Embedding theorems for classes of GBL-algebras

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Abstract

The poset product construction is used to derive embedding theorems for several classes of generalized basic logic algebras (GBL-algebras). In particular it is shown that every n-potent GBL-algebra is embedded in a poset product of finite n-potent MV-chains, and every normal GBL-algebra is embedded in a poset product of totally ordered GMV-algebras. Representable normal GBL-algebras have poset product embeddings where the poset is a root system. We also give a Conrad-Harvey-Holland-style embedding theorem for commutative GBL-algebras, where the poset factors are the real numbers extended with $-\infty$. Finally, an explicit construction of a generic commutative GBL-algebra is given, and it is shown that every normal GBL-algebra embeds in the conucleus image of a GMV-algebra.

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1. Introduction

Generalized BL-algebras (GBL-algebras for short, cf [JT02], [GT05]) are divisible residuated lattices, that is, residuated lattices such that if $x \leq y$, then there exist z, u such that $zy = yu = x$. These algebras constitute a generalization of several important classes of algebras. First of all, GBL-algebras include (zero-free subreducts of) Heyting algebras, which are the algebraic counterpart of intuitionistic logic. Moreover, as the name suggests, GBL-algebras are a generalization of (the zero free subreducts of) BL-algebras, which constitute the variety generated by the commutative and integral residuated lattices with $([0,1], \max, \min, 0, 1)$ as lattice reduct, and with a monoid operation which is continuous on $[0,1]$, called a *continuous t-norm*, cf [Haj98], [Ha98], [CEGT].

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BL-algebras have been introduced by Hàjek in [Haj98] as a general semantics for fuzzy logics. Indeed BL-algebras include Chang's MV-algebras [CDM00], product algebras [Haj98] and Gödel algebras (i.e., representable Heyting algebras, cf [Haj98]). But GBL-algebras are also a generalization of ℓ-groups, which are structures arising from classical algebra, cf [AF88] and [Gl99]. Indeed, an ℓ-group is a divisible residuated lattice, with residuals $x \backslash y = x^{-1}y$ and $y/x = yx^{-1}$. Divisibility follows from the observation that for all x, y, if $z = xy^{-1}$ and $u = y^{-1}x$ then $zy = yu = x$. Thus GBL-algebras constitute a bridge between algebraic logic and classical algebra.

In this paper we prove several embedding theorems for classes of GBL-algebras. By *embedding theorem* we mean a theorem stating that every algebra of a given class $\mathcal{C}$ embeds into an algebra in $\mathcal{C}$ having a special form. A typical example is a naive version of Stone's theorem stating that every boolean algebra *embeds* into a powerset boolean algebra. Embedding theorems are weak versions of *representation theorems*. By this terminology we mean theorems stating that every algebra of a given class $\mathcal{C}$ is *isomorphic to* an algebra in $\mathcal{C}$ having a special form. An example of a representation theorem is the strong version of Stone's theorem, which says that every boolean algebra is isomorphic to the algebra of closed and open sets of a totally disconnected and compact topological space. The list of all important representation theorems in algebraic logic (often expressed in terms of an equivalence of categories) would be too long to be included in this introduction. We will only mention a few of them, which are closely related to GBL-algebras, namely, Mundici's equivalence Γ between MV-algebras and abelian ℓ-groups with strong unit [Mu86], recently extended by Dvurečenskij [Dv02] to the non-commutative case, the categorical equivalence between integral GMV-algebras and negative cones of ell groups with a nucleus [GT05], the ordinal sum representation of totally ordered BL-algebras [AM03], also extended by Dvurečenskij [Dv07] to the non-commutative case, or even the representation of finite GBL-algebras as finite poset products of finite MV-algebras, proved in [JM09]. But in the literature of ℓ-groups we also find embedding theorems, for instance Holland's theorem stating that every ℓ-group embeds into the ℓ-group of automorphisms of a totally ordered set, with composition as group operation and with lattice operations defined pointwise, or even the Conrad-Harvey-Holland embedding of any abelian ℓ-group into the abelian ℓ-group of functions from a root system into the reals, cf [AF88], [Gl99] (in fact, the embedding is an isomorphism if the ℓ-group is divisible in the sense that for every element x and for every positive integer n there is a y such that $y^n = x$, but it is not an isomorphism in general).

Coming to the content of this paper, our aim is to generalize the ordinal sum decomposition of [AM03] or of [Dv07] to classes of GBL-algebras. To this purpose we will use the poset product construction introduced as (dual) poset sum in [JM09], which is a common generalization of ordinal sums and of direct products. The paper is organized as follows: in Section 3 we give a general sufficient condition for embeddability into a poset product of a family of GBL-algebras. Then in Section 4 we use this condition in order to prove that every n-potent GBL-algebra embeds into the poset product of a family of finite n-

potent MV-chains. Heyting algebras occur as a particular case, because they are just 1-potent bounded GBL-algebras. In Section 5 we prove that every normal[1] GBL-algebra embeds into a poset product of totally ordered GMV-algebras, and that every commutative GBL-algebra embeds into the poset product of totally ordered MV-algebras and totally ordered abelian ℓ-groups. In Section 6 we show that representable normal GBL-algebras correspond to poset products in which the poset is a root system, and we characterize various classes of GBL-algebras in terms of poset product embeddability. In Section 7 we combine the previous embedding theorems with Hahn's embedding theorem of totally ordered abelian groups, thus proving that the above mentioned classes of GBL-algebras embed into algebras of functions taking values in $\mathbf{R}\cup\{-\infty\}$, whose structure is induced only by the structure of the reals and by some orderings. Finally, in Section 8 we give an explicit construction of a strongly generic commutative GBL-algebra, that is, of a GBL-algebra which generates the full variety of commutative GBL-algebras as a quasivariety.

2. Basic notions

In this section we review some definitions and some known results about residuated lattices, GBL-algebras, GMV-algebras and ordinal sums.

2.1. Residuated lattices

Definition 2.1. A *residuated lattice* (cf e.g. [BT03], [JT02]) is an algebra of the form $(L, \vee, \wedge, \cdot, \backslash, /, e)$ where $(L, \vee, \wedge)$ is a lattice, $(L, \cdot, e)$ is a monoid and $\backslash$ and $/$ are binary operations that are left and right residuals of $\cdot$, i.e., for all $x, y, z \in L$

$$x \cdot y \leq z \text{ iff } y \leq x\backslash z \text{ iff } x \leq z/y.$$

In the sequel the symbol $\cdot$ will often be omitted. We recall briefly the terminology that is used throughout the paper.

Definition 2.2. A residuated lattice is said to be

- *commutative* if it satisfies $xy = yx$,
- *integral* if it satisfies $x \leq e$,
- *bounded* if it has a minimum m (and hence a maximum m/m) and if the signature has an additional constant symbol interpreted as m,
- *divisible* iff $x \leq y$ implies $y(y\backslash x) = (x/y)y = x$;
- *cancellative* if $uxv = uyv$ implies $x = y$, and
- *representable* if it is isomorphic to a subdirect product of totally ordered residuated lattices.

[1] a residuated lattice is said to be *normal* iff every filter of it is a normal filter

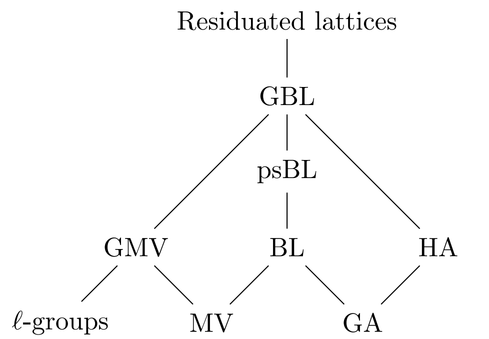

Figure 1: Inclusions between classes of residuated lattices

Note that *ℓ-groups* (cf [AF88], [Gl99]) can be presented as residuated lattices satisfying $x(x\backslash e) = e$. As mentioned in the introduction, given an ℓ-group we obtain a cancellative and divisible residuated lattice letting $x\backslash y = x^{-1}y$ and $y/x = yx^{-1}$. Conversely, from a residuated lattice satisfying $x(x\backslash e) = e$ we obtain an ℓ-group by letting $x^{-1} = x\backslash e = e/x$.

In a commutative residuated lattice the operations $x\backslash y$ and y/x coincide and are denoted by $x \to y$.

2.2. GBL-algebras and GMV-algebras

Definition 2.3. A residuated lattice is called a

- a *GBL-algebra* (cf [JT02] and [GT05]) if it is divisible,
- a *GMV-algebra* if it is a GBL-algebra that satisfies the equations $y/((x\backslash y)\wedge e) = ((y/x) \wedge e)\backslash y = x \vee y$,
- an *MV-algebra* if it is a commutative, integral and bounded GMV-algebra,
- a *pseudo BL-algebra* (*psBL-algebra* for short, cf [DGJ02]) if it is an integral and bounded GBL-algebra satisfying $(x\backslash y) \vee (y\backslash x) = (y/x) \vee (x/y) = e$,
- a *BL-algebra* (cf [Haj98]) if it is a commutative, integral, bounded and representable GBL-algebra,
- a *Heyting algebra* if it is a bounded GBL-algebra satisfying $x \cdot y = x \wedge y$,
- a *Gödel algebra* if it is a representable Heyting algebra.

Figure 1. shows the inclusion relation among the classes covered in the previous definition. In some papers, such as [GRW03], the terminology *GMV-algebra* has also been used in a more restricted sense for algebras that are in addition assumed to be bounded and integral. However we use the definition of [JT02] and [GT05] since it is more general, and hence corresponds better to the notion of generalized MV-algebra. For instance, ℓ-groups are GMV algebras in the sense of [GT05], but not in the sense of [GRW03].

Definition 2.4. The *negative cone* of a residuated lattice $\mathbf{L}$ is the algebra $\mathbf{L}^-$ whose domain is $\{x \in \mathbf{L} : x \leq e\}$, whose lattice operations and whose monoid operation are the restrictions to $\mathbf{L}^-$ of the corresponding operations in $\mathbf{L}$ and whose residuals $\backslash^-$ are and $/^-$ are defined by $x\backslash^- y = (x\backslash y) \wedge e$ and $y/^- x = (y/x) \wedge e$, where $\backslash$ and $/$ denote the residuals of $\mathbf{L}$. Thus in particular in the *negative cone of an ℓ-group* $\mathbf{G}$ the residuals are $x\backslash y = (x^{-1}y) \wedge e$ and $y/x = (yx^{-1}) \wedge e$.

In [BCGJT] it is shown that the class of negative cones of ℓ-groups, the class of cancellative and integral GMV-algebras and the class of cancellative and integral GBL-algebras coincide.

Proposition 2.5. (cf [GT05]). *Any integral GMV-algebra satisfies the equation $x\backslash y \vee y\backslash x = y/x \vee x/y = e$. Thus every integral and bounded GMV-algebra is a psBL-algebra.*

Recall that a *nucleus* on a residuated lattice $\mathbf{R}$ is a unary operation γ satisfying the following conditions:

- $x \leq y$ implies $\gamma(x) \leq \gamma(y)$,
- $x \leq \gamma(x)$,
- $\gamma(\gamma(x)) = \gamma(x)$, and
- $\gamma(xy) = \gamma(\gamma(x)\gamma(y))$.

The first three conditions state the γ is a closure operator, and the last one is called the *nuclear* condition. Nuclei were introduced for Heyting algebras and in pointfree topology to characterize congruences on frames. They also correspond to epimorphic images in the category of residuated lattices with morphisms the monoid homomorphisms that are also residuated maps. Further information can be found in [Ro90] and [MT].

The next proposition shows that any integral GMV-algebra can be represented by means of a negative cone of an ℓ-group and a nucleus.

Proposition 2.6. (cf [GT05]).

(a) *If $\mathbf{G}^-$ is the negative cone of an ℓ-group and γ is a nucleus on $\mathbf{G}^-$, then the image $\gamma(\mathbf{G}^-)$ of $\mathbf{G}^-$ under γ is a GMV-algebra with respect to the operations: $x \vee_\gamma y = \gamma(x \vee y)$, $x \wedge_\gamma y = x \wedge y$, $x \cdot_\gamma y = \gamma(x \cdot y)$, $x\backslash_\gamma y = x\backslash y$ and $x/_\gamma y = x/y$. The monoid unit is $\gamma(e)$. Moreover since $\mathbf{G}^-$ is a GMV-algebra, by [GT05], Theorem 3.4, we have that $\gamma(e) = e$ and γ preserves finite joins.*

(b) ([GT05], Theorem 3.12). *For every integral GMV-algebra $\mathbf{A}$, there are a negative cone $\mathbf{G}^-$ of an ℓ-group and a nucleus γ on $\mathbf{G}^-$, such that $\mathbf{A} = (\gamma(\mathbf{G}^-), \vee_\gamma, \wedge_\gamma, \cdot_\gamma, \backslash_\gamma, /_\gamma, \gamma(e))$, with $\cdot_\gamma, \vee_\gamma, \wedge_\gamma, /_\gamma, \backslash_\gamma$ defined as in (a). Moreover $\gamma(\mathbf{G}^-)$ is a lattice filter of $\mathbf{G}^-$, that is, it is closed upwards and it is closed under $\wedge$. Finally, by [GT05], Theorem 3.11, $\mathbf{G}^-$ is generated by $\gamma(\mathbf{G}^-)$ as a monoid.*

(c) ([GT05], Theorem 5.2). *Every GBL-algebra (hence, every GMV-algebra) is a direct product of an ℓ-group and an integral GBL-algebra (respectively GMV-algebra).*

Proposition 2.6 (c) allows us to concentrate on integral GBL-algebras.

Corollary 2.7. *Any totally ordered GMV-algebra is either an ℓ-group, or a bounded and integral GMV-algebra, or the negative cone of an ℓ-group.*

Proof. By Proposition 2.6 (c), any GMV-algebra $\mathbf{A}$ decomposes as a product of an ℓ-group and an integral GMV-algebra. Thus if $\mathbf{A}$ is totally ordered, it is either an ℓ-group or an integral GMV-algebra. In the latter case, by Proposition 2.6 (b), there are a negative cone $\mathbf{G}^-$ of an ℓ-group $\mathbf{G}$ and a nucleus γ on $\mathbf{G}^-$ such that $\mathbf{A} = \gamma(\mathbf{G}^-)$ and $\mathbf{G}^-$ is generated by $\mathbf{A}$ as a monoid. Moreover, $\gamma(G^-)$ is a lattice filter of $\mathbf{G}^-$.

