Residuated Lattices

Peter Jipsen, Chapman University, Orange, California, USA

joint work with

Nikolaos Galatos, University of Denver, Colorado, USA

ICLA, January, 2009

Outline

Part I

• Universal Algebra
• Examples of residuated lattices
• Congruences and normal filters
• The lattice of subvarieties
• Varieties generated by positive universal classes
• Direct decompositions and poset products

Part II

• Residuated Frames
• Decidability
• Enumerating finite residuated lattices

Set notation

We assume a background of basic set notation and logic

A, B, C, . . . denote sets; x ∈ A means x is an element of A

∅ = {} is the empty set; set-builder notation: {x | P(x)}

Intersection A \cap B = \{x | x ∈ A and x ∈ B\}

Union A \cup B = \{x | x ∈ A or x ∈ B\}

Set difference A − B = \{x | x ∈ A and x \notin B\}

Ordered pairs (x, y) = (u, v) \iff x = u and y = v

Cartesian product A \times B = \{(x, y) | x ∈ A and x ∈ B\}

Set of n-tuples A^n = A \times A \times \cdots \times A (n copies)

Algebras and subalgebras

An n-ary operation on a set A is a function f : A^n → A

0-ary operations are constants (fixed elements of A)

An algebra A = (A, f^A_1, f^A_2, . . .) is a set A with operations f^A_i of arity n_i

Superscript A is useful when there are several algebras, otherwise omitted

The type of an algebra is the list of arities (n_1, n_2, . . .)

E.g. a group G = (G, ·, −1, 1) is an algebra of type (2, 1, 0)

Subsets: B ⊆ A means for all x, if x ∈ B then x ∈ A

g = f|_B means for all b_i ∈ B, g(b_1, . . ., b_n) = f(b_1, . . ., b_n) ∈ B

B is a subalgebra of A if B ⊆ A and f^B_i = f^A_i|_B (all i)
Homomorphisms and isomorphisms

Let $\mathbf{A}$, $\mathbf{B}$ be algebras of the same type.

A homomorphism $h : \mathbf{A} \to \mathbf{B}$ is a function $h : A \to B$ such that for all $i$,

$$h(f_i^A(a_1, \ldots, a_n)) = f_i^B(h(a_1), \ldots, h(a_n))$$

$h$ is onto if $h[A] = \{h(a) | a \in A\} = B$

In this case $\mathbf{B} = h[\mathbf{A}]$ is called a homomorphic image of $\mathbf{A}$

$h$ is one-to-one if for all $x, y \in A$, $x \neq y$ implies $h(x) \neq h(y)$

$h$ is an isomorphism if $h$ is a one-to-one and onto homomorphism

In this case $\mathbf{A}$ is said to be isomorphic to $\mathbf{B}$, written $\mathbf{A} \cong \mathbf{B}$

Term algebras and equational classes

For a fixed type, the terms with variables from a set $X$ is the smallest set $T(X)$ such that $X \subseteq T(X)$ and

if $t_1, \ldots, t_n \in T(X)$ then "$f_i(t_1, \ldots, t_n)$" $\in T(X)$ for all $i$

The term-algebra over $X$ is $T(X) = \{ (T(X), t_1^X, t_2^X, \ldots) \}

with $f_i^X(t_1, \ldots, t_n) = "f_i(t_1, \ldots, t_n)"$ for all $i$ and $t_1, \ldots, t_n \in T(X)$

An equation is a pair of terms $(s, t)$ written "$s = t$"; often omit ""

An assignment is a pair of terms $(s, t)$ written "$s = t$"

An algebra $\mathbf{A}$ is a homomorphism $h : T(X) \to \mathbf{A}$

An algebra $\mathbf{A}$ satisfies $s = t$ if $h(s) = h(t)$ for all assignments into $\mathbf{A}$

For a set $E$ of equations, $Mod(E) = \{ \mathbf{A} | \mathbf{A}$ satisfies $s = t$ for all $s = t \in E \}$

An equational class is of the form $Mod(E)$ for some set of equations $E$

Products and HSP

The union of sets $A_j (j \in J) = \bigcup_{j \in J} A_j = \{ x | x \in A_j$ for some $j \in J \}$

