Generalizations of Boolean products for lattice-ordered algebras

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Dedicated to Franco Montagna on the occasion of his 60^{th} birthday

Abstract

It is shown that the Boolean center of complemented elements in a bounded integral residuated lattice characterizes direct decompositions. Generalizing both Boolean products and poset sums of residuated lattices, the concepts of poset product, Priestley product and Esakia product of algebras are defined and used to prove decomposition theorems for various ordered algebras. In particular, we show that FL_w-algebras decompose as a poset product over any finite set of join irreducible strongly central elements, and that bounded n-potent GBL-algebras are represented as Esakia products of simple n-potent MV-algebras.

Key words: Residuated lattices, generalized BL-algebras, basic logic, generalized MV-algebras, posets
2000 MSC: 06F05, 06D35, 03G10, 03G25,

1. Introduction

Topological dualities have been very effective tools for various classes of algebras, such as Boolean algebras with Boolean spaces as duals, distributive lattices with Priestley spaces as duals, and Heyting algebras with Esakia spaces as duals. Boolean spaces have also been applied to the representation of algebras by Boolean powers and (weak) Boolean products, where the latter are also known as algebras of global sections of sheaves of algebras over Boolean spaces [2].

In Section 2 we recall the concept of (weak) Boolean product, and define the poset product for algebras of any signature with two constants 0,1 (previously the latter notion was defined only for residuated lattices [10]). We prove that under mild assumptions on the basic operations of the algebras, the poset product is a subalgebra of the direct product. Section 3 contains general results

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about direct decompositions of integral bounded unital ℓ-groupoids, based on the Boolean center of complemented elements. In the next section we restate an embedding result, proved for integral GBL-algebras in [11], so that it applies to FL_w-algebras in general. Theorem 12 shows that an FL_w-algebra with any finite subalgebra of strongly central elements (i.e. elements c that satisfy $c \wedge x = cx = xc$ for all x) decomposes as a poset product indexed by the dual poset of join irreducible elements of the subalgebra, which generalizes a similar result of [10] for finite GBL-algebras. Finally in Section 5 we combine Boolean products and poset products by defining the concept of Priestley product and Esakia product. The latter notion is used to show that any bounded n-potent GBL-algebra is an Esakia product of simple n-potent MV-algebras.

2. Boolean products and poset products

Let $\{\mathbf{A}_i : i \in X\}$ be a family of algebras with the same fundamental operation symbols from a set $\mathcal{F}$. The *direct (cartesian) product* $\prod_{i\in X} \mathbf{A}_i$ of this family of algebras is of course the set of all functions $f : X \to \bigcup_{i\in X} A_i$ such that $f(i) \in A_i$ for all $i \in X$ (i.e. choice functions), with the operations defined pointwise, and with projections $\pi_j : \prod_{i\in X} \mathbf{A}_i \twoheadrightarrow \mathbf{A}_j$.

It is not often the case that an algebra can be expressed as a direct product of simpler algebras, so various generalizations of products are used to obtain more widely applicable representation results. E.g. Birkhoff's *subdirect product* represents algebras as subalgebras of direct products for which the projections are still surjective. Recall that a *Boolean space* is a set with a *Boolean topology*, defined as a topology that is compact and totally disconnected (i.e. distinct elements are separated by clopen sets, hence every Boolean space is Hausdorff). By Stone duality, clopen sets of a Boolean space $\mathbf{X}$ form a Boolean algebra $\mathbf{A}_{\mathbf{X}}$, and the set $\mathbf{X}_{\mathbf{A}}$ of ultrafilters of a Boolean algebra $\mathbf{A}$ carry a natural Boolean topology such that $\mathbf{X}_{\mathbf{A}_{\mathbf{X}}} \cong \mathbf{X}$ and $\mathbf{A}_{\mathbf{X}_{\mathbf{A}}} \cong \mathbf{A}$. A *weak Boolean product* is a subdirect product $\mathbf{A} \leq \prod_{i\in X} \mathbf{A}_i$ for which there exists a Boolean topology on the index set X such that for all $f, g \in A$

(i) the *equalizer* $[\![f = g]\!] = \{i \in X : f(i) = g(i)\}$ is open and
(ii) for all clopen U, $f|_U \cup g|_{X-U} \in A$

If "open" is replaced by "clopen" in (i) then $\mathbf{A}$ is a *Boolean product* of $\{\mathbf{A}_i : i \in X\}$.

The *Boolean power* of an algebra $\mathbf{B}$ over a Boolean space $\mathbf{X} = (X, \tau)$ is

$$\mathbf{B}[\mathbf{X}]^* = \{f \in B^X : f^{-1}[\{b\}] \text{ is open for all } b \in B\}$$

i.e. the set of continuous functions from $\mathbf{X}$ to $\mathbf{B}$, where $\mathbf{B}$ is considered to have the discrete topology. Every Boolean power is a Boolean product (see e.g. [2]), and if X is a finite set then both concepts reduce to the direct product (since any function on a finite domain can be constructed from a finite union of restrictions of functions in a subdirect product). Boolean products have been used in many settings to derive powerful decidability results and representation results for classes of algebras, see e.g. [3], [2] for discriminator algebras, [5] for lattices, [4] for MV-algebras, [7] for BL-algebras.

The *poset product* (introduced for residuated lattices in [10] as dual poset sum) uses a partial order on the index set to define a subset of the direct product. Specifically, let $\mathbf{X} = (X, \leq)$ be a poset, and assume the algebras $\mathbf{A}_i$ have two *distinct* constant operations denoted $0, 1$. A *labeling* of $\mathbf{X}$ is a choice function $f : X \to \bigcup_{i\in X} A_i$. An *antichain* labeling f of $\mathbf{X}$ (or *ac*-labeling for short) is a labeling that satisfies

$$f(i) = 0 \text{ or } f(j) = 1 \quad \text{ for all } i < j \text{ in } X.$$

The poset product of $\{A_i : i \in X\}$ is

$$\prod_{\mathbf{X}} A_i = \{f \in \prod_{i\in X} A_i : f \text{ is an } ac\text{-labeling}\}.$$

The poset product is distinguished visually from the direct product since the index set is a poset $\mathbf{X}$ rather than just a set X. The terminology "antichain labeling" is explained by the following observation.