We claim that $\mathbf{G}^-$ is totally ordered. First note that $\mathbf{G}^-$ is an integral GMV-algebra, therefore by Proposition 2.5 it satisfies $(x\backslash y) \vee (y\backslash x) = e$. Thus in order to prove that $\mathbf{G}^-$ is totally ordered, it suffices to show that e is join irreducible in $\mathbf{G}^-$. Now suppose $x, y \in G^-$ and $x, y < e$. Then, by Proposition 2.6 (b), x and y can be written as products of elements of A, say $x = \prod_{i=1}^n x_i$ and $y = \prod_{j=1}^m y_j$, where at least one x_i and one y_j are less than e. Moreover $x \leq x_i$ and $y \leq y_j$, since $\mathbf{G}^-$ is integral, therefore $x \vee y \leq x_i \vee y_j < e$, because $\mathbf{A}$ is totally ordered.

We continue the proof of Corollary 2.7. If $\gamma(\mathbf{G}^-) = \mathbf{G}^-$, then $\mathbf{A} = \mathbf{G}^-$ is the negative cone of an ℓ-group. Otherwise, there is c such that $c \in G^- \setminus \gamma(G^-)$. Since $\mathbf{G}^-$ is totally ordered, and A is upward closed, c is a lower bound of A. Now for all $x \in G^-$, $\gamma(x)$ is the smallest $y \in \gamma(G^-)$ such that $x \leq y$. Thus $\gamma(c)$ is the minimum of $\mathbf{A}$, and $\mathbf{A}$ is a bounded integral GMV-algebra.

Another connection between GMV-algebras and negative cones of ℓ-groups is the following: let $\mathbf{G}^-$ be the negative cone of an ℓ-group $\mathbf{G}$, let G^- be the domain of $\mathbf{G}^-$ and let $'$ be a bijection between G^- and a set G' disjoint from G^-. Let $\mathbf{GMV}(\mathbf{G}^-)$ denote the following structure:

- The domain of $\mathbf{GMV}(\mathbf{G}^-)$ is $G^- \cup G'$.
- Let $\cdot, \vee, \wedge, \backslash, /$, denote the operations of $\mathbf{G}^-$ and let e denote its neutral element. Then, observing that every element of $G^- \cup G'$ is either in G^- or has the form x' for some (uniquely determined) $x \in G^-$, the operations $\cdot', \vee', \wedge', \backslash', /'$ of $\mathbf{GMV}(\mathbf{G}^-)$ are defined as follows, for all $x, y \in G^-$:

$$x \cdot' y = x \cdot y, \quad x' \cdot' y = (y\backslash x)', \quad x \cdot' y' = (y/x)', \quad x' \cdot' y' = e';$$
$$x \vee' y = x \vee y, \quad x \vee' y' = y' \vee' x = x, \quad x' \vee' y' = (x \wedge y)';$$
$$x \wedge' y = x \wedge y, \quad x \wedge' y' = y' \wedge' x = y', \quad x' \wedge' y' = (x \vee y)';$$
$$x\backslash' y = x\backslash y, \quad x\backslash' y' = (y \cdot x)', \quad y'\backslash' x = e, \quad x'\backslash' y' = x/y;$$
$$y/' x = y/x, \quad y'/' x = (x \cdot y)', \quad x/' y' = e, \quad y'/' x' = y\backslash x.$$

  Finally, e is both the top element and the neutral element of $\mathbf{GMV}(\mathbf{G}^-)$ and e' is its bottom element.

Proposition 2.8. (cf [DDT08]). *If $\mathbf{G}^-$ is the negative cone of an ℓ-group, then $\mathbf{GMV}(\mathbf{G}^-)$ is an integral and bounded GMV-algebra. Moreover $\mathbf{GMV}(\mathbf{G}^-)$ is totally ordered iff $\mathbf{G}^-$ is totally ordered. Finally $\mathbf{G}^-$ is both a subalgebra and a normal filter of $\mathbf{GMV}(\mathbf{G}^-)$.*

2.3. Ordinal sums of integral GBL-algebras

Usually the ordinal sum of two posets $\mathbf{H}_1, \mathbf{H}_2$ is defined as the disjoint union with all elements of H_1 less than all elements of H_2 (and if $\mathbf{H}_1$ has a top and $\mathbf{H}_2$ has a bottom, these two elements are often identified). However, for integral GBL-algebras, in view of the decomposition result from [JM09] (Proposition 2.9 below), we need a slightly different definition that intuitively *replaces* the neutral element e of $\mathbf{H}_1$ by the algebra $\mathbf{H}_2$. The precise definition is as follows:

Let $\mathbf{H}_1$ and $\mathbf{H}_2$ be two integral GBL-algebras, assume that $\mathbf{H}_1 \cap \mathbf{H}_2 = \{e\}$, and that e is join irreducible in $\mathbf{H}_1$ or that $\mathbf{H}_2$ has a minimum element m. Then the *ordinal sum* $\mathbf{H}_1 \oplus \mathbf{H}_2$ of $\mathbf{H}_1$ and $\mathbf{H}_2$ has domain $H_1 \cup H_2$, and the operations in $\mathbf{H}_1 \oplus \mathbf{H}_2$ are given by

- if $x, y \in H_i$ then $x \diamond y = x \diamond_i y$ for i=1, 2 and $\diamond \in \{\cdot, \backslash, /, \wedge\}$ or i=2, $\diamond = \vee$
- if $x, y \in H_1 \setminus \{e\}$ then $x \vee y = x \vee_1 y$ if $x \vee_1 y < e$, and $x \vee y = m$ if $x \vee_1 y = e$
- if $x \in H_1 \setminus \{e\}$, $y \in H_2$ then $x \backslash y = e = y/x$, $x \cdot y = x \wedge y = x$, $x \vee y = y$
- if $y \in H_1 \setminus \{e\}$, $x \in H_2$ then $x \backslash y = y = y/x$, $x \cdot y = x \wedge y = y$, $x \vee y = x$.

It is readily seen that if $\mathbf{H}_1$ and $\mathbf{H}_2$ are integral GBL-algebras then so is $\mathbf{H}_1 \oplus \mathbf{H}_2$ (verification is left to the reader).

Note that if e is join-reducible in $\mathbf{H}_1$ and $\mathbf{H}_2$ has no minimum, then the ordinal sum of $\mathbf{H}_1$ and $\mathbf{H}_2$ cannot be defined as above. In this case, an "extended" ordinal sum may be obtained by taking the ordinal sum of $(\mathbf{H}_1 \oplus \mathbf{W}_1) \oplus \mathbf{H}_2$, where $\mathbf{W}_1$ is the MV-algebra with two elements (i.e. the 2-element Boolean algebra). The ordinal sum $\mathbf{H}_1 \oplus \mathbf{W}_1$ exists since $\mathbf{W}_1$ has a minimum element, and $(\mathbf{H}_1 \oplus \mathbf{W}_1) \oplus \mathbf{H}_2$ exists since e is join irreducible in $\mathbf{W}_1$.

A *filter* of a residuated lattice $\mathbf{A}$ is an upward closed subset F of $\mathbf{A}$ which is closed under the monoid operation and the meet operation, and which contains e. A filter F is said to be *normal* if whenever $x \in F$ and $y \in \mathbf{A}$, then $y\backslash(xy) \in F$ and $(yx)/y \in F$. A normal filter F is said to be a *value* if there exists $a \in \mathbf{A}$ such that F is maximal among all normal filters not containing a. Note that values are precisely the completely meet irreducible elements in the lattice of normal filters.

A residuated lattice is said to be *normal* if every filter of it is a normal filter. As an easy consequence of [GOR08] (Cor. 10), we have that a residuated lattice is normal iff for all x, y there is a natural number n such that $x(y \wedge e)^n \leq yx$ and $(y \wedge e)^n x \leq xy$. A residuated lattice is said to be *n-potent* if it satisfies $x^{n+1} = x^n$, where $x^n = x \cdot \ldots \cdot x$ (n times). Note that n-potent GBL-algebras are normal ([JM09] 3.6).

In every residuated lattice, the lattice of normal filters is isomorphic to the congruence lattice: to any congruence θ one associates the normal filter $F_\theta = \uparrow\{x : (x,e) \in \theta\}$. Conversely, given a normal filter F, the set θ_F of all pairs (x,y) such that $x\backslash y \in F$ and $y\backslash x \in F$ is a congruence such that the upward closure of the congruence class of e is F. In particular, the variety of residuated lattices is congruence regular at e.

Notation. Given a normal filter F of an integral residuated lattice $\mathbf{A}$, $\mathbf{A}/F$ denotes the quotient of $\mathbf{A}$ modulo the congruence θ_F determined by F and for every $a \in A$, a/F denotes the equivalence class of a modulo θ_F. Moreover for all $G \subseteq A$, G/F denotes the set $\{a/F : a \in G\}$. This notation, as well as the use of $\setminus$ to denote set-theoretic difference, conflicts with the notation used for residuals. However, we believe that this should not create confusion, as elements of a residuated lattices are usually denoted by lowercase letters and sets, filters, etc. are usually denoted by capital letters.

In [JM09] the following result is proved.

Proposition 2.9. *(i) Every subdirectly irreducible integral and normal GBL-algebra is the ordinal sum of a proper subalgebra of it and of a non-trivial integral subdirectly irreducible GMV-algebra.*

(ii) Every n-potent GBL-algebra is commutative and integral.

Note that the above result would not hold if the ordinal sum of $\mathbf{H}_1 \oplus \mathbf{H}_2$ were defined such that all elements of H_1 are below all elements of H_2. For example the unit interval $\mathbf{I} = [0,1]$ with ordinary multiplication and order is a subdirectly irreducible normal (G)BL-algebra that decomposes as $\mathbf{W}_1 \oplus (0,1]$ according to our definition of $\oplus$, but would not decompose with a subalgebra as bottom summand otherwise (since stacking $(0,1]$ on top of $\mathbf{W}_1$ is not isomorphic to $\mathbf{I}$).

Ordinal sums can be generalized in an obvious way to the case of infinitely many summands. In this case we consider a totally ordered set I of indices, and for all $i \in I$ we consider an integral GBL-algebra $\mathbf{H}_i$ such that for $i \neq j$, $\mathbf{H}_i \cap \mathbf{H}_j = \{e\}$ and for all i, e is join irreducible in $\mathbf{H}_i$. Then the ordinal sum $\bigoplus_{i\in I} \mathbf{H}_i$ is defined as follows:

- The universe of $\bigoplus_{i\in I} \mathbf{H}_i$ is $\bigcup_{i\in I} H_i$, and the monoid operation is defined by

$$x \cdot y = \begin{cases} x \cdot_i y & \text{if} \quad x, y \in H_i \ (i \in I) \\ x & \text{if} \quad x \in H_i \setminus \{e\}, y \in H_j \text{ with } i < j \\ y & \text{if} \quad y \in H_i \setminus \{e\}, x \in H_j \text{ with } i < j \end{cases}$$

- The partial order on $\bigoplus_{i\in I} \mathbf{H}_i$ is the unique partial order $\leq$ such that e is the top element with respect to $\leq$, the partial order $\leq_i$ on $\mathbf{H}_i$ is the restriction of $\leq$ to $\mathbf{H}_i$, and if $i < j$, then every element of $\mathbf{H}_i \setminus \{e\}$ precedes every element of $\mathbf{H}_j$.

- The lattice operations and the residuals are uniquely determined by $\leq$ and by $\cdot$.

The following representation theorem is proved in [AM03].

Proposition 2.10. *Every totally ordered integral commutative GBL-algebra* $\mathbf{H}$ *can be represented as an ordinal sum* $\bigoplus_{i \in I} \mathbf{H}_i$ *of commutative, integral and totally ordered GMV-algebras. Moreover* $\mathbf{H}$ *is a BL-algebra iff* I *has a minimum* i_0 *and* $\mathbf{H}_{i_0}$ *is bounded.*

Recently Dvurečenskij has shown that Proposition 2.10 extends to the non-commutative case.

Proposition 2.11. (cf [Dv07]). *Every totally ordered integral GBL-algebra* $\mathbf{H}$ *can be represented as an ordinal sum* $\bigoplus_{i \in I} \mathbf{H}_i$ *of an indexed family of integral and totally ordered GMV-algebras. Moreover* $\mathbf{H}$ *is a psBL-algebra iff* I *has a minimum* i_0 *and* $\mathbf{H}_{i_0}$ *is bounded.*

3. Poset products and a general condition for poset product embeddability

In the sequel, given a poset $\mathbf{P} = (P, \leq)$, its *dual*, denoted by $\mathbf{P}^d$, is defined as the poset $(P, \geq)$. The next definition is taken from [JM09], but we adjust the terminology to match [Ji09] and use the dual order on the index set.

Definition 3.1. Let $\mathbf{P} = (P, \leq)$ be a poset and let $(\mathbf{A}_p : p \in P)$ be a collection of residuated lattices. Up to isomorphism we can (and we will) assume that all $\mathbf{A}_p$ share the same neutral element e and that all $\mathbf{A}_p$ which are bounded share the same minimum element 0. Suppose that if p is not minimal, then $\mathbf{A}_p$ is integral and if p is not maximal then $\mathbf{A}_p$ is bounded. The *poset product* $\bigotimes_{p \in \mathbf{P}} \mathbf{A}_p$ is the algebra defined as follows.

- The domain of $\bigotimes_{p \in P} \mathbf{A}_p$ is the set of all maps h on P such that for all $p \in P$,

  (a) $h(p) \in \mathbf{A}_p$ and

  (b) if $h(p) \neq e$, then for all $q < p$, $h(q) = 0$.

- The monoid operation and the lattice operations are defined pointwise.

- The residuals are defined by

  $$(h \backslash g)(p) = \begin{cases} h(p) \backslash_p g(p) & \text{if for all } q > p \quad h(q) \leq_p g(q) \\ 0 & \text{otherwise} \end{cases}$$

  $$(g / h)(p) = \begin{cases} g(p) /_p h(p) & \text{if for all } q > p \quad h(q) \leq_p g(q) \\ 0 & \text{otherwise} \end{cases}$$

  where $\backslash_p, /_p, \leq_p$ denote the residuals and order in $\mathbf{A}_p$.

Note that the function on P that is constantly e is always an element of the poset product. Sometimes it is convenient, as in [JM09], to consider the *dual poset product*, that is, the poset product $\bigotimes_{p \in \mathbf{P}^d} \mathbf{A}_p$ of the same algebras but with respect to the dual poset $\mathbf{P}^d$. Note that in the dual poset product condition (b) must be replaced by the following condition.

(b') if $h(p) \neq e$, then for all $q > p$, $h(q) = 0$.

Moreover the definition of residuals becomes

$$(h \backslash g)(p) = \begin{cases} h(p) \backslash_p g(p) & \text{if for all } q < p \quad h(q) \leq_p g(q) \\ 0 & \text{otherwise} \end{cases}$$
$$(g/h)(p) = \begin{cases} g(p)/_p h(p) & \text{if for all } q < p \quad h(q) \leq_p g(q) \\ 0 & \text{otherwise} \end{cases} .$$

In the sequel, we will often omit subscripts when there is no danger of confusion. We first give several examples to illustrate the general applicability of poset products.