$f : J \to \bigcup_{j \in J} A_j$ is a choice function if $f(j) \in A_j$ for all $j \in J$

The cartesian product $\prod_{j \in J} A_j$ is the set of all choice functions

The direct product of algebras $\mathbf{A}_j (j \in J)$ is $\mathbf{A} = \prod_{j \in J} \mathbf{A}_j$ where $A = \prod_{j \in J} A_j$ and $f_i^A(a_1, \ldots, a_n)(j) = f_i^{A_j}(a_1(j), \ldots, a_n(j))$ for all $j \in J$

Let $\mathcal{K}$ be a class of algebras of the same type

$H\mathcal{K}$ is the class of homomorphic images of members of $\mathcal{K}$

$S\mathcal{K}$ is the class of algebras isomorphic to subalgebras of members of $\mathcal{K}$

$P\mathcal{K}$ is the class of algebras isomorphic to direct products of members of $\mathcal{K}$

$\mathcal{K}$ is a variety if $H(\mathcal{K}) = S(\mathcal{K}) = P(\mathcal{K}) = \mathcal{K}$ (Tarski’s HSP(\mathcal{K}) = \mathcal{K})

Varieties and equational logic

Exercise: Show that every equational class is a variety

**Theorem (Birkhoff 1935)**

Every variety is an equational class

For a class $\mathcal{K}$ of algebras $Eq(\mathcal{K}) = \{ s = t | \mathbf{A}$ satisfies $s = t$ for all $\mathbf{A} \in \mathcal{K} \}$

An equational theory is of the form $Eq(\mathcal{K})$ for some class of algebras $\mathcal{K}$

$t[x \mapsto r]$ is the term $t$ with all occurrences of $x$ replaced by the term $r$

**Theorem (Birkhoff 1935)**

$E$ is an equational theory if and only if for all terms $q, r, s, t$

$t = t \in E \implies t = s \in E; \quad r = s, s = t \in E \implies r = t \in E$

and $q = r, s = t \in E \implies s[x \mapsto q] = t[x \mapsto r] \in E$

I.e. the rule of algebra: “replacing all $x$ by equals in equals gives equals”
Examples of equational theories and varieties

A **binar** is an algebra \((A, \cdot)\) with one binary operation \(x \cdot y\), written \(xy\)

A **semigroup** is an **associative** binar, i.e. satisfies \((xy)z = x(yz)\)

A **band** is an **idempotent** semigroup, i.e. satisfies \(xx = x\)

A **semilattice** is a **commutative** band, i.e. satisfies \(xy = yx\)

A **unital binar** is an algebra \((A, \cdot, 1)\) that satisfies \(1x = x\) and \(x1 = x\)

A **monoid** is a unital binar that is associative, i.e. a unital semigroup

\((A, \cdot, ^{-1}, 1)\) is a **group** if \(\cdot\) is associative, \(1x = x\) and \(x^{-1} \cdot x = 1\)

**Exercise:** Show that a group satisfies \(x1 = x, xx^{-1} = 1\) and \((x^{-1})^{-1} = x\)

\[\text{Hint: } x = 1x = (x^{-1})^{-1}x^{-1}x = (x^{-1})^{-1}1x^{-1}x^{-1}1x^{-1}x^{-1}x1 = 1x1 = x1\]

Posets and meet-semilattices

A **poset** \((A, \leq)\) is a set \(A\) with a partial order \(\leq\) on \(A\)

For \(S \subseteq A\) the **meet** \(\land S\) is defined by \(x \leq \land S \iff x \leq s\) for all \(s \in S\)

**Exercise:** Prove \(\land S\) is unique (if it exists; \(\land = \) greatest lower bound)

\(\land \{x, y\}\) is denoted by \(x \land y\)

**Exercise:** Prove that in a semilattice \(ab = a \land b\) for the partial order \(\leq^A\)

A **meet-semilattice** is a poset in which \(a \land b\) exists for all \(a, b\)

A meet-semilattice is **complete** if \(\land S\) exists for all nonempty subsets \(S\)

**Exercise:** Prove if \((A, \leq)\) is a meet-semilattice then \((A, \land)\) is a semilattice

An element \(a\) has a **cover** \(b\), denoted \(a < b\), if \(\{x \mid a \leq x \leq b\} = \{a, b\}\)