Lemma 1. *Let $\mathbf{X}$ be a poset, and $\{A_i : i \in X\}$ a family of algebras with constants $0, 1$. For a labeling f of $\mathbf{X}$ the following are equivalent.*

(i) *f is an antichain labeling.*

(ii) *$\{i \in X : f(i) \notin \{0, 1\}\}$ is a (possibly empty) antichain of $\mathbf{X}$, $f^{-1}[\{0\}]$ is a downset of $\mathbf{X}$ and $f^{-1}[\{1\}]$ is an upset of $\mathbf{X}$.*

Proof. (i)⇒(ii): Assume $f : X \to \bigcup_{i\in X} A_i$ is an *ac*-labeling, and consider $i, j \in X$. If $f(i), f(j) \notin \{0, 1\}$ then they are incomparable, hence the set of all elements labeled neither 0 nor 1 is an antichain. If $f(j) = 0 \neq 1$ and $i < j$ then $f(i) = 0$ hence $f^{-1}[\{0\}]$ is a downset, and dually for $f^{-1}[\{1\}]$.

(ii)⇒(i): Assume (ii), suppose f is a labeling, and let $i < j$. If $f(i) \neq 0$ then i is in the antichain of elements labeled neither 0 nor 1, or $f(i) = 1$. In either case we must have $f(j) = 1$, hence f is an *ac*-labeling.

For every labeling f of $\mathbf{X}$ there are two "projections" $p_0(f)$ and $p_1(f)$ into the poset product defined by

$$p_0(f)(i) = \begin{cases} f(i) & \text{if } f(j) = 1 \text{ for all } j > i \\ 0 & \text{otherwise} \end{cases}$$

$$p_1(f)(i) = \begin{cases} f(i) & \text{if } f(j) = 0 \text{ for all } j < i \\ 1 & \text{otherwise} \end{cases}$$

Now each basic operation $o \in \mathcal{F}$ is defined on the poset product $\mathbf{A}$ by

$$o^{\mathbf{A}}(f_1, \ldots, f_n) = p_0(o^{\prod_{i\in X} \mathbf{A}_i}(f_1, \ldots, f_n))$$

where $o^{\prod_{i\in X}\mathbf{A}_i}$ is the usual pointwise operation on the direct product. A *poset power* is a poset product where all the factor algebras are identical to an algebra $\mathbf{B}$, in which case $\prod_{\mathbf{X}}\mathbf{A}_i$ is denoted by $\mathbf{B}^{\mathbf{X}}$.

Note that a poset product is not, in general, a subalgebra of the direct product. However, with some mild assumptions on the basic operations of the algebras, the following result shows that the projections have no effect, and hence the poset sum is closed under pointwise defined operations. An element c in an algebra $\mathbf{A}$ is an *idempotent* of the operation o if $o^{\mathbf{A}}(c, c, \ldots, c) = c$, and the operation is *strict* with respect to c if $o^{\mathbf{A}}(x_1, \ldots, x_{i-1}, c, x_{i+1}, \ldots, x_n) = c$ for all $i \in \{1, \ldots, n\}$ and all $x_1, \ldots, x_n \in A$.

Lemma 2. *Let $\mathbf{A} = \prod_{\mathbf{X}}\mathbf{A}_i$ for some poset $\mathbf{X}$ and family $\{\mathbf{A}_i : i \in X\}$. If $0, 1$ are idempotents of o and if o is strict with respect to 0 in each $\mathbf{A}_i$ or strict with respect to 1 in each $\mathbf{A}_i$ then $o^{\mathbf{A}}$ is computed pointwise in $\mathbf{A}$.*

Proof. Suppose $0, 1$ are distinct idempotents and o is strict with respect to 0 in each $\mathbf{A}_i$. For $f_1, ..., f_n \in A$, let f be the result of applying o to $f_1, \ldots, f_n$ pointwise and consider $i < j$ in X. If $f_k(i) = 0$ for some $k \in \{1, \ldots, n\}$ then $f(i) = 0$ since o is strict, and if $f_k(i) \neq 0$ for all $k \in \{1, \ldots, n\}$ then $f_k(j) = 1$ for all k and hence $f(j) = 1$ since 1 is an idempotent. Therefore f is an *ac*-labeling, and the proof for o strict with respect to 1 is similar. It follows that $p_0(f) = p_1(f) = f \in A$.

Our main application of the poset product is to bounded lattice-ordered algebras, and specifically to bounded residuated lattices. In the most general setting, a lattice-ordered algebra (or ℓ-algebra) is any universal algebra that has a lattice reduct. However, one often assumes that the operations preserve joins or meets, or interchange joins or meets, in each argument. For example, ℓ-groupoids, unital ℓ-groupoids, ℓ-monoids and ℓ-groups are defined as groupoids, unital groupoids, monoids and groups that are expanded with lattice operations and satisfy the identities $x(y \vee z) = xy \vee xz$ and $(x \vee y)z = xz \vee yz$. They are *bounded* if there are constants $\bot, \top$ denoting the bottom and top element of the lattice reduct.

