra order

A **residuated Heyting algebra** (RHA) is of the form $(A, \vee, \wedge, \to, \top, \bot, *, \backslash, /, 1)$ where $(A, \vee, \wedge, \to, \top, \bot)$ is a **Heyting algebra** and $(A, \vee, \wedge, *, \backslash, /, 1)$ is a **residuated lattice**

For example **every finite distributive residuated lattice** can be expanded to a residuated Heyting algebra

But RL congruences need not be RHA congruences:

$\cdot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	b	b
1	0	a	b	1

with $0 < a < b < 1$ has an RL congruence θ with blocks $\{0, a\}, \{b, 1\}$ but in a HA $0\theta a \Rightarrow 0\theta 1$

Duality for residuated Heyting algebras

Since RHAs have distributive RLs as reducts, many techniques from DRL can be adapted

The duality for HA is given by **Esakia spaces**, i.e. Priestley spaces for which $\downarrow U$ is open for every open set U

The elements of the space are the **prime lattice filters** of the Heyting algebra

The **monoid operation** of a RHA is captured by a **ternary Kripke relation** on the prime filters

For **finite** algebras, this reduces to **Birkhoff's duality** for distributive lattices, with a **ternary relation**

Adding divisibility: HGBL-algebras

A HGBL-algebra is a **divisible residuated Heyting algebra**

i.e., satisfies $x \leq y \implies y(y\backslash x) = (x/y)y = x$

Most of the results of J-Montagna [2006, 2009, 2010] can be lifted to HGBL-algebras

In particular, poset products completely describe the structure theory of finite HGBL-algebras

The equational decidablity of GBL-algebras is still open

Problem: Do HGBL-algebras have a decidable equational theory?

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Thank you Franco!