A **Hasse diagram** of a poset has an upward line from dot \(a\) to \(b\) if \(a < b\)

Binary relations and partial orders

\(R\) is a **binary relation on a set** \(A\) if it is a subset of \(A \times A\)

E.g. the **identity relation** \(id_A = \{(a, a) \mid a \in A\}\) is a binary relation on \(A\)

\(aRb\) means \((a, b) \in R\)

\(R\) is **reflexive** if \(xRx\) for all \(x \in A\)

\(R\) is **antisymmetric** if \(xRy\) and \(yRx\) implies \(x = y\)

\(R\) is **transitive** if \(xRy\) and \(yRz\) implies \(xRz\)

\(R\) is a **partial order** if it is reflexive, antisymmetric and transitive

For a semilattice \(A\) define \(a \leq^A b \iff ab = a\)

**Exercise:** Prove that \(\leq^A\) is a partial order on \(A\)

(Dually)nonisomorphic connected posets with \(\leq 5\) elements

**P Jipsen (Chapman), N Galatos (DU) Residuated Lattices ICLA, January, 2009 9 / 72**

**P Jipsen (Chapman), N Galatos (DU) Residuated Lattices ICLA, January, 2009 10 / 72**

**P Jipsen (Chapman), N Galatos (DU) Residuated Lattices ICLA, January, 2009 11 / 72**

**P Jipsen (Chapman), N Galatos (DU) Residuated Lattices ICLA, January, 2009 12 / 72**
Lattices

For a relation $\leq$, define the dual $\geq$ by $b \geq a \iff a \leq b$

$A^\theta = (A, \geq)$ is the dual poset of $A = (A, \leq)$

Every partial order concept has a dual, obtained by interchanging $\leq$ and $\geq$

The join $\lor$ is defined dually to the meet $\land$ ($\lor$ = least upper bound)

A join-semilattice is a poset where $a \lor b = \lor\{a, b\}$ exists for all $a, b$

A lattice is a poset that is a meet-semilattice and a join-semilattice

Note $x \leq y$ is definable by $x \lor y = y$, as well as by $x \land y = x$

Exercise: Show $A = (A, \land, \lor)$ is a lattice iff $\land, \lor$ are associative, commutative and absorbive, i.e. $x \land (x \lor y) = x$ and $x \lor (x \land y) = x$

Hint: $x \land x = x \land (x \lor (x \land y)) = x$

Examples of lattices

A lattice is complete if $\land, \lor$ exist for all subsets $S$

A lattice is bounded if it has a top element $\top$ and a bottom element $\bot$

Exercise: Show that every complete lattice is bounded

Exercise: Show any complete meet-semilattice with $\top$ is a complete lattice

The powerset $P(X)$ of all subsets of $X$ is a complete lattice with $\cap, \cup$

The collection $\Lambda^\lor$ of subvarieties of $\mathcal{V}$ is a complete lattice with $\land = \cap$

Any linear order (i.e. if $x \leq y$ or $y \leq x$ for all $x, y$) is a lattice

A lattice is distributive if $x \land (y \lor z) = (x \land y) \lor (x \land z)$ holds ($\iff$ dual)

E.g. $P(X)$ and any linear order are distributive lattices

Equivalence relations and congruences

Let $A$ be an algebra and $R$ a binary relation on $A$

$R$ is symmetric if $xRy$ implies $yRx$ (implicitly quantified)

$R$ is an equivalence relation if it is reflexive, symmetric and transitive

$R$ is a congruence on $A$ if it is an equivalence relation and $xRy$ implies $f_i(a_1, \ldots, x, \ldots, a_n) R f_i(a_1, \ldots, y, \ldots, a_n)$ (all args, $i$)

The set $\text{Con}(A)$ of all congruences on $A$ is a complete lattice with $\land = \cap$

$\bot = \text{id}_A$ and $\top = A^2$; $\text{con}(a, b) = \cap\{R \in \text{Con}(A) \mid aRb\}$

A congruence class is a set of the form $[a]_R = \{x \mid aRx\}$

$\{C_i : i \in I\}$ is a partition of $A$ if $A = \bigcup_{i \in I} C_i$ and $C_i \cap C_j = \emptyset$ or $C_i = C_j$