A bounded residuated lattice $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1, \bot, \top)$ is a lattice-ordered monoid $(A, \wedge, \vee, \cdot, 1)$ such that for all $x, y, z \in A$

$$x \cdot y \leq z \quad \text{iff} \quad x \leq z/y \quad \text{iff} \quad y \leq x\backslash z$$

and $\bot, \top$ are the bottom and top element of $\mathbf{A}$ (see e.g. [8]). For bounded residuated lattices the operations $\wedge, \vee, \cdot$ satisfy the assumption of the previous lemma (with $0, 1$ replaced by $\bot, \top$), while $\backslash, /$ do not. The next result implies that the poset product of a family of bounded residuated lattices is again a bounded residuated lattice, and this motivates our choice of p_0 (rather than p_1) in the definition of operations on poset products.

Lemma 3. *Let f be a labeling of a poset $\mathbf{X}$ and assume that the algebras $\mathbf{A}_i$ are partially ordered with 0 and 1 as bottom and top elements respectively. Then $p_0(f)$ is the largest element of $\prod_{\mathbf{X}}\mathbf{A}_i$ that is pointwise less or equal to f, and likewise $p_1(f)$ is the smallest element that is pointwise greater or equal to f.*

3. Direct decompositions and Boolean products of $\mathbf{FL}_w$-algebras

Mostly we consider *integral* bounded unital ℓ-groupoids (or *ibuℓ*-groupoids for short), i.e. they have the identity element 1 as top element, and in this case the bottom element is denoted by 0. A *residuated* ℓ-groupoid (or *rℓ*-groupoid) is an ℓ-groupoid for which the residuals $\backslash, /$ exist relative to the groupoid operation. A *FL_w-algebra* is a residuated integral bounded ℓ-monoid (see e.g. [8]).

A subset F of a residuated lattice $\mathbf{A}$ is a *filter* if F is up-closed, $1 \in F$ and F is closed under the monoid operation and the meet operation. A filter F is *normal* if it is closed under *conjugation*, i.e.

$$x \in F \text{ and } y \in A \quad \text{imply} \quad y\backslash(xy),\ (yx)/y \in F.$$

For $a \in A$ and $S \subseteq A$, we let $\downarrow_{\mathbf{A}} a = \{x \in A : x \le a\}$, $\downarrow_{\mathbf{A}} S = \bigcup\{\downarrow_{\mathbf{A}} a : a \in S\}$, and $\uparrow_{\mathbf{A}} a$, $\uparrow_{\mathbf{A}} S$ are defined dually ($\mathbf{A}$ is often omitted). For any residuated lattice, the lattice of normal filters is isomorphic to the congruence lattice via $\theta \mapsto \uparrow\{x : (x,1) \in \theta\}$ and $F \mapsto \{(x,y) : x\backslash y,\ y\backslash x \in F\}$. The congruence class of an element $x \in A$ with respect to the congruence induced by the filter F is denoted by x/F. A *normal* residuated lattice is one in which every filter is normal. For example every commutative residuated lattice is normal.

Before characterizing poset decompositions we consider some results about direct decompositions. An element c in an *ibuℓ*-groupoid $\mathbf{A}$ is *complemented* if there exists $c' \in A$ such that $c \wedge c' = 0$ and $c \vee c' = 1$. The *Boolean center* of $\mathbf{A}$ is the set $B(\mathbf{A})$ of all complemented elements. The next result generalizes similar results for MV-algebras [4] and BL-algebras [7]. The first part is essentially from [1].

Lemma 4. *Let $\mathbf{A}$ be an ibuℓ-groupoid and let $c \in B(\mathbf{A})$. Then*

(i) *$x \wedge c = xc = cx$ for all $x \in A$, hence the Boolean center is a Boolean sublattice of central idempotent elements.*

(ii) *If $\mathbf{A}$ is a residuated ibuℓ-groupoid then $B(\mathbf{A})$ is also closed under the residuals, the complement of c is $-c = 0/c = c\backslash 0$ and $c\backslash x = x/c = -c \vee x$ for all $c \in B(\mathbf{A})$ and $x \in A$.*

Proof. (i) Suppose A is an *ibuℓ*-groupoid and $c \wedge d = 0$, $c \vee d = 1$. By integrality

$$cx \le c \wedge x = (c \vee d)(c \wedge x) = c(c \wedge x) \vee d(c \wedge x) \le cx \vee 0 = cx,$$

and similarly $xc \le x \wedge c \le xc$. Suppose we also have $a \wedge b = 0$, $a \vee b = 1$. To see that $B(\mathbf{A})$ is a sublattice of $\mathbf{A}$, it suffices to show that $a \vee c$ and $b \wedge d$ are complements: $(a \vee c) \wedge (b \wedge d) = (a \vee c)bd = abd \vee cbd = 0$ and $(a \vee c) \vee (b \wedge d) = a \vee c \vee bd = a \vee c \vee bc \vee bd = a \vee c \vee b(c \vee d) = a \vee c \vee b = 1$.

Now $B(\mathbf{A})$ is complemented by definition, and it is a distributive lattice since $\cdot$ distributes over $\vee$, hence it is a Boolean lattice.

(ii) For complements c, d and any $x \in A$ we have $c\backslash x = (c \vee d)(c\backslash x) = c(c\backslash x) \vee d(c\backslash x) \le x \vee d$. On the other hand $c(x \vee d) = cx \vee cd \le x$ implies $x \vee d \le c\backslash x$. Hence $c\backslash x = d \vee x$, and for $x = 0$ we obtain $-c = c\backslash 0 = d$. Therefore $c\backslash x = -c \vee x$ for all $x \in A$. The results for $/$ follow similarly.

For an $ib(r)u\ell$-groupoid $\mathbf{A}$ and an element $c \in B(\mathbf{A})$, define the *relativized subalgebra* $\mathbf{A}c$ with universe $Ac = \downarrow c$, unit $1^{\mathbf{A}c} = c$, operations $\wedge, \vee, \cdot$ restricted from $\mathbf{A}$, and $a\backslash b = (a\backslash^{\mathbf{A}} b) \wedge c$, $a/b = (a/^{\mathbf{A}} b) \wedge c$ for all $a, b \in \downarrow c$.