Examples. (1). Suppose that $\leq$ is just equality on P. Then, every element of P is both maximal and minimal. Therefore, the poset product $\bigotimes_{p \in \mathbf{P}} \mathbf{A}_p$ is defined for any family $(\mathbf{A}_p : p \in P)$ of residuated lattices. Moreover, every element of $\prod_{p \in P} A_p$ is in $\bigotimes_{p \in \mathbf{P}} \mathbf{A}_p$, and all operations (including residuals) are pointwise. Hence, $\bigotimes_{p \in \mathbf{P}} \mathbf{A}_p$ is simply the direct product $\prod_{p \in P} \mathbf{A}_p$.

(2) Suppose that $(P, \leq)$ is totally ordered and finite, say, $P = \{p_1, \ldots, p_n\}$ with $p_1 > p_2 > \cdots > p_n$. Suppose further that $\mathbf{A}_{p_1}, \ldots, \mathbf{A}_{p_n}$ are integral and bounded residuated lattices. Then, $\bigotimes_{p \in \mathbf{P}} \mathbf{A}_p$ is isomorphic to the ordinal sum $\mathbf{A}_{p_1} \oplus \cdots \oplus \mathbf{A}_{p_n}$ (warning: the minimum elements of all $\mathbf{A}_{p_i}$ are identified in $\bigotimes_{p \in \mathbf{P}} \mathbf{A}_p$, but not in $\mathbf{A}_{p_1} \oplus \cdots \oplus \mathbf{A}_{p_n}$, whilst the neutral elements are identified in both constructions). An isomorphism Φ is defined as follows: for every $h \in \bigotimes_{p \in P} \mathbf{A}_p$, if h is not constantly equal to e, then let $p^h \in P$ be maximal such that $h(p^h) < e$ (if $p^h = p_1$, then $h(p) < e$ for all $p \in P$). Now if h is constantly equal to e, let $\Phi(h) = e$. Otherwise, let

$$\Phi(h) = \begin{cases} h(p^h) & \text{if} \quad h(p^h) > 0 \\ 0_i & \text{if} \quad h(p^h) = 0 \end{cases} ,$$

where 0_i is the minimum of $\mathbf{A}_{p_i}$. It is easy to check that Φ is an isomorphism.

(3) If $(P, \leq)$ is totally ordered but P is infinite, then it is still true that the ordinal sum of all $\mathbf{A}_p$ embeds into their dual poset product under the embedding Ψ defined by

$$\Psi(x)(p) = \begin{cases} x_p & \text{if} \quad x \in A_p \\ e & \text{if} \quad \exists q > p \ (x \in A_q \setminus \{e\}) \\ 0 & \text{if} \quad \exists q < p \ (x \in A_q \setminus \{e\}) \end{cases}$$

where $x_p = x$ if $x \neq \min(A_p)$ and $x_p = 0$ if $x = \min(A_p)$.

However, the ordinal sum of all $\mathbf{A}_p$ and their dual poset product are not isomorphic in general: consider e.g. the poset $\mathbf{Q}$ of rational numbers with the usual order $\leq$, and let all algebras $\mathbf{A}_p$ be boolean algebras (considered as residuated lattices) with two elements. Then the ordinal sum of all $\mathbf{A}_p$ is countable, while their dual poset product is not: for every downward closed subset X of Q, the function h_X defined by

$$h_X(p) = \begin{cases} e & \text{if } p \in X \\ 0 & \text{otherwise} \end{cases}$$

is in $\bigotimes_{p \in \mathbf{Q}^d} \mathbf{A}_p$, and hence $\bigotimes_{p \in \mathbf{Q}^d} \mathbf{A}_p$ has the same cardinality as the reals.

(4). Let $\mathbf{X} = (X, \leq)$ be a poset, and let $P_\uparrow(X)$ be the set of upward closed subsets of X. Then $P_\uparrow(X)$ becomes a Heyting algebra with respect to the constants $\emptyset$ (bottom) and X (top) and with respect to the operations $\cup$, $\cap$ and $\Rightarrow$, where for all $Y, Z \in P_\uparrow(X)$, $Y \Rightarrow Z = \{x : \forall y \geq x (\text{if } y \in Y, \text{ then } y \in Z)\}$. We denote such Heyting algebra by $\mathbf{P}_\uparrow(X)$. For the readers who are familiar with Kripke frames for intuitionistic logic, the poset $\mathbf{X}$ is the Kripke frame associated with the algebra $\mathbf{P}_\uparrow(X)$. Note that every Heyting algebra embeds into one of the form $\mathbf{P}_\uparrow(X)$.

For every $x \in X$, let $\mathbf{A}_x$ denote the two element boolean algebra (considered as a residuated lattice). Then the poset product $\bigotimes_{x \in \mathbf{X}} \mathbf{A}_x$ is isomorphic to $\mathbf{P}_\uparrow(X)$ under the isomorphism Φ defined, for all $Y \in P_\uparrow(X)$ and for all $x \in X$, by

$$\Phi(Y)(x) = \begin{cases} e & \text{if } x \in Y \\ 0 & \text{otherwise.} \end{cases}$$

In [JM09] the following is shown:

Proposition 3.2. *(a) The poset product of a collection of residuated lattices is a residuated lattice, which is integral (divisible, bounded respectively) when all factors are integral (divisible, bounded respectively).*

(b) Every finite GBL-algebra can be represented as the poset product of a finite family of finite MV-chains.

Our aim is to extend Proposition 3.2 (b) to larger classes of GBL-algebras. As we could not obtain a general representation theorem, we will present some embedding theorems. To begin with, in this section we give a sufficient condition for poset product embeddability. Recall that by Corollary 2.7, a totally ordered integral GMV-algebra $\mathbf{A}$ is either bounded or the negative cone of an ℓ-group. In the first case we set $\mathbf{A}^* = \mathbf{A}$ and in the second case we set $\mathbf{A}^* = \mathbf{GMV}(\mathbf{A})$. Note that in either case $\mathbf{A}^*$ is a totally ordered and bounded GMV-algebra and that $\mathbf{A}$ is a subalgebra of $\mathbf{A}^*$, cf Proposition 2.8.

Theorem 3.3. *Let $\mathbf{A}$ be an integral GBL-algebra, let Δ be a collection of normal filters of $\mathbf{A}$, let $\preceq$ be a partial order on Δ, and let $\mathbf{\Delta} = (\Delta, \preceq)$. Suppose that the following conditions are satisfied.*

(a) For every $F \in \Delta$, $\mathbf{A}/F$ decomposes as an ordinal sum $\mathbf{B}_F \oplus \mathbf{W}_F$, where $\mathbf{B}_F$ is an integral GBL-algebra and $\mathbf{W}_F$ is a totally ordered and integral GMV-algebra.

(b) For every $F, G \in \Delta$, if $F \prec G$, then $\{a : a/F \in \mathbf{W}_F\} \subseteq G$.

(c) For every $F \in \Delta$ and for every $a \notin F$ there exists $G \in \Delta$ such that $F \preceq G$ and $a/G \in \mathbf{W}_G \setminus \{e\}$.

(d) $\bigcap \Delta = \{e\}$.

Then $\mathbf{A}$ *embeds into the poset product* $\mathbf{A}^{\mathbf{\Delta}} = \bigotimes_{F\in\mathbf{\Delta}} \mathbf{W}_F^*$.

Proof. First of all, note that if conditions (a), (b), (c) and (d) hold, then $F \preceq G$ implies $F \subseteq G$. The claim is clear if $F = G$. If $F \prec G$, then by (b), $\{a : a/F \in \mathbf{W}_F\} \subseteq G$. But if $a \in F$, then $a/F = e \in \mathbf{W}_F$. Thus $a \in F$ implies $a \in G$ and the claim follows.

Now for every $a \in \mathbf{A}$, let h_a be the function on Δ defined by

$$h_a(F) = \begin{cases} a/F & \text{if} \quad a/F \in \mathbf{W}_F \\ 0 & \quad\text{otherwise} \end{cases} .$$

We claim that the map $\Phi : a \mapsto h_a$ is an embedding of $\mathbf{A}$ into $\mathbf{A}^{\mathbf{\Delta}}$. We start from the following observation. For $F \in \Delta$ and for $h, k \in A^{\Delta}$, let $h \preceq^{\uparrow F} k$ iff $h(G) \leq k(G)$ for all $G \succ F$.

Lemma 3.4. *For all* $a, b \in \mathbf{A}$ *and for all* $F \in \Delta$ *we have*

(i) $a/F \in \mathbf{W}_F$ *iff for all* $G \in \Delta$ *with* $F \prec G$, $h_a(G) = e$.
(ii) $h_a \preceq^{\uparrow F} h_b$ *iff* $(a\backslash b)/F \in \mathbf{W}_F$ *(iff* $(b/a)/F \in \mathbf{W}_F$*).*

Proof. (i) If $a/F \in \mathbf{W}_F$, then by (b), $F \prec G$ implies $a \in G$, therefore $h_a(G) = a/G = e$. Conversely, if $a/F \notin \mathbf{W}_F$, then $a \notin F$, and by (c) there exists $F \preceq G$ such that $a/G \in \mathbf{W}_G \setminus \{e\}$. Clearly, $G \neq F$, as $a/F \notin \mathbf{W}_F$ and $a/G \in \mathbf{W}_G$. Thus $F \prec G$ and $h_a(G) = a/G < e$.

(ii) If $(a\backslash b)/F \in \mathbf{W}_F$, then by (b) we have $a\backslash b \in G$ for every $G \succ F$. Thus for all $G \succ F$, since G is a normal filter, we have $a/G \leq b/G$ and hence $h_a(G) \leq h_b(G)$. Conversely, suppose that $(a\backslash b)/F \notin \mathbf{W}_F$. Then by the argument used in the proof of (i) we see that there exists $G \succ F$ such that $(a\backslash b)/G \in \mathbf{W}_G \setminus \{e\}$. Since G is a normal filter, $(a\backslash b)/G = (a/G)\backslash(b/G)$, so by the definition of ordinal sum, we must have $b/G \in \mathbf{W}_G \setminus \{e\}$, $a/G \in \mathbf{W}_G$ and $a/G \not\leq b/G$. Hence $h_a(G) \not\leq h_b(G)$. This concludes the proof of Lemma 3.4.

Continuing with the proof of Theorem 3.3, we verify the following facts.

(1) For $a \in \mathbf{A}$, $\Phi(a) = h_a \in A^{\Delta}$. Indeed, for $F \in \Delta$, $h_a(F)$ is either an element of $\mathbf{W}_F$ or 0, therefore $h_a(F) \in \mathbf{W}_F^*$. Moreover, if $h_a(F) > 0$, then $a/F \in \mathbf{W}_F$, and by Lemma 3.4 (i) $h_a(G) = e$ for all $G \succ F$. Thus if $h_a(G) < e$, then $h_a(F) = 0$ for all $F \prec G$.

(2) Φ is one-one. Indeed, suppose $a \neq b$. Without loss of generality, we may assume $a\backslash b < e$. Since $\bigcap \Delta = \{e\}$, there exists $G \in \Delta$ such that $a\backslash b \notin G$. Thus by (c) there exists $H \succeq G$ such that $(a\backslash b)/H \in \mathbf{W}_H \setminus \{e\}$, therefore $h_a(H) \not\leq h_b(H)$ and $\Phi(a) \neq \Phi(b)$.

(3) Φ preserves $\vee$, $\wedge$ and $\cdot$. Let us verify first that Φ preserves $\vee$. Let $a, b \in \mathbf{A}$ and let $F \in \Delta$. If $(a \vee b)/F \in \mathbf{W}_F \setminus \{0\}$, then either $a/F \in \mathbf{W}_F \setminus \{0\}$ or $b/F \in \mathbf{W}_F \setminus \{0\}$, and recalling that every element of $\mathbf{W}_F$ is an upper bound of $(\mathbf{A}/F)\backslash \mathbf{W}_F$, we get

$$\Phi(a \vee b)(F) = h_{a\vee b}(F) = (a \vee b)/F = a/F \vee b/F = \Phi(a)(F) \vee \Phi(b)(F).$$

If either $(a \vee b)/F \notin \mathbf{W}_F$ or $(a \vee b)/F = 0$, then $\Phi(a \vee b)(F) = \Phi(a)(F) = \Phi(b)(F) = (\Phi(a) \vee \Phi(b))(F) = 0$.

We verify that Φ preserves $\cdot$. If $a/F, b/F \in \mathbf{W}_F$, then $(a \cdot b)/F \in \mathbf{W}_F$, and $\Phi(a \cdot b)(F) = (a \cdot b)/F = a/F \cdot b/F = (\Phi(a) \cdot \Phi(b))(F)$. Otherwise, if e.g. $a/F \notin \mathbf{W}_F$, then $(a \cdot b)/F \notin \mathbf{W}_F$ and $\Phi(a \cdot b)(F) = \Phi(a)(F) = (\Phi(a) \cdot \Phi(b))(F) = 0$.

The proof for $\wedge$ is similar.

(4) Φ preserves $\backslash$ and $/$. We prove the claim for $\backslash$, the proof for $/$ being quite similar. Suppose first that $a \backslash b \in F$. Then $\Phi(a \backslash b) = e$, $a/F \leq b/F$ and $\Phi(a)(F) \leq \Phi(b)(F)$. Moreover, if $F \preceq G$, then by the observation made at the beginning of the proof, $F \subseteq G$, therefore $a \backslash b \in G$ and $a/G \leq b/G$. Thus for all $G \succ F$, $\Phi(a)(G) \leq \Phi(b)(G)$, and by the definition of $\backslash$ in a poset product, $(\Phi(a) \backslash \Phi(b))(F) = \Phi(a)(F) \backslash \Phi(b)(F) = e$.
Next assume $a \backslash b \notin F$ and $(a \backslash b)/F \in \mathbf{W}_F$. Then $\Phi(a \backslash b)(F) = (a \backslash b)/F$. Moreover $(a \backslash b)/F \in \mathbf{W}_F \setminus \{e\}$, therefore by the argument used in the proof of Lemma 3.4, (ii), $a/F, b/F \in \mathbf{W}_F$. Also, by Lemma 3.4, (ii), we have that for all $G \succ F$, $\Phi(a)(G) \leq \Phi(b)(G)$, therefore

$$(\Phi(a) \backslash \Phi(b))(F) = \Phi(a)(F) \backslash \Phi(b)(F) = (a/F) \backslash (b/F) = \Phi(a \backslash b)(F).$$

Finally, if $(a \backslash b)/F \notin \mathbf{W}_F$, then $\Phi(a \backslash b)(F) = 0$. On the other hand, by Lemma 3.4, (ii), there is $G \succ F$ such that $\Phi(a)(G) \not\leq \Phi(b)(G)$, therefore by the definition of $\backslash$ in a poset product, $(\Phi(a) \backslash \Phi(b))(F) = 0$. This ends the proof.