The set $A/R$ of all congruence classes is a partition of $A$
Homomorphic images and quotient algebras

The quotient algebra $A/R = (A/R, f_1, f_2, \ldots)$ is defined by

$$f_i([a_1]_R, \ldots, [a_n]_R) = [f_i(a_1, \ldots, a_n)]_R$$

**Exercise:** Show that $f_i$ is well-defined if and only if $R$ is a congruence

For a homom. $h : A \rightarrow B$, define the **kernel** $\ker h = \{(x, y) \mid h(x) = h(y)\}$

**Exercise:** Show that $\ker h$ is a congruence on $A$ and that

the **natural map** $[\cdot]_R : A \rightarrow A/R$ is a homomorphism

**Theorem (First Isomorphism Theorem)**

$k : A/\ker h \rightarrow h[A]$ defined by $k([a]_\ker h) = h(a)$ is an isomorphism

**Theorem (Second Isomorphism Theorem)**

If $R \subseteq S$ are congruences on $A$ and $T = \{([a]_R, [b]_R) \mid aSb\}$ then

$T \in \text{Con}(A/R)$ and $(A/R)/T \cong A/S$

Birkhoff’s Theorem says that every algebra is a subalgebra of a product of
subdirectly irreducible algebras (s.i. algebras for short)

So the s.i. algebras are **building blocks** of varieties

For an element $a$ in a poset, the **principal filter** of $a$ is $\uparrow a = \{x \mid a \leq x\}$

A subset $S$ of a poset is an **upset** if for all $a \in S$ we have $\uparrow a \subseteq S$

$S$ is **down-directed** if for all $a, b \in S$ there is a $c \in S$ with $c \leq a$ and $c \leq b$

A **filter** $F$ is a down-directed upset

An **ideal** is the dual concept of a filter, i.e. an **up-directed downset**

**Exercise:** Show that the 2-element semilattice is the only s.i. semilattice
and that the 2-element lattice is the only s.i. **distributive** lattice

Hint: Every congruence is the intersection of congruences with two blocks

Every semilattice is a subalgebra of a product of two-element semilattices

Subdirectly irreducible algebras

An algebra is **directly decomposable** if it is isomorphic to a direct product
of nontrivial algebras (happens rarely)

Let $R_j \in \text{Con}(A)$ and define $h : A \rightarrow \prod_{j \in J} A/R_j$ by $h(a)(j) = [a]_{R_j}$

**Exercise:** Show that $h$ is one-to-one if and only if $\bigcap_{j \in J} R_j = id_A$

In this case $h$ is called a **subdirect decomposition** of $A$

An element $c$ in a complete lattice is completely meet irreducible if $c = \bigwedge S$ implies $c \in S$ for all subsets $S$; equiv. if $c$ has a unique cover

$A$ is **subdirectly irreducible** if $id_A$ is completely meet irreducible in $\text{Con}(A)$

**Theorem (Birkhoff 1944)**

Every algebra $A$ has a subdirect decomposition using only subdirectly
irreducible homomorphic images of $A$

Introductory references for further background reading


Brian Davey and Hilary Priestley, "Introduction to Lattices and Order", 2nd ed, Cambridge University Press, 2002

Residuated maps

A function \( f : (A, \leq) \rightarrow (B, \leq) \) is residuated if some \( g : B \rightarrow A \) satisfies
\[
f(x) \leq y \iff x \leq g(y)
\]
for all \( x \in A, \ y \in B \)
g is called the residual of \( f \) and \( g(y) = \max \{ x \mid f(x) \leq y \} \) (if it exists)

Exercise: \( f \) preserves all existing joins and \( g \) preserves all existing meets

Exercise: Show that \( f \) is residuated \( \iff f, g \) are order-preserving and
\[
f(g(y)) \leq y \quad \text{and} \quad x \leq g(f(x))
\]
for all \( x \in A, \ y \in B \)

Exercise: If \( A, B \) are lattices then \( f \) is residuated with residual \( g \) \( \iff \)
\[
f(x \land g(y)) \lor y = y \quad \text{and} \quad x = x \land g(f(x) \lor y)
\]
Hence residuation can be expressed by equations (also in semilattices)