Lemma 5. *For any $ib(r)u\ell$-groupoid $\mathbf{A}$ and any $c \in B(\mathbf{A})$, the relativized subalgebra $\mathbf{A}c$ is an $ib(r)u\ell$-groupoid. If $\mathbf{A}$ is an FL_w-algebra then the map $f : \mathbf{A} \to \mathbf{A}c$ given by $f(a) = ac$ is a homomorphism, hence $\mathbf{A}c$ satisfies all identities that hold in $\mathbf{A}$.*

Proof. By (i) of the preceding lemma, $\mathbf{A}c$ has c as a unit and is closed under $\wedge, \vee, \cdot$, hence it is an $ibu\ell$-groupoid. If $\mathbf{A}$ has residuals then for all $a, b, x \in \mathbf{A}c$ we have

$$ax \le b \quad \text{iff} \quad x \le^{\mathbf{A}} a\backslash^{\mathbf{A}} b \text{ and } x \le^{\mathbf{A}} c \quad \text{iff} \quad x \le a\backslash b,$$

and similarly $a/b = (a/^{\mathbf{A}} b) \wedge c$, whence $\backslash, /$ are residuals of $\cdot$ in $\mathbf{A}c$.

Now $f(1) = 1c = 1^{\mathbf{A}c}$, $(a \wedge b)c = a \wedge b \wedge c = ac \wedge bc$ and $(a \vee b)c = ac \vee bc$ hence f preserves $\wedge, \vee$. If $\cdot$ is associative then $(ab)c = abcc = (ac)(bc)$. In any residuated lattice $x\backslash y \le zx\backslash zy$, hence $f(a\backslash^{\mathbf{A}} b) \le f(a)\backslash f(b)$. For the opposite inequality, we have $ac(ac\backslash^{\mathbf{A}} bc) \le bc \le b$, so $c(ac\backslash^{\mathbf{A}} bc) \le a\backslash^{\mathbf{A}} b$, and therefore $(ac\backslash^{\mathbf{A}} bc) \wedge c \le (a\backslash^{\mathbf{A}} b)c$. This shows $f(a)\backslash f(b) \le f(a\backslash^{\mathbf{A}} b)$.

Theorem 6. *If $\mathbf{A}$ is an FL_w-algebra and if $c, d \in B(\mathbf{A})$ are complements then $\mathbf{A} \cong \mathbf{A}c \times \mathbf{A}d$.*

Proof. Consider the map $h : \mathbf{A} \to \mathbf{A}c \times \mathbf{A}d$ defined by $h(a) = (a \wedge c, a \wedge d)$. The preceding two lemmas show that h is a homomorphism, and h has an inverse given by $(x, y) \mapsto x \vee y$ since $ac \vee ad = a(c \vee d) = a$ and for $x \le c$, $y \le d$ we have $((x \vee y)c, (x \vee y)d) = (xc \vee yc, xd \vee yd) = (x, y)$.

Conversely, any direct decomposition of an $ib(r)u\ell$-groupoid is obtained in this way, since the elements $(0, 1)$, $(1, 0)$ are complements.

Corollary 7. *An FL_w-algebra is directly indecomposable iff its Boolean center contains only the elements $\{0, 1\}$.*

The preceding results about direct decompositions are useful for a characterization of (weak) Boolean products of FL_w-algebras. We first recall a general characterization of weak Boolean products in terms of Boolean algebras of factor congruences from [13]. A *(weak) Boolean decomposition* of $\mathbf{A}$ is an isomorphism from $\mathbf{A}$ to a (weak) Boolean product. A pair θ, ψ of congruences of $\mathbf{A}$ are called *factor congruences* if $\theta \cap \psi = \mathrm{id}_A$ and $\theta \circ \psi = A^2$. A *Boolean algebra of factor congruences* is a set of factor congruences that is a Boolean algebra, with $\cap$ and $\circ$ as lattice operations.

Theorem 8. *Let $\mathbf{A}$ be an algebra.*

(i) *Suppose $\mathbf{K}$ is a Boolean algebra of factor congruences on $\mathbf{A}$. For each prime filter F of $\mathbf{K}$, let $\theta_F = \bigcup(K - F)$ and define $\varepsilon : \mathbf{A} \to \prod_{F \in X_{\mathbf{K}}} \mathbf{A}/\theta_F$ by $\varepsilon(a)(F) = a/\theta_F$. Then ε is a weak Boolean decomposition of $\mathbf{A}$.*

(ii) *If* $\mathbf{X}$ *is a Boolean space and* $\varepsilon' : \mathbf{A} \to \prod_{i\in X} \mathbf{A}_i$ *is any weak Boolean decomposition then there exists a unique Boolean algebra* $\mathbf{K}$ *of factor congruences, a homeomorphism* $k : \mathbf{X} \to \mathbf{X_K}$*, and isomorphisms* $h_i : \mathbf{A}_i \cong \mathbf{A}/\theta_{k(i)}$ *such that* $h_i\pi_i\varepsilon' = \pi_{k(i)}\varepsilon$*, where* $\varepsilon : \mathbf{A} \to \prod_{i\in X} \mathbf{A}/\theta_{k(i)}$ *s given by* $\varepsilon(a)(i) = a/\theta_{k(i)}$.

The algebra $\mathbf{K}$ in (ii) is the set of congruences $\psi_U = \cap\{\ker(\pi_i\varepsilon') : i \in U\}$ where U ranges over the clopen sets of $\mathbf{X}$. For an FL_w-algebra $\mathbf{A}$ the algebra of all factor congruences is isomorphic to $B(\mathbf{A})$. The following result generalizes Theorem 2.1 in [7].