4. A poset product embedding theorem for n-potent GBL-algebras

In [DL03], Di Nola and Lettieri prove a representation theorem for finite BL-algebras. These algebras can be presented as finite trees whose nodes are labeled by finite MV-algebras. This result is extended to finite GBL-algebras in [JM09] (in this case one has to take posets instead of trees). In the current section we partially extend the result to n-potent GBL-algebras. In fact we will prove the following embedding theorem:

Theorem 4.1. *Every n-potent GBL-algebra embeds into the poset product of a family of finite and n-potent MV-chains.*

Proof. Let $\Delta(\mathbf{A})$ be the set of all values of $\mathbf{A}$, and let $\mathbf{\Delta}(\mathbf{A})$ denote the poset $(\Delta(\mathbf{A}), \subseteq)$, that is, $\Delta(\mathbf{A})$ ordered with respect to set-theoretic inclusion. Then for $F \in \Delta(\mathbf{A})$, $\mathbf{A}/F$ is subdirectly irreducible, because if $b \in \mathbf{A}$ is such that F is maximal among the filters not containing b, then the minimum non-trivial filter of $\mathbf{A}/F$ is the filter generated by b/F. By Proposition 2.9, $\mathbf{A}/F$ decomposes as an ordinal sum $\mathbf{A}/F = \mathbf{B}_F \oplus \mathbf{W}_F$, where $\mathbf{B}_F$ is a proper subalgebra of $\mathbf{A}/F$ and $\mathbf{W}_F$ is a non-trivial subdirectly irreducible n-potent GMV-algebra. Now by [JM09], Lemma 18, $\mathbf{W}_F$, being n-potent and subdirectly irreducible, is (the reduct of) an n-potent MV-chain with $\leq n+1$ elements. In particular, $\mathbf{W}_F$ is bounded and $\mathbf{W}_F^* = \mathbf{W}_F$. Now consider the poset product $\mathbf{A}^{\mathbf{\Delta}(\mathbf{A})} = \bigotimes_{F \in \mathbf{\Delta}(\mathbf{A})} \mathbf{W}_F$. Then by Proposition 2.9, $\mathbf{A}^{\mathbf{\Delta}(\mathbf{A})}$ is a commutative and integral

GBL-algebra. Moreover since product is defined pointwise in a poset product, it is readily seen that $\mathbf{A}^{\mathbf{\Delta}(\mathbf{A})}$ is n-potent. We claim that $\mathbf{A}$ embeds into $\mathbf{A}^{\mathbf{\Delta}(\mathbf{A})}$. To this end, it suffices to verify that $\mathbf{\Delta}(\mathbf{A})$ and the indexed family $(\mathbf{B}_F, \mathbf{W}_F : F \in \Delta(\mathbf{A}))$ satisfy the assumptions (a), (b), (c) and (d) of Theorem 3.3.

(a) Clear.

(b) For every $F \in \Delta(\mathbf{A})$, $\mathbf{W}_F$ is simple and is a filter of $\mathbf{A}/F$. Hence it is the minimum filter of $\mathbf{A}/F$. Thus if $F \subset G$, then $G/F \supseteq \mathbf{W}_F$, therefore G contains all elements a such that $a/F \in \mathbf{W}_F$.

(c) Let $F \in \Delta(\mathbf{A})$ and $a \notin F$. Let G be a filter which is maximal with respect to the properties $G \supseteq F$ and $a \notin G$ (such a filter exists by Zorn's Lemma). Then $G \in \Delta(\mathbf{A})$ and $G \supseteq F$. Moreover since $\mathbf{W}_G$ is the minimum filter of $\mathbf{A}/G$ and a/G belongs to this filter, $a/G \in \mathbf{W}_G \setminus \{e\}$.

(d) By Zorn's Lemma, for every $a < e$, there is a filter F which is maximal with respect to the property that $a \notin F$. Then $F \in \Delta(\mathbf{A})$ and $a \notin F$, therefore $a \notin \bigcap \Delta(\mathbf{A})$, and the claim is proved.

This ends the proof.

Remark. Theorem 4.1 is an embedding theorem but not a representation theorem, in the sense that not all n-potent GBL-algebras are isomorphic to a poset product of n-potent MV-algebras. Indeed, any such poset product has a minimum (the constantly zero function), whereas not all n-potent GBL-algebras are bounded. More generally, any poset product of bounded residuated lattices is bounded, and this fact imposes a limitation on the class of GBL-algebras which are representable as a poset product.

Remark. Clearly, 1-potent GBL-algebras are commutative, integral and idempotent residuated lattices, that is, subreducts of Heyting algebras that may omit the constant 0, called zer0-free subreducts (note that in 1-potent residuated lattices, product and meet coincide). Thus Theorem 4.1 reduces to an embedding theorem for Heyting algebras and its zero-free subreducts, that is, every Heyting algebra embeds into a poset product of a family of two-elements Boolean algebras.

5. A poset product embedding theorem for integral and normal GBL-algebras and for commutative GBL-algebras

The poset product construction in the previous section does not extend to arbitrary integral and normal GBL-algebras $\mathbf{A}$. Indeed, it is possible that for some $F, G \in \Delta(\mathbf{A})$ with $F \subset G$ and for some $a \in \mathbf{A}$, one has $a/F \in \mathbf{W}_F \setminus \{0, e\}$ and $a/G \in \mathbf{W}_G \setminus \{0, e\}$ (this is the case if $\mathbf{W}_F$ is not simple), therefore $F \subset G$, $h_a(G) < e$ and $h_a(F) > 0$, which is incompatible with the definition of poset product.

In order to overcome this problem, we will still consider the set $\Delta(\mathbf{A})$ of values of $\mathbf{A}$, but with a different partial order. More precisely, we set $F \preceq G$ iff either $F = G$ or $G \supseteq \{a : a/F \in \mathbf{W}_F\}$. Clearly $\preceq$ is a partial order. In the sequel $\mathbf{\Delta}(\mathbf{A})$ will denote the poset $(\Delta(\mathbf{A}) \preceq)$. (This notation does not conflict with the notation used for n-potent GBL-algebras, because if $\mathbf{A}$ is an n-potent GBL-algebra, then the relations $\preceq$ and $\subseteq$ on $\Delta(\mathbf{A})$ coincide).

Another difference with the n-potent case is that in general if F is a value and we decompose $\mathbf{A}/F$ as an ordinal sum $\mathbf{A}/F = \mathbf{B}_F \oplus \mathbf{W}_F$, it is possible that $\mathbf{W}_F$ is unbounded and therefore $\mathbf{W}_F^* \neq \mathbf{W}_F$. But with the adjustment to $\boldsymbol{\Delta}(\mathbf{A})$ introduced above, it is still possible to get a poset product embedding theorem. We begin with the following result.

Lemma 5.1. *Every subdirectly irreducible and normal GMV-algebra is totally ordered.*

Proof. Let $\mathbf{C}$ be a subdirectly irreducible normal GMV-algebra. We first prove that e is join irreducible in $\mathbf{C}$. Indeed, suppose by contradiction that $a, b < e$ and $a \vee b = e$. Let $c < e$ be a generator of the minimum non-trivial filter F of $\mathbf{C}$. Then c belongs both to the filter generated by a and to the filter generated by b (note that such filters are normal, because $\mathbf{C}$ is normal). Then for some n, $a^n \leq c$ and $b^n \leq c$. Now $(a \vee b)^{2n} \leq a^n \vee b^n$, because $\cdot$ distributes over $\vee$, and therefore $(a \vee b)^{2n}$ is a join of products of $2n$ factors of which, for some $i \leq 2n$, i factors are equal to a and $2n - i$ are equal to b. Since either $i \geq n$ or $2n - i \geq n$, we have that each factor is bounded above by $a^n \vee b^n$. Then we deduce $e = (a \vee b)^{2n} \leq a^n \vee b^n \leq c < e$, which is a contradiction. Thus e is join irreducible.

By Proposition 2.6 (c) $\mathbf{C}$ is either an integral GMV-algebra or an ℓ-group. In the first case, by Proposition 2.5, $\mathbf{C}$ satisfies the identity $x\backslash y \vee y\backslash x \geq e$, and in the second case the same identity holds since ℓ-groups satisfy $y\backslash x = (x\backslash y)^{-1}$ and $z \vee z^{-1} \geq e$ (cf [Gl99]). Since e is join irreducible and $e = (x\backslash y \vee y\backslash x) \wedge e = (x\backslash y \wedge e) \vee (y\backslash x \wedge e)$, we conclude that either $e \leq x\backslash y$ or $e \leq y\backslash x$. Therefore $x \leq y$ or $y \leq x$, and hence $\mathbf{C}$ is totally ordered.

Note that there are totally ordered GMV-algebras that are not normal, see for example Clifford's o-group (cf [Da95], p. 57).

Lemma 5.2. *For any variety $\mathcal{V}$ of residuated lattices, the class of normal members of $\mathcal{V}$ is closed under quotients, subalgebras and finite products.*

Proof. By Corollary 10 of [GOR08], a residuated (semi)lattice is normal iff for all x, y there exists n such that $x(y \wedge e)^n \leq yx$ and $(y \wedge e)^n x \leq xy$. This property is clearly preserved under taking quotients, subalgebras and finite products.

An ℓ-group example showing that normality is not preserved under arbitrary products can be found in [Da95] (p. 325; note that the property of normality for ℓ-groups is called *Hamiltonian*).

Theorem 5.3. *(i) Let $\mathbf{A}$ be any integral normal GBL-algebra, and let $\boldsymbol{\Delta}(\mathbf{A})$ and $\mathbf{W}_F$ ($F \in \Delta(\mathbf{A})$) have the usual meaning. Then $\mathbf{W}_F$ is totally ordered and $\mathbf{A}$ embeds into the poset product $\mathbf{A}^{\boldsymbol{\Delta}(\mathbf{A})} = \bigotimes_{F \in \boldsymbol{\Delta}(\mathbf{A})} \mathbf{W}_F^*$. Thus every integral normal GBL-algebra embeds into a poset product of totally ordered, integral and bounded GMV-algebras.*

(ii) Every normal GBL-algebra embeds into a poset product of totally ordered integral and bounded GMV-algebras and totally ordered ℓ-groups.

Proof. (i) By Proposition 2.9 (i) for all $F \in \Delta(\mathbf{A})$, $\mathbf{A}/F$ decomposes as $\mathbf{A}/F = \mathbf{B}_F \oplus \mathbf{W}_F$, where $\mathbf{B}_F$ is an integral GBL-algebra and $\mathbf{W}_F$ is a non-trivial subdirectly irreducible integral GMV-algebra. Note that $\mathbf{W}_F$ is normal, because a filter of $\mathbf{W}_F$ is also a filter of $\mathbf{A}/F$, which is normal, since $\mathbf{A}$ is normal and normality is preserved under quotients. Thus by Lemma 5.1, $\mathbf{W}_F$ is totally ordered. By Corollary 2.7, it is either a totally ordered integral and bounded GMV-algebra or the negative cone of a totally ordered ℓ-group. In both cases, $\mathbf{W}_F^*$ is a totally ordered, integral and bounded GMV-algebra. Thus in order to derive the claim it suffices to prove that the poset $\mathbf{\Delta}(\mathbf{A})$ and the indexed family $(\mathbf{B}_F, \mathbf{W}_F : F \in \Delta(\mathbf{A}))$ defined above satisfy conditions (a), (b), (c) and (d) of Theorem 3.3.

(a) Clear.

(b) This follows from the definition of $\preceq$.

(c) Let $F \in \Delta(\mathbf{A})$ and $a \notin F$. If $a/F \in \mathbf{W}_F$, then $a/F \in \mathbf{W}_F \setminus \{e\}$ and we are done. Otherwise, let $G_0 = \{x \in \mathbf{A} : x/F \in \mathbf{W}_F\}$. Then G_0 is a normal filter, $F \subset G_0$ and $a \notin G_0$. By Zorn's Lemma there is a normal filter G which is maximal with respect to the property that $G_0 \subseteq G$ and $a \notin G$. Then $G \in \Delta(\mathbf{A})$, $a/G \neq e$ and $F \preceq G$. Moreover a/G is in the minimum normal filter of $\mathbf{A}/G$. Since $\mathbf{W}_G$ is a normal filter of $\mathbf{A}/G$, $a/G \in \mathbf{W}_G \setminus \{e\}$.

(d) Clear.

This ends the proof of (i).

(ii) By Proposition 2.6, any normal GBL-algebra $\mathbf{A}$ decomposes as a product of a normal integral GBL-algebra $\mathbf{B}$ and an ℓ-group $\mathbf{G}$. Now by (i), $\mathbf{B}$ embeds into an algebra of the form $\bigotimes_{F \in \mathbf{\Delta}(\mathbf{B})} \mathbf{W}_F^*$, where $\mathbf{\Delta}(\mathbf{B})$ is the set of values of $\mathbf{B}$ partially ordered by the relation $\preceq$ defined just before Theorem 5.3 and each $\mathbf{W}_F^*$ is a totally ordered integral and bounded GMV-algebra. Moreover $\mathbf{G}$, being a quotient of $\mathbf{A}$, is normal, as normality is preserved under quotients. By Lemma 5.1, $\mathbf{G}$ has a subdirect embedding into an algebra of the form $\prod_{i \in I} \mathbf{G}_i$, where each $\mathbf{G}_i$ is a totally ordered ℓ-group. Without loss of generality we may assume that $I \cap \Delta(\mathbf{B}) = \emptyset$. Now consider the poset $\mathbf{P} = (\Delta(\mathbf{B}) \cup I, \sqsubseteq)$, where $\sqsubseteq$ is defined by $x \sqsubseteq y$ iff either $x = y$ or $x, y \in \Delta(\mathbf{B})$ and $x \preceq y$. Thus every element of I is comparable only with itself. Now let for $x \in \Delta(\mathbf{B}) \cup I$, $\mathbf{A}_x = \mathbf{W}_x^*$ if $x \in \Delta(\mathbf{B})$ and $\mathbf{A}_x = \mathbf{G}_x$ if $x \in I$. Then it is readily seen that $\bigotimes_{x \in \mathbf{P}} \mathbf{A}_x = (\bigotimes_{F \in \mathbf{\Delta}(\mathbf{B})} \mathbf{W}_F) \times (\prod_{i \in I} \mathbf{G}_i)$, therefore $\mathbf{A}$ embeds into $\bigotimes_{x \in \mathbf{P}} \mathbf{A}_x$. This ends the proof.

Corollary 5.4. *(i) Every commutative and integral GBL-algebra embeds into a poset product of an indexed family of totally ordered MV-algebras.*

(ii) Every commutative GBL-algebra embeds into a poset product of an indexed family of totally ordered MV-algebras and totally ordered abelian ℓ-groups.