Heyting algebras and Boolean algebras

\[ A = (A, \land, \lor, \to, 0, 1) \] is a Heyting algebra (HA) if
\[
\bullet (A, \land, \lor, 0, 1) \text{ is a bounded lattice and}
\bullet \text{for all } a, b, c \in A, \quad a \land b \leq c \iff b \leq a \to c \quad (\land\text{-residuation})
\]
The unary negation operation \( \neg \) is defined by \( \neg x = x \to 0 \)

A Boolean algebra (BA) is a Heyting algebra that satisfies \( \neg \neg x = x \)

Exercise: Heyting algebras are distributive lattices and \( x \land \neg x = 0 \)

BAs are also defined by \( x \lor \neg x = 1 \), i.e. \( \neg \) is a complement
\( \land \text{-residuation can be written equationally} \) (hint: \( a \land \neg \) has residual \( a \to \))

BAs give algebraic semantics for classical propositional logic \( x \to y = \neg x \lor y \)

HAs give algebraic semantics for intuitionistic propositional logic

Algebras of relations

For binary relations \( R \) and \( S \) on a set \( X \), we denote by
\[
\bullet R' \text{ the complement } X^2 - R \text{ and by } R^\sim \text{ the converse } \{(y, x) \mid xRy\}
\]
\[
\bullet R \cdot S \text{ the composition } \{(x, z) \mid xR(y, z) \in S \text{ for some } z\}
\]
\[
\bullet R \setminus S = (R^\sim \cdot S')' \text{ and } S/R = (S' \cdot R^\sim)'
\]
\[
\bullet R \to S = R' \cup S
\]

Exercise: Check that
\[
\bullet (P(X^2), \cap, \cup, \to, 0, X^2) \text{ is a Boolean algebra}
\]
\[
\bullet (P(X^2), \cdot, id_X) \text{ is a monoid}
\]
\bullet for all \( R, S, T \subseteq X^2, \)
\[
R \cdot S \subseteq T \iff S \subseteq R \setminus T \iff R \subseteq T / S
\]

Relation algebras

A relation algebra is of the form \( A = (A, \land, \lor, \cdot, \setminus, /, 1, ') \) with
\[
\bullet x \to y = y' \lor y, \quad \bot = 1 \land 1' \text{ and } T = 1 \lor 1' \text{ such that}
\bullet (A, \land, \lor, \to, \bot, T) \text{ is a Boolean algebra}
\bullet (A, \cdot, 1) \text{ is a monoid}
\bullet \text{for all } a, b, c \in A,
\[
a \cdot b \leq c \iff b \leq a \setminus c \iff a \leq c / b \quad \text{(residuation)}
\]
\[
x^\sim = x \text{ and } (x \cdot y)^\sim = y' \cdot x^\sim \text{ where } x^\sim \text{ is defined as } (x \setminus 1')'
\]

Exercise: Show that \( x^\sim = (1' / x)' \) and \( x^\sim (xy)' \leq y' \)
**Residuated Lattices**

A lattice-ordered group is a lattice with an order-pres. group operation

Alternatively, a **lattice-ordered group** is an algebra \( \mathbb{L} = (L, \wedge, \vee, \cdot, \backslash, /, 1) \) such that

- \((L, \wedge, \vee)\) is a lattice
- \((L, \cdot, 1)\) is a monoid
- for all \(a, b, c \in L\),
  \[
  a \cdot b \leq c \iff b \leq a/c \iff a \leq c/b
  \]
- and \(L\) satisfies \((1/x) \cdot x = 1\)

**Exercise:** Show that \((L, \cdot, 1^{-1}, 1)\) is a group, where \(x^{-1} = 1/x = x \backslash 1\)

**Example.** The reals \(\mathbb{R}\) under the usual order, addition and subtraction

Let \(\mathbb{M} = (M, \cdot, e)\) be a monoid and \(X, Y \subseteq M\) and define

\[
X \cdot Y = \{ x \cdot y \mid x \in X \text{ and } y \in Y \}
\]

**Exercise:** Show that the powerset \(\mathcal{P}(M)\) satisfies

\[
\begin{align*}
(\mathcal{P}(M), \cap, \cup) & \text{ is a lattice} \\
(\mathcal{P}(M), \cdot, \{e\}) & \text{ is a monoid} \\
\text{for all } X, Y, Z \subseteq M,
\end{align*}
\]

\[
X \cdot Y \subseteq Z \iff Y \subseteq X \backslash Z \iff X \subseteq Z \backslash Y
\]