Corollary 9. *Let* $\mathbf{A}$ *be a weak Boolean product of a nonempty family* $\{\mathbf{A}_i : i \in X\}$ *of non-trivial* FL_w*-algebras over a Boolean space* $\mathbf{X}$ *and let* $C = \{f \in A : f[X] \subseteq \{0,1\}\}$*. Then*

(i) $\mathbf{C}$ *is a subalgebra of* $B(\mathbf{A})$*,*
(ii) *the map* $k(i) = \{f \in C : f(i) = 1\}$ *is a homeomorphism from* $\mathbf{X}$ *onto* $\mathbf{X_C}$*,*
(iii) $\mathbf{A}_i$ *is isomorphic to* $\mathbf{A}/{\uparrow}k(i)$*, and*
(iv) $\mathbf{C}$ *coincides with* $B(\mathbf{A})$ *iff all algebras* $\mathbf{A}_i$ *are directly indecomposable.*

Conversely, suppose $\mathbf{A}$ *is a nontrivial* FL_w*-algebra and* $\mathbf{C}$ *is a subalgebra of* $B(\mathbf{A})$*. Then* $\mathbf{A}$ *is isomorphic to a weak Boolean product of* $\{\mathbf{A}/{\uparrow}F : F \in X_{\mathbf{C}}\}$.

PROOF. (i) holds since $f \in C$ implies $f\backslash 0$ is a complement of f, and (ii) follows from the observation that the algebra $\mathbf{A_X}$ of clopen subsets of $\mathbf{X}$ is isomorphic to $\mathbf{C} \cong \mathbf{A_{X_C}}$. The isomorphism in (iii) follows from (ii) of the preceding theorem, and the converse is from part (i) of the same result.

4. Embeddings and representations via poset products

A *generalized BL-algebra* or (GBL-algebra for short) is a residuated lattice that is *divisible*, i.e. satisfies

$$x \le y \quad \Rightarrow \quad x = (x/y)y = y(y\backslash x).$$

This property is equivalent to an identity (replace x by $x \wedge y$), and implies that there are no idempotent elements above 1. Hence any bounded GBL-algebra is integral, and we again denote the bottom element by 0. As examples we list the following subvarieties:

- *BL-algebras* are bounded GBL-algebras that satisfy *commutativity* ($xy = yx$) and *prelinearity* ($x\backslash y \vee y\backslash x = 1$),
- *Heyting algebras* are bounded GBL-algebras in which all elements are idempotent (whence $xy = x \wedge y$),
- *GMV-algebras* are GBL-algebras that satisfy $x \wedge y = x/(y\backslash x) = (x/y)\backslash x$),
- *pseudo MV-algebras* are bounded GMV-algebras,

- *MV-algebras* in addition satisfy commutativity $xy = yx$, and
- Boolean algebras are the intersection of Heyting algebras and (pseudo-)MV-algebras.

We now recall a result from [11] that gives sufficient conditions for an algebra to be embeddable into a poset product. There it is proved for integral GBL-algebras, and the factors are assumed to be totally ordered GMV-algebras. Since they need not have a lower bound, the factors are first embedded into pseudo MV-algebras. Here we state the result for FL_w-algebras in general, but note that the proof is essentially the same. The *ordinal sum* of two algebras $\mathbf{B}_0, \mathbf{B}_1$, each with constants $0, 1$, is defined as $\mathbf{B}_0 \oplus \mathbf{B}_1 = \prod_{\mathbf{2}^\partial} \mathbf{B}_i$, where $\mathbf{2}^\partial = \{0, 1\}$ is the two element poset with $1 < 0$. For *ib*(r)*ul*-groupoids this agrees with the usual definition of (amalgamated) ordinal sum where all elements of $\mathbf{B}_0$ are less or equal to all elements of $\mathbf{B}_1$.

Theorem 10. *Let $\mathbf{A}$ be a FL_w-algebra, $\mathbf{X}$ a poset, and $\{F_i : i \in X\}$ a family of normal filters of $\mathbf{A}$ such that for all $i \in X$*

(i) $\mathbf{A}/F_i = \mathbf{B}_i \oplus \mathbf{C}_i$ *where $\mathbf{B}_i, \mathbf{C}_i$ are FL_w-algebras,*
(ii) $c \subseteq F_j$ *for all $c \in C_i$ and all $j > i$,*
(iii) *for all $a \notin F_i$ there exists $j \geq i$ such that $a/F_j \in C_j - \{1/F_j\}$,*
(iv) $\bigcap_{i \in X} F_i = \{1\}$.

Then $\mathbf{A}$ embeds into the poset product $\prod_{\mathbf{X}} \mathbf{C}_i$.

In [11] this theorem is used to prove that every integral normal GBL-algebra *embeds* into a poset product of totally ordered integral bounded GMV-algebras. The key result that enables this application is the Blok-Ferreirim decomposition theorem for subdirectly irreducible integral normal GBL-algebras proved in [10]: every such algebra is isomorphic to an ordinal sum $\mathbf{B} \oplus \mathbf{W}$ where $\mathbf{W}$ is a non-trivial totally ordered integral GMV-algebra and $\mathbf{B}$ is an integral GBL-algebra.

An algebra $\mathbf{A}$ is *poset indecomposable* if whenever $\mathbf{A}$ is isomorphic to a poset product $\prod_{\mathbf{X}} \mathbf{A}_i$ there exists $i \in X$ such that $\mathbf{A} \cong \mathbf{A}_i$.

In [9] it is shown that every finite GBL-algebra is *isomorphic* to a (uniquely determined) poset product of totally ordered integral GMV-algebras, which are poset indecomposable. In the next section we augment poset products with a Boolean topology on the index poset, with the aim of extending the representation of finite GBL-algebras to a larger class of algebras.