6. Poset product embedding theorems for classes of GBL-algebras

It is clear that a normal GBL-algebra is integral iff it embeds into a poset product of integral totally ordered, normal and bounded GMV-algebras and that a GBL-algebra is commutative iff it embeds into a poset product of totally ordered MV-algebras and of totally ordered abelian ℓ-groups. In this section we give similar characterizations for other classes of GBL-algebras. We start from the class of representable GBL-algebras. Our characterization is again in terms of poset product embeddability and involves the following notion. A *root system* is a poset $(P, \leq)$ such that for all $p \in P$ the set $\{q \in P : q \geq p\}$ is totally ordered. The dual of a root system is called a *forest*.

Theorem 6.1. *Let* $\mathbf{A}$ *be a GBL-algebra. The following are equivalent:*
(i) $\mathbf{A}$ *is representable.*
(ii) $\mathbf{A}$ *is embeddable into a poset product* $\bigotimes_{x \in \mathbf{P}} \mathbf{A}_x$ *such that each* $\mathbf{A}_x$ *is a totally ordered GMV-algebra and the poset* $\mathbf{P}$ *is a root system.*

Proof. By Proposition 2.6 and along the lines of the proof of Theorem 5.3, (ii), we can prove the theorem separately for integral GBL-algebras and for ℓ-groups. Thus suppose first that $\mathbf{A}$ is integral.

(i) $\Rightarrow$ (ii) Let us decompose $\mathbf{A}$ as a subdirect product of totally ordered integral GBL-algebras $(\mathbf{A}_i : i \in I)$. Next let us apply Dvurečenskij's Theorem 2.11, thus getting an ordinal sum decomposition $\mathbf{A}_i = \bigoplus_{j \in J_i} \mathbf{W}_{i,j}$, where each $\mathbf{W}_{i,j}$ is a totally ordered integral GMV-algebra. Thus $\mathbf{W}^*_{i,j}$ is a totally ordered, integral and bounded GMV-algebra. Now let $P = \{(i,j) : i \in I,\, j \in J_i\}$. Define a partial order $\preceq$ on P by $(i,j) \preceq (i',j')$ iff $i = i'$ and $j \geq j'$. Clearly $\mathbf{P} = (P, \preceq)$ is a root system. We associate to each $a \in \mathbf{A}$ the function h_a on P, defined by

$$h_a(i,j) = \begin{cases} e & \text{if } a_i \in W_{i,k} \quad \text{for some } k < j \\ a_i & \text{if } a_i \in W_{i,j} \\ 0 & \text{if } a_i \in W_{i,k} \setminus \{e\} \quad \text{for some } k > j \end{cases}$$

It is readily seen that the map $a \mapsto h_a$ is an embedding of $\mathbf{A}$ into $\bigotimes_{(j,i) \in \mathbf{P}} \mathbf{W}^*_{j,i}$ and this shows (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i) Since representability is preserved under taking subalgebras, it suffices to show that if for all $p \in P$, $\mathbf{A}_p$ is totally ordered and $\mathbf{P} = (P, \leq)$ is a root system, then the algebra $\mathbf{A}^{\mathbf{P}} = \bigotimes_{p \in \mathbf{P}} \mathbf{A}_p$ is representable. For $h, k \in \mathbf{A}^{\mathbf{P}}$ and for $p \in P$, define $h \equiv_p k$ iff for all $q \geq p$, $h(q) = k(q)$. Note that in a poset product, for every operation $\circ$, $(h \circ k)(p)$ only depends on the restrictions of h and k to the set $\{q \in P : q \geq p\}$. It follows that $\equiv_p$ is a congruence of $\mathbf{A}^{\mathbf{P}}$. Moreover $\bigcap \{\equiv_p : p \in P\}$ is the minimum congruence, because if $h \equiv_p k$ for all $p \in P$, then h and k coincide. Thus $\mathbf{A}^P$ has a subdirect embedding into $\prod_{p \in P} (\mathbf{A}^{\mathbf{P}} / \equiv_p)$, and it suffices to prove that each $\mathbf{A}^{\mathbf{P}} / \equiv_p$ is totally ordered. In other words, it suffices to prove that for every $p \in P$ and for every $h, k \in A^P$, either $h(q) \leq k(q)$ for all $q \geq p$ or $h(q) \geq k(q)$ for all $q \geq p$. Suppose not. Then there are $q, r \geq p$ such that $h(q) < k(q)$ and $k(r) < h(r)$. Since $\mathbf{P}$ is a root system, the set $\{q \in P : q \geq p\}$ is totally ordered, therefore either $q > r$ or $r > q$. Suppose e.g. $q > r$. Then $h(q) < k(q) \leq e$, therefore by the definition of poset product, $h(s) = 0$ for all $s < q$. In particular, $h(r) = 0 \leq k(r)$, and a contradiction has been reached.

The case where $\mathbf{A}$ is an ℓ-group is easy:

(i) $\Rightarrow$ (ii) Suppose that $\mathbf{A}$ is representable. Consider a subdirect embedding of $\mathbf{A}$ into $\prod_{i \in I} \mathbf{A}_i$, where each $\mathbf{A}_i$ is a totally ordered ℓ-group. Define for $i, j \in I$, $i \leq j$ iff $i = j$. Then $\mathbf{I} = (I, \leq)$ is a root system and $\mathbf{A}$ embeds into $\bigotimes_{i \in \mathbf{I}} \mathbf{A}_i = \prod_{i \in I} \mathbf{A}_i$.

(ii) $\Rightarrow$ (i) If an ℓ-group $\mathbf{A}$ is the poset product $\bigotimes_{i \in \mathbf{I}} \mathbf{A}_i$ of an indexed family of totally ordered GBL-algebras then it is readily seen that each $\mathbf{A}_i$ must be an ℓ-group. Now a non-trivial ℓ-group is not integral, therefore the definition of poset product implies that every $i \in I$ must be minimal. Hence for all $i, j \in I$ one has $i \leq j$ iff $i = j$. Thus $\bigotimes_{i \in \mathbf{I}} \mathbf{A}_i = \prod_{i \in I} \mathbf{A}_i$, which is a representable ℓ-group. This ends the proof.

Several classes of representable GBL-algebras, arising from many-valued logic, have a simple characterization in terms of poset product embeddability. We collect all of them in the next theorem, whose easy proof is left to the reader.

Theorem 6.2. *A GBL-algebra is*

- *a BL-algebra iff it is isomorphic to a subalgebra* $\mathbf{A}$ *of a poset product* $\bigotimes_{p \in \mathbf{P}} \mathbf{A}_p$ *such that*
  (a) *each* $\mathbf{A}_p$ *is a totally ordered MV-algebra,*
  (b) $\mathbf{P} = (P, \leq)$ *is a root system and*
  (c) *the function on* P *which is constantly equal to* 0 *is in* $\mathbf{A}$*;*
- *an MV-algebra iff it is isomorphic to a subalgebra* $\mathbf{A}$ *of a poset product* $\bigotimes_{p \in \mathbf{P}} \mathbf{A}_p$ *such that conditions* (a) *and* (c) *above hold and*
  (d) $\mathbf{P} = (P, \leq)$ *is a poset such that* $\leq$ *is the identity on* P*;*
- *an abelian* ℓ*-group iff it is isomorphic to a subalgebra* $\mathbf{A}$ *of a poset product* $\bigotimes_{p \in \mathbf{P}} \mathbf{A}_p$ *such that each* $\mathbf{A}_p$ *is a totally ordered abelian* ℓ*-group and condition* (d) *above holds;*
- n*-potent iff it is embeddable into a poset product of totally ordered* n*-potent MV-algebras;*
- *a Heyting algebra iff it is isomorphic to a subalgebra* $\mathbf{A}$ *of a poset product* $\bigotimes_{p \in \mathbf{P}} \mathbf{A}_p$ *where condition* (c) *holds and in addition*
  (e) *every* $\mathbf{A}_p$ *is the two-element MV-algebra* $\mathbf{W}_1$*;*
- *a Gödel algebra iff it is isomorphic to a subalgebra* $\mathbf{A}$ *of a poset product* $\bigotimes_{p \in \mathbf{P}} \mathbf{A}_p$ *where* (b), (c) *and* (e) *hold;*

- *a boolean algebra iff it is isomorphic to a subalgebra* $\mathbf{A}$ *of a poset product* $\bigotimes_{p\in\mathbf{P}}\mathbf{A}_p$ *where* (c), (d) *and* (e) *hold.*

7. Conrad-Harvey-Holland-style embedding theorems for commutative GBL-algebras

A simplified version of the Conrad-Harvey-Holland theorem says that every abelian ℓ-group can be embedded into an ℓ-group of functions from a root system into the set $\mathbf{R}$ of reals, with pointwise addition as group operation. In this section we aim to extend the result to commutative GBL-algebras.

Definition 7.1. Let $\mathbf{\Delta} = (\Delta, \leq)$ be a root system and for every function f from Δ into $\mathbf{R}$, let $Supp(f) = \{\delta \in \Delta : f(\delta) \neq 0\}$. We define a structure $V(\mathbf{\Delta}, \mathbf{R})$ as follows:
(a) The universe of $V(\mathbf{\Delta}, \mathbf{R})$ is the set of all functions f from Δ into $\mathbf{R}$ such that every non-empty subset of $Supp(f)$ has a maximal element.
(b) The group operation is pointwise addition (hence the neutral element is the constantly 0 function $\overline{0}$ and the inverse operation $^{-1}$ is defined, for $f \in V(\mathbf{\Delta}, \mathbf{R})$ and for $\delta \in \Delta$, by $(f^{-1})(\delta) = -f(\delta)$).
(c) The positive cone of $V(\mathbf{\Delta}, \mathbf{R})$ consists of $\overline{0}$ together with all $f \in V(\mathbf{\Delta}, \mathbf{R})$ such that $f(\delta) > 0$ for each maximal element $\delta \in Supp(f)$.

Then we have:

Proposition 7.2. *(Conrad-Harvey-Holland, simplified version, cf [Gl99]).*

(a) The algebra $V(\mathbf{\Delta}, \mathbf{R})$ *is an* ℓ*-group with respect to the operations and to the positive cone introduced in Definition 7.1.*
(b) Every abelian ℓ*-group* $\mathbf{G}$ *embeds into an* ℓ*-group of the form* $V(\mathbf{\Delta}, \mathbf{R})$ *for a suitable root system* $\mathbf{\Delta} = (\Delta, \leq)$.

Note that lattice operations in $V(\mathbf{\Delta}, \mathbf{R})$ are induced by its positive cone. They may be explicitly defined as follows: let $f, g \in V(\mathbf{\Delta}, \mathbf{R})$ and let $\delta \in \Delta$. If for all $\rho \geq \delta$ we have $f(\rho) = g(\rho)$, then $(f \vee g)(\delta) = (f \wedge g)(\delta) = f(\delta) = g(\delta)$. Otherwise, since $\mathbf{\Delta}$ is a root system and $f, g \in V(\mathbf{\Delta}, \mathbf{R})$, the set $\{\rho \in Supp(g - f) : \delta \leq \rho\}$ has a maximum element, δ_0 say. Then if $g(\delta_0) < f(\delta_0)$ we have $(f \vee g)(\delta) = f(\delta)$ and $(f \wedge g)(\delta) = g(\delta)$. Otherwise we have $(f \vee g)(\delta) = g(\delta)$ and $(f \wedge g)(\delta) = f(\delta)$.

For totally ordered ℓ-groups $\mathbf{G}$, the result was shown first by Hahn:

Proposition 7.3. *(Hahn, simplified version, cf [Gl99]).*

(a) If $\mathbf{\Delta} = (\Delta, \leq)$ *is totally ordered, then* $V(\mathbf{\Delta}, \mathbf{R})$ *is a totally ordered abelian* ℓ*-group.*
(b) Every totally ordered abelian ℓ*-group* $\mathbf{G}$ *embeds into an* ℓ*-group of the form* $V(\mathbf{\Delta}, \mathbf{R})$ *for a suitable totally ordered set* $\mathbf{\Delta}$.

Note that the proofs of both Hahn's theorem and of the Conrad-Harvey-Holland theorem provide for an explicit construction of the root system $\mathbf{\Delta}$. More precisely, recall that a *convex subgroup* of an ℓ-group $\mathbf{G}$ is an ℓ-subgroup $\mathbf{H}$ of $\mathbf{G}$ such that for all $h, g \in \mathbf{G}$, if $h \in \mathbf{H}$ and $g \vee g^{-1} \leq h \vee h^{-1}$, then $g \in \mathbf{H}$. We also recall that a *value* of an abelian ℓ-group $\mathbf{G}$ is a convex subgroup $\mathbf{H}$ of $\mathbf{G}$ for which there exists $a \in \mathbf{G}$ such that $\mathbf{H}$ is maximal among all convex subgroups not containing a. Then $\mathbf{\Delta}$ may be assumed to be the set $\Delta(\mathbf{G})$ of all values of $\mathbf{G}$, partially ordered by set-theoretic inclusion.

In order to extend the (simplified version of the) Conrad-Harvey-Holland embedding theorem to commutative GBL-algebras, it suffices to extend it to poset products of totally ordered MV-algebras and totally ordered abelian ℓ-groups. To begin with, we give some embedding theorems for the factors of such poset products. We already have an embedding theorem for totally ordered abelian ℓ-groups, namely, Hahn's theorem. For totally ordered MV-algebras we will use a variant of Mundici's functor Γ. This functor allows us to represent any MV-algebra as an interval $[e, u]$ of an abelian ℓ group $\mathbf{G}$ such that u is a strong unit of $\mathbf{G}$. However, since integral residuated lattices are regarded as negative cones and not as positive cones, we prefer to represent MV-algebras as intervals of the form $[u^{-1}, e]$ and not of the form $[e, u]$.

We start from an analogue of Hahn's theorem for negative cones.

Definition 7.4. Let $\mathbf{\Delta} = (\Delta, \leq)$ be a totally ordered set. Let $0{\downarrow}$ be the set of all $f \in V(\mathbf{\Delta}, \mathbf{R})$ such that either $f = \overline{0}$ or $f(\max(Supp(f))) < 0$. We define a structure $V^-(\mathbf{\Delta}, \mathbf{R})$ as follows:

- The domain of $V^-(\mathbf{\Delta}, \mathbf{R})$ is $0{\downarrow}$.
- The monoid operation is pointwise addition and the lattice operations are the restrictions of the lattice operations on $V(\mathbf{\Delta}, \mathbf{R})$.
- The residual $\to$ is defined as follows: if $g - f \in 0{\downarrow}$ (here $g - f$ denotes the pointwise difference of g and f), then $f \to g = g - f$. Otherwise, $f \to g = \overline{0}$.

Hahn's theorem immediately gives the following result.