**Ideals of a ring**

Let \(\mathbb{R}\) be a ring with unit and let \(\mathcal{I}(\mathbb{R})\) be the set of (two-sided) ideals of \(\mathbb{R}\)

For \(I, J \in \mathcal{I}(\mathbb{R})\) define
- \(I \cdot J = \{ a_1 b_1 + \cdots + a_n b_n \mid a_i \in I, b_i \in J\}\)
- \(I \setminus J = \{ a \in R \mid Ia \subseteq J\}\)
- \(I \vee J = \{ a + b \mid a \in I \text{ and } b \in J\}\)

**Exercise:** Show that the set of ideal \(\mathcal{I}(\mathbb{R})\) is closed under \(\setminus, /\) and

\[
\begin{align*}
\mathcal{I}(\mathbb{R}), & \cap, \vee \text{ is a lattice} \\
\mathcal{I}(\mathbb{R}), & \cdot, \mathbb{R} \text{ is a monoid} \\
\text{for all ideals } I, J, K \text{ of } \mathbb{R}
\end{align*}
\]

\[
I \cdot J \subseteq K \iff J \subseteq I \setminus K \iff I \subseteq K / J
\]

**Residuated lattices**

A **residuated lattice** is an algebra \(\mathbb{L} = (L, \wedge, \vee, \cdot, \backslash, /, 1)\) such that

- \((L, \wedge, \vee)\) is a lattice
- \((L, \cdot, 1)\) is a monoid and
- for all \(a, b, c \in L\), \(ab \leq c \iff b \leq a/c \iff a \leq c/b\)

A **Full Lambek algebra** is a residuated lattice with a new constant \(0\)

In an FL-algebra, define two **linear negations** \(\sim x = x \setminus 0\) and \(- x = 0 \backslash x\)

A FL-algebra or residuated lattice is called

- **commutative** if \((L, \cdot, 1)\) is commutative (\(xy = yx\))
- **distributive** if \((L, \wedge, \vee)\) is distributive
- **integral** if it satisfies \(x \leq 1\)
- **contractive** if it satisfies \(x \leq x^2\)
- **involutive** if it satisfies \(\sim \sim x = x = \sim x\)
Properties

\[ x(y \lor z) = xy \lor xz \quad \text{and} \quad (y \lor z)x = yx \lor zx \]
\[ x(y \land z) = (x'y) \land (x'z) \quad \text{and} \quad (y \land z)/x = (y/x) \land (z/x) \]
\[ x/(y \lor z) = (x'/y) \lor (x'/z) \quad \text{and} \quad (y \lor z)x = (y'x) \lor (z'x) \]
\[ (x/y)y \leq x \quad \text{and} \quad y(y'x) \leq x \]
\[ x(y/z) \leq (xy)/z \quad \text{and} \quad (z'x)x \leq z'(yx) \]
\[ (x/y)/z = x/(zy) \quad \text{and} \quad z'(y'x) = (yz)'x \]
\[ x(y/z) = (x'y)/z \]
\[ x/1 = x = 1/x \]
\[ 1 \leq x/x \quad \text{and} \quad 1 \leq x'x \]
\[ 1 \leq x/x \quad \text{and} \quad y/x \leq (y/x)'y \]
\[ y/(y/x)'y = y/x \quad \text{and} \quad y/(x'y) \leq y \]
\[ x/(x'x) = x \quad \text{and} \quad x/x = x \]
\[ (z/y)(y/x) \leq z/x \quad \text{and} \quad (x/y)(y/z) \leq x/z \]

Multiplication is order preserving in both arguments. Each division operation is order preserving in the numerator and order reversing in the denominator.