For a residuated lattice $\mathbf{A}$ we define the set of strongly central elements $I_{\mathbf{A}} = \{a \in A : a \wedge x = ax = xa \text{ for all } x \in A\}$. Recall from [9] that if $\mathbf{A}$ is a GBL-algebra then $\mathbf{I_A}$ is a subalgebra of $\mathbf{A}$. For bounded GBL-algebras, $\mathbf{I_A}$ is in fact a Heyting algebra, and $B(\mathbf{A})$ is the subalgebra of complemented elements of $\mathbf{I_A}$. For MV-algebras $B(\mathbf{A}) = \mathbf{I_A}$.

Lemma 11. *Let $\mathbf{A}$ be a FL_w-algebra and let $a, b \in I_{\mathbf{A}}$ with $a \leq b$. Then the interval $[a, b] = \{x \in A : a \leq x \leq b\}$ is a FL_w-algebra, with $0 = a$, $1 = b$, $\wedge$, $\vee$, $\cdot$ inherited from $\mathbf{A}$, and $x\backslash y = (x\backslash^{\mathbf{A}} y) \wedge b$, $x/y = (x/^{\mathbf{A}} y) \wedge b$. If $\mathbf{A}$ is a GBL-algebra, then so is $[a, b]$.*

Proof. As in Lemma 5, $h(x) = xb$ is a homomorphism from $\mathbf{A}$ to $\mathbf{A}b$. For any integral residuated lattice $\mathbf{B}$ and idempotent $a \in B$, the principal filter $\uparrow a$ is a subalgebra of $\mathbf{B}$ (see [8] Lemma 3.40). Therefore the GBL identity holds in $[a, b]$ if it holds in $\mathbf{A}$.

We now generalize the poset decomposition result of [9] from finite GBL-algebras to FL_w-algebras. Recall that an element c in a lattice $\mathbf{L}$ is *completely join irreducible* if, for any subset S of L, $c = \bigvee S$ implies $c \in S$. Equivalently, c is completely join irreducible if there exists a unique element $c_* < c$, called a *lower cover* of c, such that no element of $\mathbf{L}$ is strictly between c_* and c.

Theorem 12. *Consider a FL_w-algebra $\mathbf{A}$ with a finite subalgebra $\mathbf{C}$ such that $C \subseteq I_{\mathbf{A}}$, and let $\mathbf{X}$ be the dual of the partially ordered set of completely join irreducible elements of $\mathbf{C}$. If $\mathbf{A}c = \downarrow c_* \oplus [c_*, c]$ for all $c \in X$ then $\mathbf{A} \cong \prod_{\mathbf{X}} [c_*, c]$, where c_* is the unique lower cover of c in $\mathbf{C}$ and $[c_*, c]$ is an interval in $\mathbf{A}$.*

Proof. Let $\mathbf{A}$ be a FL_w-algebra with a subalgebra $\mathbf{C}$ that satisfies the assumptions of the theorem. We define the map $h : \mathbf{A} \to \prod_{\mathbf{X}} [c_*, c]$ by $h(a)(c) = ac \vee c_*$ (this is an element of $[c_*, c]$ since $c_* \le ac \vee c_* \le C$). To see that $f = h(a)$ is an element of the poset product, we have to show that if $c < d$ in $\mathbf{X}$ (hence $c > d$ in $\mathbf{C}$) then $f(c) = c_*$ (the 0 of $[c_*, c]$) or $f(d) = d$ (the 1 of $[d_*, d]$). Assuming $f(c) \ne c_*$, we have $a \wedge c > c_*$ since $\mathbf{A}c = \downarrow c_* \oplus [c_*, c]$. Therefore $a > c_* \ge d$, and it follows that $f(d) = ad \vee d_* = d$.

We claim that h is a FL_w-algebra isomorphism. It suffices to show that h is an order-isomorphism that preserves the monoid structure (since order-isomorphisms always preserve the first-order definable lattice operations and residuals). We have $h(1) = \mathbf{1}$ since $1c \vee c_* = c$, and the preservation of $\cdot$ follows from $(ac \vee c_*)(bc \vee c_*) = acbc \vee acc_* \vee bcc_* \vee c_* = (ab)c \vee c_*$.

The map h is clearly order-preserving, and to show it is a bijection, we define $g : \prod_{\mathbf{X}} [c_*, c] \to \mathbf{A}$ by

$$g(f) = \bigvee \{f(c) : f(k) = k \text{ for all } k \in X \text{ with } k <^{\mathbf{C}} c\}.$$

Then g is also order-preserving, and it remains to show that it is the inverse of h. For $a \in A$, note that $g(h(a)) = \bigvee \{ac \vee c_* : ak = k \text{ for all } k \in X \text{ with } k <^{\mathbf{C}} c\} = \bigvee \{ac : c_* \le a\}$, since we have $c_* = \bigvee \{k \in X : k < c\}$. Moreover, because $\mathbf{C}$ is finite, there is a smallest $m \in C$ such that $a \le m$. For any $c \in X$ with $c \le m$ we have $c_* < ca$ or $ca \le c_*$ since $\mathbf{A}c = \downarrow c_* \oplus [c_*, c]$. But $ca \le c_*$ implies $a \le c \backslash c_* \in C$ (since $\mathbf{C}$ is a subalgebra of $\mathbf{A}$), and by choice of m it follows that $m \le c \backslash c_*$, so $c = cm \le c_*$, contradicting $c_* < c$. Hence $c_* < ca \le a$, and we obtain $g(h(a)) = a(\bigvee \{c : c_* \le a\}) = am = a$.