Proposition 7.5. *(a) If $\mathbf{\Delta}$ is a totally ordered set, then $V^-(\mathbf{\Delta}, \mathbf{R})$ is the negative cone of a totally ordered abelian ℓ-group.*
(b) For every negative cone, $\mathbf{G}^-$, of a totally ordered abelian ℓ-group $\mathbf{G}$, there is a totally ordered set $\mathbf{\Delta}$ such that $\mathbf{G}^-$ embeds into $V^-(\mathbf{\Delta}, \mathbf{R})$.

Now we treat totally ordered MV-algebras. Recall that a *strong unit* of a lattice ordered abelian group $\mathbf{G}$ with group operation $+$ is an element $u \in \mathbf{G}$ such that for all $g \in \mathbf{G}$ there is a positive integer n such that $g \leq u + \cdots + u$ (n times). Then after reversing the order, Mundici's Γ functor can be rewritten as follows:

Definition 7.6. Let u be a strong unit of an abelian ℓ-group $\mathbf{G}$ with group operation $+$, with neutral element 0 and with inverse operation $-x$. Then $\Gamma(\mathbf{G}, -u)$ denotes the algebra $\mathbf{A} = (A, \odot, \to, \vee, \wedge, -u, 0)$ where:

- $A = \{x \in \mathbf{G} : -u \leq x \leq 0\}$.
- The lattice operations $\vee$ and $\wedge$ are the restriction of the lattice operations in $\mathbf{G}$.
- For $x, y \in A$, $x \odot y = (x + y) \vee (-u)$ and $x \to y = (y - x) \wedge 0$.

After reversing the order and restricting our attention to totally ordered MV-algebras, Mundici's equivalence [Mu86] between MV-algebras and lattice ordered abelian groups with a strong unit immediately implies the following result.

Proposition 7.7. *For every totally ordered MV-algebra* $\mathbf{A}$ *there are a totally ordered abelian* ℓ*-group* $\mathbf{G}$ *and a strong unit* u *of* $\mathbf{G}$ *such that* $\mathbf{A}$ *is isomorphic to* $\Gamma(\mathbf{G}, -u)$*. Hence for every totally ordered MV-algebra* $\mathbf{A}$ *there are a totally ordered set* $\mathbf{\Delta}$ *and a strong unit* $u \in V(\mathbf{\Delta}, \mathbf{R})$ *such that* $\mathbf{A}$ *embeds into* $\Gamma(V(\mathbf{\Delta}, \mathbf{R}), -u)$*.*

Note that for every totally ordered abelian ℓ-group $\mathbf{G}$ it is possible to choose a totally ordered set $\mathbf{\Delta}$ such that $\mathbf{G}$ is cofinal in $V(\mathbf{\Delta}, \mathbf{R})$, that is, every element of $V(\mathbf{\Delta}, \mathbf{R})$ has an upper bound in $\mathbf{G}$. This property implies that every strong unit of $\mathbf{G}$ is a strong unit of $V(\mathbf{\Delta}, \mathbf{R})$. Moreover $V(\mathbf{\Delta}, \mathbf{R})$ has a strong unit iff $\mathbf{\Delta}$ has a maximum element. Indeed, if u is a strong unit and $\delta = \max(Supp(u))$, then δ must be the maximum of Δ, otherwise if $\sigma > \delta$, then the function f such that $f(\sigma) = 1$ and $f(\rho) = 0$ for $\rho \neq \sigma$ is such that $f \in V(\mathbf{\Delta}, \mathbf{R})$ and for every positive integer n, $f > u + \cdots + u$ (n times). Moreover, if $\delta = \max(\Delta)$, then any $u \in V(\mathbf{\Delta}, \mathbf{R})$ such that $u(\delta) > 0$ is a strong unit of $V(\mathbf{\Delta}, \mathbf{R})$.

Remark. For a more general representation theorem of GMV-algebras by means of an algebra of real-valued functions, the reader is invited to consult [GRW03].

Since we want that all non-minimal factors in a poset product share the same minimum, and since 0 is already booked (it is the neutral element of the group of the reals), we will replace $-u$ by $-\infty$ (where we assume that $-\infty \notin \mathbf{R}$ and that $-\infty \notin V(\mathbf{\Delta}, \mathbf{R})$), and we will call the resulting structure $\Gamma'(V(\mathbf{\Delta}, \mathbf{R}), -u)$. Thus $\Gamma'(V(\mathbf{\Delta}, \mathbf{R}), -u)$ is defined as follows:

Definition 7.8. Let $\mathbf{\Delta}$ be a totally ordered set with maximum and let u be a strong unit of $V(\mathbf{\Delta}, \mathbf{R})$. Let $0{\downarrow}$ be as in Definition 7.4 and let $(-u){\uparrow} = \{f \in V(\mathbf{\Delta}, \mathbf{R}) \setminus \{-u\} : \max(Supp(f + u)) > 0\}$. Then $\Gamma'(V(\mathbf{\Delta}, \mathbf{R}), -u)$ is defined as follows:

- The domain of $\Gamma'(V(\mathbf{\Delta}, \mathbf{R}), -u)$ is $(0{\downarrow} \cap (-u){\uparrow}) \cup \{-\infty\}$.
- Lattice operations on $0{\downarrow} \cap (-u){\uparrow}$ are the restrictions of lattice operations on $V(\mathbf{\Delta}, \mathbf{R})$, and for all $f \in \Gamma'(V(\mathbf{\Delta}, \mathbf{R}), -u)$, $f \wedge -\infty = -\infty \wedge f = -\infty$ and $f \vee -\infty = -\infty \vee f = f$.

- For all $f \in \Gamma'(V(\mathbf{\Delta}, \mathbf{R}), -u)$, $f \cdot -\infty = -\infty \cdot f = -\infty$. Moreover, if $f, g \in 0{\downarrow} \cap (-u){\uparrow}$, then
$f \cdot g = \begin{cases} f+g & \text{if} \quad f+g \in (-u){\uparrow} \\ -\infty & \text{otherwise} \end{cases}$.

- For all $f \in \Gamma'(V(\mathbf{\Delta}, \mathbf{R}), -u)$, $-\infty \to f = \overline{0}$ and if $f \neq \overline{0}$ and $f \neq -\infty$, then $f \to -\infty = -u - f$. Moreover $\overline{0} \to -\infty = -\infty$, and if $f, g \in 0{\downarrow} \cap (-u){\uparrow}$, then
$f \to g = \begin{cases} g-f & \text{if} \quad g-f \in 0{\downarrow} \\ \overline{0} & \text{otherwise} \end{cases}$.

Then we have:

Proposition 7.9. *Every totally ordered MV-algebra embeds into an algebra of the form $\Gamma'(V(\mathbf{\Delta}, \mathbf{R}), -u)$ for some totally ordered set $(\Delta, \leq)$ with maximum and for some strong unit u of $V(\mathbf{\Delta}, \mathbf{R})$.*

It follows from Corollary 5.4 and from Propositions 7.3 and 7.9 that every commutative GBL-algebra embeds into a poset product $\bigotimes_{p \in \mathbf{P}} \mathbf{A}_p$ of algebras $\mathbf{A}_p$ having one of the forms $V(\mathbf{\Delta}_p, \mathbf{R})$ or $\Gamma'(V(\mathbf{\Delta}_p, \mathbf{R}), -u_p)$ for some totally ordered set $\mathbf{\Delta}_p$ and for some strong unit u_p of $V(\mathbf{\Delta}_p, \mathbf{R})$. Such poset products are uniquely determined by the poset $\mathbf{P} = (P, \leq)$, by the totally ordered sets $\mathbf{\Delta}_p$, by the choice, for each p, of one of the forms $V(\mathbf{\Delta}_p, \mathbf{R})$ or $\Gamma'(V(\mathbf{\Delta}_p, \mathbf{R}), -u_p)$ and in the last case, by the choice of the strong unit u_p of $V(\mathbf{\Delta}_p, \mathbf{R})$, that is, of a function u_p from Δ_p into $\mathbf{R}$ such that every non-empty subset of $Supp(u_p)$ has a maximum and $u_p(\max(\Delta_p)) > 0$. Thus the only algebraic structure in the definition of such algebras is the group structure of the reals, the rest of the construction essentially depends on order. The algebras of the form shown above will be called *real-valued GBL-algebras*.

We want to describe real-valued GBL-algebras more closely. First of all, every element F of a real-valued GBL-algebra $\bigotimes_{p \in \mathbf{P}} \mathbf{A}_p$ is a function which associates to every $p \in P$ either $-\infty$ or a function F_p from Δ_p into $\mathbf{R}$. Up to isomorphism we may safely replace such a function F by the function H from the set $P_\Delta = \{(p, \delta) : p \in P, \delta \in \Delta_p\}$ into $\mathbf{R} \cup \{-\infty\}$ defined by

$$H(p, \delta) = \begin{cases} -\infty & \text{if } F(p) = -\infty \\ (F(p))(\delta) & \text{otherwise.} \end{cases}$$

In the sequel, given a function $H(p, \delta)$ on P_Δ such that for all $p \in P$, either for all $\delta \in \Delta_p$, $H(p, \delta) = -\infty$, or for all $\delta \in \Delta_p$, $H(p, \delta) \in \mathbf{R}$, we define H_p as follows: if for all $\delta \in \Delta_p$, $H(p, \delta) = -\infty$, then we set $H_p = -\infty$; otherwise we set H_p to be the function on Δ_p defined by $H_p(\delta) = H(p, \delta)$. Then real-valued GBL-algebras can be defined as follows:

Definition 7.10. Let $\mathbf{P} = (P, \leq)$ be a poset and let $\{P_G, P_{MV}\}$ be a partition of P such that every $p \in P_G$ is incomparable with the other elements with respect to $\leq$. Let us label each element of $p \in P_G$ by a totally ordered set $\mathbf{\Delta}_p = (\Delta_p, \leq_p)$ and each $q \in P_{MV}$ by a totally ordered set $\mathbf{\Delta}_q = (\Delta_q, \leq_q)$ with maximum δ_q and by a function $u_q \in V(\mathbf{\Delta}_q, \mathbf{R})$ such that $u(\delta_q) > 0$. Then the *real-valued GBL-algebra* associated to the poset $\mathbf{P}$, to the partition $\{P_G, P_{MV}\}$ and to the labeling $\Lambda_G = (\mathbf{\Delta}_p : p \in P_G)$ and $\Lambda_{MV} = (\mathbf{\Delta}_q, u_q : q \in P_{MV})$ is the algebra $\mathbf{A} = \mathbf{GBL}(\mathbf{P}, P_G, P_{MV}, \Lambda_G, \Lambda_{MV})$ defined as follows:

- The domain of $\mathbf{A}$ is the set of all functions H from P_Δ into $\mathbf{R} \cup -\{\infty\}$ such that
  - for all $p \in P_G$, $H(p, \delta) \in \mathbf{R}$ for all $\delta \in \Delta_p$;
  - for all $p \in P_{MV}$ we have that either $H(p, \delta) = -\infty$ for all $\delta \in \Delta_p$ or $H(p, \delta) \in \mathbf{R}$ for all $\delta \in \Delta_p$;
  - if $p \in P_G$ ($p \in P_{MV}$ respectively) then $H_p \in V(\mathbf{\Delta}_p, \mathbf{R})$ ($H_p \in \Gamma'(V(\mathbf{\Delta}_p, \mathbf{R}), -u_p)$ respectively), and
  - if for some $\delta \in \Delta_p$, $H(p, \delta) \neq 0$, then $H(q, \sigma) = -\infty$ for all $q < p$ and for all $\sigma \in \Delta_q$.
- For every operation $\circ$ of commutative GBL-algebras, let $\circ_p$ denote its realization in $V(\mathbf{\Delta}_p, \mathbf{R})$ if $p \in P_G$ and in $\Gamma'(V(\mathbf{\Delta}_p, \mathbf{R}), -u_p)$ if $p \in P_{MV}$. Then
  - for $\circ \in \{\vee, \wedge, \cdot\}$, for $H, K \in \mathbf{A}$ and for $(p, \delta) \in P_\Delta$, $(H \circ K)(p, \delta) = (H_p \circ_p K_p)(\delta)$;
  - if for all $q > p$ we have that $H_q = -\infty$ implies $K_q = -\infty$ and $H_q, K_q \neq -\infty$ implies that either $H_q = K_q$ or $\max(Supp(H_q - K_q)) < 0$, then $(H \to K)(p, \delta) = (H_p \to_p K_p)(\delta)$; otherwise $(H \to K)(p, \delta) = -\infty$.

The next theorem is a almost a rephrasing of the results of the previous section for commutative GBL-algebras, in terms of embeddability into real-valued GBL-algebras. We use the notation $\mathbf{A} \subseteq \mathbf{B}$ to indicate that $\mathbf{A}$ is a subalgebra of $\mathbf{B}$.

Theorem 7.11. *Every commutative GBL-algebra embeds into a real-valued GBL-algebra of the form* $\mathbf{GBL}(\mathbf{P}, P_G, P_{MV}, \Lambda_G, \Lambda_{MV})$*, cf Definition 7.10.*

Moreover, a commutative GBL-algebra is

- *integral iff it embeds into an algebra* $\mathbf{GBL}(\mathbf{P}, P_G, P_{MV}, \Lambda_G, \Lambda_{MV})$ *in which*

  (a) $P_G = \Lambda_G = \emptyset$*;*
- *an ℓ-group iff it embeds into some* $\mathbf{GBL}(\mathbf{P}, P_G, P_{MV}, \Lambda_G, \Lambda_{MV})$ *in which*

  (b) $P_{MV} = \Lambda_{MV} = \emptyset$*;*
- *representable iff it embeds into some* $\mathbf{GBL}(\mathbf{P}, P_G, P_{MV}, \Lambda_G, \Lambda_{MV})$ *in which*

  (c) $\mathbf{P}$ *is a forest;*
- *a BL-algebra iff it is isomorphic to some* $\mathbf{A} \subseteq \mathbf{GBL}(\mathbf{P}, P_G, P_{MV}, \Lambda_G, \Lambda_{MV})$ *in which* (a) *and* (c) *hold and*

  (d) *the constantly $-\infty$ function is in* $\mathbf{A}$*;*

- *an MV-algebra iff it is isomorphic to some* $\mathbf{A} \subseteq \mathbf{GBL}(\mathbf{P}, P_G, P_{MV}, \Lambda_G, \Lambda_{MV})$ *in which* (a) *and* (d) *hold and*

  (e) *any two distinct elements of* P *are incomparable with respect to* $\leq$*;*

- *a Heyting algebra iff it is isomorphic to some* $\mathbf{A} \subseteq \mathbf{GBL}(\mathbf{P}, P_G, P_{MV}, \Lambda_G, \Lambda_{MV})$ *in which* (a) *and* (d) *hold and*

  (f) *for all* $H \in \mathbf{A}$ *and for all* $(p, \delta) \in P_\Delta$*,* $H(p, \delta) \in \{-\infty, 0\}$*;*

- *a Gödel algebra iff it is isomorphic to some* $\mathbf{A} \subseteq \mathbf{GBL}(\mathbf{P}, P_G, P_{MV}, \Lambda_G, \Lambda_{MV})$ *in which* (a), (c), (d) *and* (f) *are satisfied;*

- *a boolean algebra iff it is isomorphic to some* $\mathbf{A} \subseteq \mathbf{GBL}(\mathbf{P}, P_G, P_{MV}, \Lambda_G \Lambda_{MV})$ *in which* (a), (d), (e) *and* (f) *are satisfied.* $\square$

8. Explicit constructions of generic commutative GBL-algebras

Recall that a *quasivariety* is a class of algebras that can be axiomatized by quasi-identities (i.e. strict universal Horn formulas). An algebra $\mathbf{A}$ in a variety $\mathcal{V}$ is said to be *generic* for $\mathcal{V}$ if it generates $\mathcal{V}$ as a variety, and *strongly generic* for $\mathcal{V}$ if it generates $\mathcal{V}$ as a quasivariety. In this section we present a commutative and integral countable GBL-algebra which is strongly generic for the variety of commutative and integral GBL-algebras and a commutative GBL-algebra which is strongly generic for the variety of commutative GBL-algebras. We start with the integral case.