Proofs of some of these properties

\[ x(y \lor z) \leq w \quad \iff \quad y \lor z \leq x'w \]
\[ y \leq x'w \quad \text{and} \quad z \leq x'w \]
\[ xy \leq w \quad \text{and} \quad xz \leq w \]
\[ xy \lor xz \leq w \]

\[ x/y \leq x/y \quad \Rightarrow \quad (x/y)y \leq x \]

\[ x(y/z)x \leq x(y/z) \quad \Rightarrow \quad x(y/z) \leq (xy)/z \]

\[ ([x/y]/(zy)/x) \leq x \quad \Rightarrow \quad (x/y)/z \leq x/(zy) \]

\[ [x/(zy)/](xz) \leq x \quad \Rightarrow \quad x/(zy) \leq (x/y)/z \]

\[ w \leq x/(y/z) \quad \iff \quad xw \leq y/z \]
\[ xwz \leq y \]
\[ wz \leq x'y \]
\[ w \leq (x'y)/z \]

Lattice/monoid properties

\[ (z/y)(y/x) \leq (z/y)y \leq z \quad \Rightarrow \quad (z/y)(y/x) \leq z/x \]

RL’s satisfy no special purely lattice-theoretic or monoid-theoretic property

Every lattice can be embedded in a (cancellative) residuated lattice

Every monoid can be embedded in a (distributive) residuated lattice

Congruences in groups and Boolean algebras

Recall that a congruence on an algebra \( A \) is an equivalence relation on \( A \) that is compatible with the operations of \( A \)

A subgroup \( N \) of a group \( G \) is normal if \( a \in N \) implies \( x^{-1}ax \in N \)

Congruences in groups correspond to normal subgroups:

For a congruence \( R \) on \( G \), the congruence class \([1]_R\) is a normal subgroup

Given a normal subgroup \( N \) of a group \( G \), the relation \( R_N \) is a congruence, where \((a,b) \in R_N \) iff \( a \setminus b \in N \) iff \( \{a,b,a\} \subseteq N \)

Congruences in Boolean algebras correspond to filters:

Given a congruence \( R \) on a BA, the congruence class \([1]_R \) is a filter of \( A \)

Given a filter \( F \) of a Boolean algebra \( A \), \( R_F \) is a congruence, where \((a,b) \in R_F \) iff \( a \leftrightarrow b \in F \) iff \( \{a \rightarrow b, b \rightarrow a\} \subseteq F \)
Congruences on rings and monoids

Congruences on rings correspond to (two-sided) ideals
Congruences on ℓ-groups correspond to convex ℓ-subgroups
Congruences on monoids do not correspond to any particular kind of subset
Do congruences on residuated lattices correspond to certain subsets?

Correspondence

If $R$ is a congruence on $A$ and $F$ is a normal filter of $A$ then
- $F(R) = \uparrow[1]_R$ is a normal filter of $A$ and
- $R(F) = \{(a, b) \mid a/\ell b, \ b/\ell a \in F\}$ is a congruence of $A$

Normality of $F(R)$: If $a \geq c \in [1]_R$ and $b \in A$ then
$$ba/b \geq (bc/b \wedge 1) \ R (b1/b \wedge 1) = 1$$

Compatibility of $R(F)$: If $a/\ell b, \ b/\ell a \in F$ then
$$(a \wedge c)(a/\ell b \wedge 1) \leq a(a/\ell b) \wedge c1 \leq b \wedge c \text{ so } (a \wedge c)(b \wedge c) \in F \ (\forall \text{ same})$$
$$a/\ell b \leq ca/\ell cb \in F, \ c/(a/\ell b)c \leq ac/\ell bc \in F \text{ and } a/\ell b \leq (c/a)(c/\ell b) \in F$$
$$a/\ell b \leq (a/\ell c)/(b/\ell c) \in F \text{ so } (b/\ell c)((a/\ell c)/(b/\ell c))(b/\ell c) \leq (a/\ell c)/(b/\ell c) \in F$$

The normal filter lattice is isomorphic to $Con(A)$

The normal filters of $A$ form a lattice, denoted by $N\text{Fil}(A)$

This lattice is isomorphic to the lattice $Con(A)$ via the maps $F$ and $R$

Claim: $F$ and $R$ are inverse maps
$$F = \uparrow[1]_R(F): \ a \in F \text{ implies } a \wedge 1 \in F. \text{ Also } a \wedge 1 \leq 1 \text{ implies } 1 \leq (a \wedge 1)1 \in F, \text{ hence } a \wedge 1 \in [1]_R(F).$$
$$a \in [1]_R(F) \text{ implies } a \geq b \in [1]_R(F), \text{ hence } 1 \wedge b = b \in F \text{ so } a \in F.$$ 

$$R = R(\uparrow[1]_R): \text{ If } aR([1]_R)b, \text{ then } (a/\ell b) \wedge 1 \in [1]_R \text{ since } [1]_R \text{ is convex. Hence } a \wedge b R a(a/\ell b \wedge 1) \lor b = b, \text{ and similarly } a \wedge b R a, \text{ so } aRb.$$ 