Now let f be an element of the poset product, and consider $h(g(f))$. We have $h(g(f))(c) = g(f)c \vee c_* = \bigvee \{f(d)c \vee c_* : f(k) = k \text{ for all } k \in X \text{ with } k <^{\mathbf{C}} d\} \in [c_*, c]$. The elements $f(d)c \vee c_*$ are refered to as the *joinands* of the join. If $f(c) = c$ then $f(k) = k$ for all $k < c$, so $f(c)$ is one of the joinands, hence $h(g(f))(c) = c = f(c)$. On the other hand, if $f(c) < c$ then $h(g(f))(c) \ge f(c)$ (even in case $f(k) < k$ for some $k < c$, since then $f(c) = c_*$). We need to show that $f(d)c \leq f(c)$ for all joinands, i.e. whenever $f(k) = k$ for all $k < d$. In this case we know $d \not> c$ since $f(c) < c$. If $d < c$ then $f(d) \leq d \leq c_* \leq f(c)$, and if c, d are incomparable then $c \wedge d \leq c_*$, hence $f(d)c \leq dc \leq c_* \leq f(c)$. This concludes the proof that $h(g(f)) = f$.

5. Combining Boolean products and poset products

As observed in the previous sections, both Boolean products and poset products are a generalization of direct products. Even if all the factors of a Boolean product are complete lattices, the resulting algebra need not be complete. However for poset products the completeness of the factors implies the completeness of the poset product. So it is not possible to represent incomplete algebras by poset products of finite (or complete) algebras, without generalizing the poset product to include topological aspects.

A *Priestley space* $\mathbf{X} = (X, \leq, \tau)$ is a poset $(X, \leq)$ such that τ is a compact totally order disconnected topology on X, i.e. for all $i \not\leq j$ in X there is a clopen upset U such that $i \in U$ and $j \notin U$. By the well known Priestley duality [6], the collection $\mathbf{D}_{\mathbf{X}}$ of clopen upsets of $\mathbf{X}$ forms a bounded distributive lattice under intersection and join, and from any distributive lattice $\mathbf{D}$ one can obtain a Priestley space $\mathbf{X}_{\mathbf{D}} = (X_{\mathbf{D}}, \subseteq, \tau)$ by considering the set $X_{\mathbf{D}}$ of prime filters of $\mathbf{D}$, ordered by inclusion, and with τ given by a basis $\{U_a \cap (X_{\mathbf{D}} - U_b) : a, b \in D\}$ where $U_a = \{F \in X_{\mathbf{D}} : a \in F\}$. Moreover, $\mathbf{X}_{\mathbf{D}_{\mathbf{X}}} \cong \mathbf{X}$ and $\mathbf{D}_{\mathbf{X}_{\mathbf{D}}} \cong \mathbf{D}$.

An *Esakia space* $\mathbf{X}$ is a Priestley space that satisfies the additional requirement that $\downarrow U$ is clopen for every clopen set U. By compactness $\downarrow K$ is closed for any closed set K, so it suffices to require that the downset generated by any open set is open. The Esakia duality states that the clopen upsets of an Esakia space form a Heyting algebra $\mathbf{A}_{\mathbf{X}}$ (with $U \to V = X - \downarrow(U - V)$), and the Priestley space $\mathbf{X}_{\mathbf{A}}$ of any Heyting algebra $\mathbf{A}$ is in fact an Esakia space. As before, $\mathbf{X}_{\mathbf{A}_{\mathbf{X}}} \cong \mathbf{X}$ and $\mathbf{A}_{\mathbf{X}_{\mathbf{A}}} \cong \mathbf{A}$.

Let $\mathbf{X}$ be a Priestley space, and consider a family $\{\mathbf{A}_i : i \in X\}$ of algebras with constants $0, 1$. A *weak Priestley product* is a subalgebra $\mathbf{A}$ of the poset product $\prod_{(X,\leq)} \mathbf{A}_i$ such that for all $f, g \in A$

(i) $[\![f = g]\!]$ is open,
(ii) for all clopen U, if $f|_U \cup g|_{X-U}$ is an ac-labeling then $f|_U \cup g|_{X-U} \in A$, and
(iii) for each $i \in X$ the projection π_i restricted to $\mathbf{A}$ is surjective.

As for Boolean products, $\mathbf{A}$ is a *Priestley product* $\mathbf{A}$ is obtained if "open" in (i) is replaced by "clopen". The *Priestley power* $\mathbf{B}^{\mathbf{X}}$ of an algebra $\mathbf{B}$ over a Priestley space $\mathbf{X}$ is the set of continuous functions in the poset power $\mathbf{B}^{(X,\leq)}$. Note that, in general, a Priestley power may not be closed under the operations of the poset power (consider, for example, a Priestley power of 2-element Heyting algebras over a non-Esakia space).

Lemma 13. *If* $0, 1$ *are idempotents of* $\mathbf{B}$*, and each operation is strict with respect to* 0 *or* 1 *then every Priestley power of* $\mathbf{B}$ *is a Priestley product, hence a subalgebra of the poset product.*

PROOF. Under the assumptions on $0, 1$, the poset product is a subalgebra of the direct product, so a Priestley product is just the intersection of a Boolean product and a poset product, and likewise for the Priestley power. Since every Boolean power is a Boolean product and every poset power is (by definition) a poset product, the result follows.

Priestley products or powers can be used to give representations for many algebras that cannot be represented by Boolean products or powers since e.g. finite Priestley products are poset products rather than direct products.

However, the assumptions in the preceding lemma are too strong for an application to residuated lattices, which motivates the following refinement. A (weak) *Esakia product* is a (weak) Priestley product $\mathbf{A}$ such that

(iv) $p_0(f|_U \cup g|_{X-U}) \in A$ for all $f, g \in A$ and all clopen U.