Lemma 8.1. *Every finite GBL-algebra* $\mathbf{A}$ *embeds into a poset product* $\bigotimes_{p\in\mathbf{P}} \mathbf{A}_p$ *where each* $\mathbf{A}_p$ *is a finite MV-chain and* $\mathbf{P} = (P, \preceq)$ *is a finite forest.*

Proof. By Proposition 3.2, we know that $\mathbf{A}$ is isomorphic to an algebra of the form $\bigotimes_{d\in\mathbf{D}} \mathbf{B}_d$ where $\mathbf{D} = (D, \leq)$ is a finite poset and for all $d \in D$, $\mathbf{B}_d$ is a finite MV-chain. Now let P be the set of all finite non-empty sequences $(d_1, \ldots, d_n)$ of elements of D such that d_1 is a minimal element of D and for $i = 1, \ldots, n-1$, d_{i+1} is a cover of d_i, that is, $d_i < d_{i+1}$ and for all z if $d_i \leq z \leq d_{i+1}$, then either $z = d_i$ or $z = d_{i+1}$. For $p, p' \in P$, define $p' \preceq p$ iff p is an end extension of p', that is, if either $p = p'$ or there is a finite sequence σ of elements of D such that p is the juxtaposition of p' and σ. Clearly, $\mathbf{P} = (P, \preceq)$ is a forest. Now for $p = (d_1, \ldots, d_n) \in P$, let $\mathbf{A}_p = \mathbf{B}_{d_n}$. We define a map Φ from $\bigotimes_{d\in\mathbf{D}} \mathbf{B}_d$ into $\bigotimes_{p\in\mathbf{P}} \mathbf{A}_p$ letting for $h \in \bigotimes_{d\in\mathbf{D}} \mathbf{B}_d$ and for $p = (d_1, \ldots, d_n) \in P$, $\Phi(h)(p) = h(d_n)$. We claim that Φ is an embedding of $\bigotimes_{d\in\mathbf{D}} \mathbf{B}_d$ into $\bigotimes_{p\in\mathbf{P}} \mathbf{A}_p$. The proof follows from the claims listed below.

Claim (a) If $h \in \bigotimes_{d\in\mathbf{D}} \mathbf{B}_d$, then $\Phi(h) \in \bigotimes_{p\in\mathbf{P}} \mathbf{A}_p$.

Proof of claim (a). For $p = (d_1, \ldots, d_n) \in P$, $\Phi(h)(p) = h(d_n) \in \mathbf{B}_{d_n} = \mathbf{A}_p$. Moreover, if $\Phi(h)(p) = h(d_n) < e$ and $p \succ p' = (d_1, \ldots, d_i)$, then $d_i < d_n$, therefore $\Phi(h)(p') = h(d_i) = 0$. This ends the proof of claim (a).

Claim (b) Φ is one-one and preserves $\cdot$, $\vee$ and $\wedge$.

Proof of claim (b). If $h, k \in \bigotimes_{d \in \mathbf{D}} \mathbf{B}_d$ and $h \neq k$, then $h(d) \neq k(d)$ for some $d \in D$. Clearly there is $p = (d_1, \ldots, d_n) \in P$ such that $d_n = d$. Therefore $\Phi(h)(p) = h(d) \neq k(d) = \Phi(k)(p)$. Thus Φ is one-one. Moreover for $\circ \in \{\cdot, \vee, \wedge\}$ we have that for $p = (d_1, \ldots, d_n) \in P$, $\Phi(h \circ k)(p) = (h \circ k)(d_n) = h(d_n) \circ k(d_n) = \Phi(h)(p) \circ \Phi(k)(p)$. This ends the proof of claim (b).

Claim (c). Φ preserves $\to$.

Proof of claim (c). Let $h, k \in \bigotimes_{d \in \mathbf{D}} \mathbf{B}_d$ and let $p = (d_1, \ldots, d_n) \in P$. We first compute $\Phi(h \to k)(p)$. Distinguish two cases:

(c1) If for all $d > d_n$ $h(d) \leq k(d)$, then $\Phi(h \to k)(p) = (h \to k)(d_n) = h(d_n) \to k(d_n)$;

(c2) Otherwise, $\Phi(h \to k)(p) = 0$.

Now we compute $(\Phi(h) \to \Phi(k))(p)$. Again, distinguish two cases:

(c1') If for all $p' \succ p$, $\Phi(h)(p') \leq \Phi(k)(p')$, then $(\Phi(h) \to \Phi(k))(p) = \Phi(h)(p) \to \Phi(k)(p) = h(d_n) \to k(d_n)$.

(c2') Otherwise $(\Phi(h) \to \Phi(k))(p) = 0$.

Thus it suffices to show that (c1) and (c1') are equivalent. Now (c1') reads: for all $p' = (d_1, \ldots, d_n, \ldots, d) \in P$, $h(d) \leq k(d)$, which is clearly equivalent to: for all $d \in D$ with $d > d_n$, $h(d) \leq k(d)$, that is, to (c1). This concludes the proof of Lemma 8.1.

Definition 8.2. An *initial segment* of a poset $\mathbf{P} = (P, \leq)$ is a subset I of P such that if $x \in I$, $y \in P$ and $y \leq x$, then $y \in I$.

Notation. In the sequel we denote by $\mathbf{MV}(\mathbf{Q})$ the MV-algebra with domain $[-1, 0] \cap \mathbf{Q}$ ($\mathbf{Q}$ is the set of rationals), with max and min as lattice operations, with monoid operation $x \cdot y = \max\{x + y, -1\}$ and residual $x \to y = \min\{y - x, 0\}$.

Lemma 8.3. *Let $\mathbf{A} = \bigotimes_{p \in \mathbf{P}} \mathbf{A}_p$ be a poset product of integral residuated lattices, and let I be an initial segment of $\mathbf{P}$. Let $\mathbf{I}$ be the subposet of $\mathbf{P}$ determined by I, and let $\mathbf{B} = \bigotimes_{p \in \mathbf{I}} \mathbf{A}_p$. Then:*

(a) The map Φ defined, for all $h \in \mathbf{B}$ and for all $p \in P$, by

$$\Phi(h)(p) = \begin{cases} h(p) & \text{if } p \in I \\ e & \text{otherwise} \end{cases}$$

is an embedding of $\mathbf{B}$ into $\mathbf{A}$.

(b) For $h \in \mathbf{A}$, let $N_h = \{p \in P : h(p) \neq e\}$ and $A_{\text{fin}} = \{h \in \mathbf{A} : N_h \text{ is finite}\}$. Then A_{fin} is the domain of a subalgebra of $\mathbf{A}$.

(c) If for $p \in P$, $\mathbf{B}_p$ is a subalgebra of $\mathbf{A}_p$, then $\bigotimes_{p \in \mathbf{P}} \mathbf{B}_p$ is a subalgebra of $\bigotimes_{p \in \mathbf{P}} \mathbf{A}_p$.

Proof. (a) First of all, we prove that Φ maps $\mathbf{B}$ into $\mathbf{A}$. Let $h \in \mathbf{B}$ and $p \in P$. Then clearly $\Phi(h)(p) \in \mathbf{A}_p$. Moreover, if $p > q$ and $\Phi(h)(p) < e$, then $p \in I$ and $q \in I$, since I is an initial segment. Hence $\Phi(h)(q) = h(q) = 0$. It follows that

$\Phi(h) \in \mathbf{A}$. That Φ is one-one and that it preserves $\cdot$, $\vee$ and $\wedge$ is clear, as these operations are defined pointwise. Now we prove that Φ preserves $\backslash$. Let $h, k \in \mathbf{A}$ and $p \in P$ be given. If $p \notin I$, then $\Phi(h\backslash k)(p) = (\Phi(h)\backslash\Phi(k))(p) = e$. If $p \in I$ and for all $q \in I$ such that $q > p$ we have $h(q) \leq k(q)$, then for all $q \in P$ with $q > p$ we have $\Phi(h)(q) \leq \Phi(k)(q)$, because if $q \notin I$, then $\Phi(h)(q) = \Phi(k)(q) = e$. Thus in this case, $\Phi(h\backslash k)(p) = (\Phi(h)\backslash\Phi(k))(p) = \Phi(h)(p)\backslash\Phi(k))(p)$. If there is $q \in I$ such that $q > p$ and $h(q) \not\leq k(q)$, then $\Phi(h\backslash k)(p) = (\Phi(h)\backslash\Phi(k))(p) = 0$. This shows compatibility with $\backslash$. The proof that Φ is compatible with $/$ is symmetric, and part (a) is proved.

(b) Just note that $N_{h\cdot k} = N_{h\wedge k} = N_h \cup N_k$, $N_{h\vee k} = N_h \cap N_k$, $N_{h\backslash k} \subseteq N_k$ and $N_{k/h} \subseteq N_k$. Thus if $h, k \in A_{\text{fin}}$, then $h \cdot k$, $h \wedge k$, $h \vee k$, $h\backslash k$ and k/h are in A and $N_{h\cdot k}$, $N_{h\wedge k}$, $N_{h\vee k}$, $N_{h\backslash k}$ and $N_{k/h}$ are all finite. Thus $h \cdot k$, $h \wedge k$, $h \vee k$, $h\backslash k$ and k/h are in A_{fin}. Clearly the function which is constantly equal to e is in A_{fin} and (b) is proved.

(c) Almost trivial.

Remark. The image of $\mathbf{B}$ under the embedding Φ defined in the proof of Lemma 8.3 (a) is the subalgebra of $\mathbf{A}$ consisting of all $h \in \mathbf{A}$ such that $h(p) = e$ for all $p \notin I$. This subalgebra will be denoted by $\mathbf{A}(I)$ and will be called *the relativization of* $\mathbf{A}$ *to* I.

Notation. In the sequel, given a poset $\mathbf{P}$, $\mathbf{Q}(\mathbf{P})$ will denote the algebra $\bigotimes_{p\in\mathbf{P}} \mathbf{A}_p$ with $\mathbf{A}_p = \mathbf{MV}(\mathbf{Q})$ for every $p \in P$. Moreover, given a poset product $\mathbf{A} = \bigotimes_{p\in\mathbf{P}} \mathbf{A}_p$, $\mathbf{A}_{\text{fin}}$ will denote the subalgebra of $\mathbf{A}$ whose domain is the set of all $h \in \mathbf{A}$ such that N_h is finite, cf Lemma 8.3, (b).

Theorem 8.4. *Let $\mathbf{P} = (P, \preceq)$ be a poset such that every finite forest is isomorphic to an initial segment of it. Then every finite GBL-algebra embeds into $\mathbf{Q}(\mathbf{P})_{\text{fin}}$. Therefore by Proposition 2.9 (iii), $\mathbf{Q}(\mathbf{P})_{\text{fin}}$ generates the variety of commutative and integral GBL-algebras as a quasivariety.*

Proof. Let $\mathbf{A}$ be any finite GBL-algebra. By Proposition 3.2 and by Lemma 8.1, $\mathbf{A}$ embeds into a poset product $\bigotimes_{d\in\mathbf{D}} \mathbf{A}_d$ such that $\mathbf{D} = (D, \leq)$ is a finite forest, and for $d \in D$, $\mathbf{A}_d$ is a finite MV-chain. Now $\mathbf{D}$ is isomorphic to an initial segment of $\mathbf{P}$. Since any finite MV-chain embeds into $\mathbf{MV}(\mathbf{Q})$, by Lemma 8.3 (a) and (c), $\bigotimes_{d\in\mathbf{D}} \mathbf{A}_d$ is a subalgebra of $\mathbf{Q}(\mathbf{P})$. Moreover, after identifying each element $h \in \bigotimes_{d\in\mathbf{D}} \mathbf{A}_d$ with its image under the embedding Φ defined in Lemma 8.3, we have that for $h \in \bigotimes_{d\in\mathbf{D}} \mathbf{A}_d$, $N_h \subseteq D$, therefore N_h is finite and $\bigotimes_{d\in\mathbf{D}} \mathbf{A}_d$ is a subalgebra of $\mathbf{Q}(\mathbf{P})_{\text{fin}}$. Since $\mathbf{A}$ is a subalgebra of $\bigotimes_{d\in\mathbf{D}} \mathbf{A}_d$, the claim is proved.

By Theorem 8.4, a strongly generic algebra for the variety $\mathcal{CIGBL}$ of commutative and integral GBL-algebras is given by $\mathbf{Q}(\mathbf{P})_{\text{fin}}$, where $\mathbf{P}$ is a poset such that every finite forest embeds into it as an initial segment. An example of such a poset is given by the set $\omega^{<\omega}$ of all finite non-empty sequences of natural numbers, partially ordered by the relation $\preceq$ defined by $\sigma \preceq \tau$ iff either $\sigma = \tau$ or

τ is an end extension of σ. Let $\mathbf{\Omega} = (\omega^{<\omega}, \preceq)$. Then, recalling that the variety of abelian ℓ-groups is generated as a quasivariety by the ℓ-group $\mathbf{Z}$ of integers, by Theorem 8.4 and by Proposition 2.6, we have:

Theorem 8.5. *(a)* $\mathbf{Q}(\mathbf{\Omega})_{\mathrm{fin}}$ *is a countable strongly generic algebra for the variety* $\mathcal{CIGBL}$.

(b) $\mathbf{Q}(\mathbf{\Omega})_{\mathrm{fin}} \times \mathbf{Z}$ *is a countable strongly generic algebra for the variety* $\mathcal{CGBL}$ *of commutative GBL-algebras.* □

We now investigate strongly generic models for some notable subvarieties of $\mathcal{CIGBL}$. Strongly generic models for the variety of MV-algebras and for the variety of BL-algebras are easy to obtain: for the variety of MV-algebras, just take $\mathbf{MV}(\mathbf{Q})$, which corresponds to $\mathbf{Q}_{\mathrm{fin}}(\mathbf{P})$ with $\mathbf{P}$ the one-element poset. For the variety of BL-algebras, it follows from [AM03] that a strongly generic model is given by the ordinal sum of ω copies of $\mathbf{MV}(\mathbf{Q})$. This ordinal sum corresponds to the dual poset product $\mathbf{Q}(\mathbf{N})_{\mathrm{fin}}$, where $\mathbf{N} = (\omega, \leq)$ is the poset of natural numbers with the usual order. We now consider the variety of Heyting algebras. This variety is also generated as a quasivariety by their finite members. These are poset products of copies of the two-element MV-algebra $\mathbf{W}_1$. Now let for every $n > 0$, $\mathbf{W}_n$ denote the MV-chain with $n+1$ elements, and let $\mathbf{W}_n(\mathbf{\Omega}) = \bigotimes_{\sigma \in \mathbf{\Omega}} \mathbf{A}_\sigma$ with $\mathbf{A}_\sigma = \mathbf{W}_n$. Then by Lemma 8.3 we have that every finite Heyting algebra embeds into $\mathbf{W}_1(\mathbf{\Omega})_{\mathrm{fin}}$, therefore $\mathbf{W}_1(\mathbf{\Omega})_{\mathrm{fin}}$ is strongly generic for the variety of Heyting algebras. By a similar argument we have that the variety $\mathcal{GBL}_2$ of 2-potent GBL-algebras is generated as a quasivariety by $\mathbf{W}_2(\mathbf{\Omega})_{\mathrm{fin}}$. This depends on the fact that every 2-potent MV-chain is a subalgebra of $\mathbf{W}_2$. However, it is not true that for every n the algebra $\mathbf{W}_n(\mathbf{\Omega})_{\mathrm{fin}}$ is strongly generic for the variety $\mathcal{GBL}_n$ of n-potent GBL-algebras. For instance, let $x' = x \to x^3$ and $2x = (x' \cdot x')'$. Then the identity $x \vee x' = 2(x \vee x')^2$ is not valid in the 3-potent MV-algebra $\mathbf{W}_2$, but is valid in $\mathbf{W}_3(\mathbf{\Omega})_{\mathrm{fin}}$.

A countable strongly generic algebra for $\mathcal{GBL}_n$ is obtained as follows: let for every natural number k, $r(k)$ denote the remainder of the division of k by n, and let $w(k) = r(k) + 1$. Let for every $\sigma = (k_1, \ldots, k_n) \in \omega^{<\omega}$, $\mathbf{A}_\sigma = \mathbf{W}_{w(k_n)}$ and let $\mathbf{W}_{\leq n}(\mathbf{\Omega}) = \bigotimes_{\sigma \in \mathbf{\Omega}} \mathbf{A}_\sigma$. Then we have:

Theorem 8.6. $\mathbf{W}_{\leq n}(\mathbf{\Omega})_{\mathrm{fin}}$ *is strongly generic for* $\mathcal{GBL}_n$.

Proof. It suffices to show that any finite n-potent GBL-algebra $\mathbf{A}$ embeds into $\mathbf{W}_{\leq n}(\mathbf{\Omega})_{\mathrm{fin}}$. By Proposition 3.2 and by Lemma 8.1, we can embed $\mathbf{A}$ into a poset product $\bigotimes_{p \in \mathbf{P}} \mathbf{A}_p$ where $\mathbf{P}$ is a finite forest and for $p \in P$, $\mathbf{A}_p$ is an MV-chain with cardinality $\leq n+1$. We prove by induction on the cardinality n of P that there is a one-one map Ψ from P into $\omega^{<\omega}$ such that for every $p \in P$ the following conditions hold:

(a) If p is minimal with respect to $\leq$, then $\Psi(p)$ has length 1 (hence it is a minimal element in $\mathbf{\Omega}$).

(b) If p' is a cover of p, then $\Psi(p')$ is a cover of $\Psi(p)$ (thus in particular, $p \leq p'$ iff $\Psi(p) \preceq \Psi(p')$).

(c) Let m be the last element of the sequence $\Psi(p)$. Then $\mathbf{A}_p = \mathbf{W}_{w(m)}$.

For $n = 1$, the claim is easy: let p be the unique element of P, let $h \leq n$ be such that $\mathbf{A}_p = \mathbf{W}_h$, and let $\Psi(p) = (h-1)$ (the sequence whose unique element is $h-1$). Since $w(h-1) = h$, (a), (b) and (c) are satisfied.

Now suppose that the claim is true for every forest of cardinality less than n (with $n > 1$) and consider a forest $\mathbf{P}$ of cardinality n. Let p be a maximal element of P, and consider the subposet $(P', \leq)$ with domain $P' = P \setminus \{p\}$. By the induction hypothesis there is a map Ψ' on $(P', \leq)$ satisfying (a), (b) and (c). We distinguish two cases:

(i) If p is also minimal (thus p is incomparable with the remaining elements), then let h such that $\mathbf{A}_p = \mathbf{W}_h$, let k be big enough such that the one-element sequence $(kn + h - 1)$ is not in the range of Ψ' and extend Ψ' to a function Ψ on P letting

$$\Psi(x) = \begin{cases} \Psi'(x) & \text{if} \quad x \neq p \\ (kn + h - 1) & \text{if} \quad x = p \end{cases} .$$

It is readily seen that Ψ meets our requirements.

(ii) If p is not minimal, then since $\mathbf{P}$ is a finite forest, there is a unique element p' that is covered by p. Let $\Psi(p') = (k_1, \ldots, k_r)$ and let k be big enough so that the sequence $(k_1, \ldots, k_r, kn + h - 1)$ is not in the range of Ψ'. Now extend Ψ' to a function Ψ on P letting

$$\Psi(x) = \begin{cases} \Psi'(x) & \text{if} \quad x \neq p \\ (k_1, \ldots, k_r, kn + h - 1) & \text{if} \quad x = p \end{cases} .$$

It is readily seen that Ψ meets our requirements.

Now by (a) and (b) the image $\Psi[P]$ of P under Ψ is an initial segment of $\mathbf{\Omega}$ which is isomorphic to $\mathbf{P}$. Moreover by (c) we have $\mathbf{A}_p = \mathbf{A}_{\Psi(p)}$, therefore the relativization $\mathbf{W}_{\leq n}(\mathbf{\Omega})(\Psi[P])$ of $\mathbf{W}_{\leq n}(\mathbf{\Omega})$ to $\Psi([P])$ (cf Lemma 8.3) is a subalgebra of $\mathbf{W}_{\leq n}(\mathbf{\Omega})_{\mathrm{fin}}$ which is isomorphic to $\bigotimes_{p \in \mathbf{P}} \mathbf{A}_p$. Therefore $\mathbf{A}$ embeds into $\mathbf{W}_{\leq n}(\mathbf{\Omega})_{\mathrm{fin}}$. This ends the proof.

9. Normal GBL-algebras and GMV-algebras with a conucleus.

It is well known that every Heyting algebra can be represented as the algebra of open elements of a boolean algebra with an interior operator. In this section we partially extend this result to normal GBL-algebras. More precisely, we show that every normal GBL-algebra embeds into the image of a GMV-algebra under a conucleus.

Definition 9.1. A *conucleus* on a residuated lattice $\mathbf{A}$ is a unary operation σ on $\mathbf{A}$ such that for all $x, y \in A$ the following conditions hold:

- $x \leq y$ implies $\sigma(x) \leq \sigma(y)$,
- $\sigma(x) \leq x$,

- $\sigma(x) = \sigma(\sigma(x))$,
- $\sigma(\sigma(x) \cdot \sigma(y)) = \sigma(x) \cdot \sigma(y)$, and
- $\sigma(e) = e$.

Definition 9.2. Let $\mathbf{A}$ be a residuated lattice and σ be a conucleus on $\mathbf{A}$. Then $\sigma(\mathbf{A})$ denotes the structure $(\sigma(A), \cdot_\sigma, \vee_\sigma, \wedge_\sigma, \backslash_\sigma, /_\sigma, e)$, where $\sigma(A)$ is the image of A under σ, and for all $x, y \in \sigma(A)$, the operations $\cdot_\sigma, \vee_\sigma, \wedge_\sigma, \backslash_\sigma$ and $/_\sigma$ are defined as follows:
$(x \cdot_\sigma y) = x \cdot y$, $x \vee_\sigma y = x \vee y$, $x \wedge_\sigma y = \sigma(x \wedge y)$, $x \backslash_\sigma y = \sigma(x \backslash y)$ and $x /_\sigma y = \sigma(x/y)$.

The next lemma is proved in [MT].

Lemma 9.3. (cf [MT]). *If $\mathbf{A}$ is a residuated lattice and σ is a conucleus on $\mathbf{A}$, then $\sigma(\mathbf{A})$ is a residuated lattice (in particular, $\sigma(A)$ is closed under $\cdot_\sigma, \vee_\sigma, \wedge_\sigma, \backslash_\sigma$ and $/_\sigma$).*

Lemma 9.4. *Let $\mathbf{A} = \bigotimes_{p \in \mathbf{P}} \mathbf{A}_p$ be a poset product of a family of integral and bounded residuated lattices with common top element e and with common bottom element 0, and let $\mathbf{B} = \prod_{p \in P} \mathbf{A}_p$. Define for all $f \in B$ and for all $p \in P$*

$$\sigma(f)(p) = \begin{cases} f(p) & \text{if } f(q) = e \text{ for all } q > p \\ 0 & \text{otherwise.} \end{cases}$$

Then σ is a conucleus and $\mathbf{A} = \sigma(\mathbf{B})$.

Proof. Clearly, properties (1), (2), (3) and (5) of conuclei are satisfied by σ. We verify property (4), that is, we prove that for all $f, g \in B$ and for all $p \in P$ we have $\sigma(\sigma(f) \cdot \sigma(g))(p) = (\sigma(f) \cdot \sigma(g))(p)$. The claim is clear if either $\sigma(f)(p) = 0$ or $\sigma(g)(p) = 0$. If $\sigma(f)(p) \neq 0$ and $\sigma(g)(p) \neq 0$, then for all $q > p$ we have $(\sigma(f) \cdot \sigma(g))(q) = e$, therefore by the definition of σ it follows that $\sigma(\sigma(f) \cdot \sigma(g))(p) = (\sigma(f) \cdot \sigma(g))(p)$. Thus σ is a conucleus. Now note that for all $f \in B$ we have that $f \in A$ iff $f = \sigma(f)$. It follows that $\bigotimes_{p \in \mathbf{P}} A_p = \sigma(B)$ and for all $f \in B$, $\sigma(f)$ is the greatest element g of A such that $g \leq f$. Thus since the order on $\mathbf{A}$ is the restriction to A of the order on $\mathbf{B}$, $\sigma(\mathbf{B})$ and $\mathbf{A}$ have the same order, and therefore they have the same lattice operations. Moreover, the monoid operation is defined pointwise in both $\sigma(\mathbf{B})$ and $\mathbf{A}$. Hence $\sigma(\mathbf{B})$ and $\mathbf{A}$ coincide as lattice ordered monoids. It follows that residuals in $\sigma(\mathbf{B})$ and $\mathbf{A}$ also coincide, and the claim is proved.

Theorem 9.5. *Every normal GBL-algebra $\mathbf{A}$ embeds into a GBL-algebra of the form $\sigma(\mathbf{B})$ for some GMV-algebra $\mathbf{B}$ and for some conucleus σ on $\mathbf{B}$.*

Proof. By Proposition 2.6, $\mathbf{A}$ can be represented as $\mathbf{A} = \mathbf{C} \times \mathbf{G}$ for some integral and normal GBL-algebra $\mathbf{C}$ and for some ℓ-group $\mathbf{G}$. Moreover $\mathbf{C}$ embeds into a poset product of the form $\mathbf{D} = \bigotimes_{p \in \mathbf{P}} \mathbf{D}_p$ where for every $p \in P$,

$\mathbf{D}_p$ is an integral GMV-algebra. Now by Lemma 9.4, there is an integral GMV-algebra $\mathbf{H}$ and a conucleus τ on $\mathbf{H}$ such that $\mathbf{D} = \tau(\mathbf{H})$. Clearly $\mathbf{A}$ embeds into $\mathbf{D} \times \mathbf{G}$. Now let $\mathbf{F} = \mathbf{H} \times \mathbf{G}$ and let for $(x, y) \in H \times G$, $\sigma(x, y) = (\tau(x), y)$. Clearly $\mathbf{F}$ is a GMV-algebra, σ is a conucleus on $\mathbf{F}$, $\mathbf{D} \times \mathbf{G} = \sigma(\mathbf{F})$ and $\mathbf{A}$ embeds into $\sigma(\mathbf{F})$, as desired.

Note that the converse of Theorem 9.5 does not hold, that is, the image $\sigma(\mathbf{B})$ of a GMV-algebra $\mathbf{B}$ under a conucleus σ need not be a GBL-algebra. For instance, let $\mathbf{B}$ be the algebra $\mathbf{MV}(\mathbf{Q})$ defined in Section 8. Define a map σ on $\mathbf{B}$ as follows:

$$\sigma(x) = \begin{cases} 1 & \text{if} \quad x = 1 \\ x \wedge \frac{1}{2} & \quad\text{otherwise} \end{cases}.$$

It is readily seen that σ is a conucleus on $\mathbf{B}$. However $\sigma(\mathbf{B})$ is not a GBL-algebra, because $\frac{1}{4} = \frac{1}{2} \wedge_\sigma \frac{1}{4}$, but $\frac{1}{2} \cdot_\sigma (\frac{1}{2} \to_\sigma \frac{1}{4}) = \frac{1}{2} \cdot (\frac{3}{4} \wedge \frac{1}{2}) = \frac{1}{2} \cdot \frac{1}{2} = 0$. Of course for every GMV-algebra $\mathbf{B}$ and for every conucleus σ on $\mathbf{B}$, we have that $\sigma(\mathbf{B})$ is a GBL-algebra iff it satisfies the translation of the divisibility condition, namely the equation

$$(\mathrm{div}_\sigma) \qquad \sigma(x) \cdot_\sigma (\sigma(x) \backslash_\sigma \sigma(y)) \wedge_\sigma e) = (\sigma(y) /_\sigma \sigma(x)) \wedge_\sigma e) \cdot_\sigma \sigma(x) = \sigma(x) \wedge_\sigma \sigma(y).$$

This remark and Theorem 9.5 can be summarized as follows:

Theorem 9.6. 1. *Let $\mathbf{A}$ be a normal residuated lattice. Then the following are equivalent:*
(a) $\mathbf{A}$ is a GBL-algebra.
(b) $\mathbf{A}$ embeds into an algebra of the form $\sigma(\mathbf{B})$ where $\mathbf{B}$ is a GMV-algebra and σ is a conucleus on $\mathbf{B}$ such that (div_σ) holds.

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