Note that if the partial order on the Priestley space is an antichain, then both Priestley products and Esakia products reduce to Boolean products. Furthermore, Priestley powers over the 2-element distributive lattice are isomorphic to the distributive lattice of clopen upsets of the Priestley space, and similarly Esakia powers of the 2-element Heyting algebra are isomorphic to the Heyting algebra of clopen upsets of the given Esakia space.

Lemma 14. *For a weak Esakia product, the Priestley space $\mathbf{X}$ is necessarily an Esakia space.*

PROOF. Suppose U is a clopen set of $\mathbf{X}$. We need to show that (iv) implies $\downarrow U$ is open. Note that the constant functions $\mathbf{0}, \mathbf{1}$ are in $\mathbf{A}$, and let $f = \mathbf{0}|_U \cup \mathbf{1}|_{X-U}$. Then $p_0(f) \in A$ by (iv). We claim that $[\![p_0(f) = \mathbf{0}]\!] = \downarrow U$, hence by (i) $\downarrow U$ is open. The claim follows from the following equivalent statements: $i \in [\![p_0(f) = \mathbf{0}]\!]$ iff $p_0(f)(i) = 0$ iff $f(j) = 0$ for some $j \geq i$ iff $i \in \downarrow U$.

A GBL-algebra is *n-potent* if it satisfies the identity $x^{n+1} = x^n$. Note that simple MV-algebras are n-potent iff they are totally ordered and contain at most n elements. The following result is from [10].

Lemma 15. *Every n-potent GBL-algebra is integral and commutative, hence normal. It is subdirectly irreducible if and only if it has a maximal idempotent below* 1.

It is proved in [9] that the poset product of GBL-algebras is again a GBL-algebra. The result below generalizes the representation of finite GBL-algebras as poset products of simple MV-algebras.

Theorem 16. *Let $\mathbf{A}$ be a weak Esakia product of a family $\{\mathbf{A}_i : i \in X\}$ of simple n-potent MV-algebras, and let $C = \{f \in A : f[X] \subseteq \{0, 1\}\}$. Then*

(i) $\mathbf{C} = \mathbf{I}_{\mathbf{A}}$,

(ii) the map $k : X \to X_{\mathbf{C}}$ defined by $k(i) = \{f \in C : f(i) = 1\}$ is an order-preserving homeomorphism and

(iii) for all $i \in X$, $\mathbf{A}/{\uparrow_{\mathbf{A}}}k(i)$ is subdirectly irreducible and its minimal nontrivial filter is isomorphic to $\mathbf{A}_i$.

Conversely, suppose $\mathbf{A}$ is a nontrivial bounded n-potent GBL-algebra and let $\mathbf{C} = \mathbf{I_A}$. Then for each $F \in X_{\mathbf{C}}$, $\mathbf{A}/{\uparrow_{\mathbf{A}}}F$ is subdirectly irreducible and its minimal nontrivial filter A_F is a simple n-potent MV-algebra. Furthermore $\mathbf{A}$ is isomorphic to an Esakia product of $\{\mathbf{A}_F : F \in X_{\mathbf{C}}\}$.

Proof. (i) For $g \in C$, we have $f \wedge g = fg = gf$ for all $f \in \mathbf{A}$ since $x \wedge 0 = x0 = 0x = 0$ and $1 \wedge x = 1x = x1 = x$ in any FL_w-algebra, hence $C \subseteq I_{\mathbf{A}}$. On the other hand, if $g \notin C$ then $g(i) \notin \{0, 1\}$ for some $i \in X$, and since simple MV-algebras only have $0, 1$ as idempotents, we have $g(i)^2 \neq g(i)$, whence $g \notin I_{\mathbf{A}}$.

(ii) holds because by Esakia duality $\mathbf{A_X} \cong \mathbf{2^X} = \mathbf{C}$ (where $\mathbf{2^X}$ denotes the Esakia power) and k gives the correspondence between prime filters in these two isomorphic algebras.

(iii) Since $k(i)$ is a prime filter of the Heyting algebra $\mathbf{C}$, the quotient $\mathbf{C}/k(i)$ is subdirectly irreducible and hence has a coatom $f/k(i)$, where $f \in C$. Letting $F_i = {\uparrow_{\mathbf{A}}}k(i)$ it follows that $f/F_i < \mathbf{1}/F_i$ is a maximal idempotent of $\mathbf{A}/F_i$, and hence A/F_i is subdirectly irreducible. Note that $f \in C - k(i)$, whence $f(i) = 0$. The isomorphism between $\mathbf{A}_i$ and the finite chain above f/F_i is given by $b \mapsto f_b/F_i$ where f_b agrees with f except that $f_b(i) = b$.

For the converse, note that $\mathbf{C}$ is a Heyting algebra and let $F \in X_{\mathbf{C}}$. Then F is a prime filter of $\mathbf{C}$, so as before $\mathbf{A}/{\uparrow}F$ is subdirectly irreducible and has a maximal idempotent $f/{\uparrow}F$ below the top 1. The algebra $\mathbf{A}_F$ is the interval $[f/{\uparrow}F, 1]$ which by n-potence is a finite simple MV-algebra. Using Theorem 10 it follows that the map $\varepsilon : A \to \prod_{F \in X_{\mathbf{C}}} A_F$ given by

$$\varepsilon(a)(i) = \begin{cases} a/{\uparrow}F & \text{if } a/{\uparrow}F \in A_F \\ 0 & \text{otherwise} \end{cases}$$

is an embedding into the poset product $\prod_{\mathbf{X_C}} A_F$, and by construction $\varepsilon[A]$ is an Esakia product.

As mentioned earlier, if $\mathbf{A}$ is an MV-algebra then $I_{\mathbf{A}} = B(\mathbf{A})$, so a representation of $\mathbf{A}$ as an Esakia product is in fact a Boolean product. Hence for MV-algebras the preceding result reduces to the representation of n-potent MV-algebras as Boolean products of simple MV-algebras (see [4]).

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