If $aRb$, then $1 = (a/\ell a \wedge b/\ell b \wedge 1)R(a/\ell a \wedge b/\ell b)R(a/\ell a \wedge b/\ell b) \wedge 1)$, hence $aR([1]_R)b$
Generation of normal filters

- The normal filter $F(X)$ of $A$ generated by $X \subseteq A$ is
  \[ \uparrow \{ \gamma_1(x_1) \gamma_2(x_2) \cdots \gamma_n(x_n) \mid n \geq 0, x_i \in X, \gamma_i \text{ iterated conjugates} \} \]
- The congruence $R(Y)$ on $A$ generated by $Y \subseteq A^2$ is
  \[ R(F(\{ a \land b, a \land 1 \mid (a, b) \in Y \})) \]

For a finite residuated lattice any filter is generated by an idempotent $\leq 1$

An element $c$ is central if $cx = xc$ for all $x \in A$

The normal filters are the ones generated by a central idempotent $\leq 1$

A is subdirectly irreducible if it has a maximal central idempotent $< 1$

Exercise: A finite RL $A \cong B \times C$ with $B, C$ nontrivial if and only if there are central idempotents $b, c < 1$ such that $b \land c = \bot$ and $b \lor c = 1$

Applications to subvarieties of residuated lattices

RL and FL denote the variety of residuated lattices and FL-algebras resp.

We view RL as the subvariety of FL axiomatized by $0 = 1$

Let’s take a look at a map of the subvariety lattices $\Lambda_{FL}$ and $\Lambda_{RL}$

Some subvarieties of FL ordered by inclusion

- D = distributive
- w = bounded by 0, 1
- e = commutative
- R = representable
- ps = pseudo
- MTL = monoidal t-norm
- BL = basic logic algebras
- x $\land y = x(x \rightarrow y)$
- HA = Heyting algebras
- GA = Gödel algebras
- MV = Multivalued algs
- x $\lor y = (x \rightarrow y) \rightarrow y$
- Π = Product algebras
- BA = Boolean algebras
- O = Trivial algebras

Some subvarieties of RL ordered by inclusion

- D = distributive
- I = integral $x \leq 1$
- C = commutative
- R = representable
- GBL = generalized BL
- x $\land y = x(x \land y)$ & mi
- GMV = generalized MV
- x $\lor y = x((x \land y) \land x)$ & mi
- BH = basic hoops
- Br = Brouwerian algebras
- WH = Wajsberg hoops
- LG = lattice-ordered groups
- N = normal valued
- O = Trivial algebras
Ultraproducts

\( \mathcal{F} \) is a filter over a set \( J \) if \( \mathcal{F} \) is a filter in \( (\mathcal{P}(J), \subseteq) \).

\( \mathcal{F} \) defines a congruence on \( \prod_{j \in J} A_j \) via \( a \equiv_F b \iff \{ j \in J | a(j) = b(j) \} \in \mathcal{F} \).

\( \prod_{j \in J} A_j/\equiv_F \) is called an ultraproduct if \( J \) is a maximal filter.

\( \mathcal{P}_u \mathcal{K} \) is the class of all ultraproducts of members of \( \mathcal{K} \).

\( \mathcal{K} \) is finitely axiomatizable if \( \mathcal{K} = \text{Mod}(E) \) for a finite set \( E \).

Exercise: Show that \( V(\mathcal{K}) \) is the smallest variety containing \( \mathcal{K} \).

Congruence distributivity and Jónsson’s Theorem

\( \mathcal{A} \) is congruence distributive (CD) if \( \text{Con}(\mathcal{A}) \) is a distributive lattice.

A class \( \mathcal{K} \) of algebras is CD if every algebra in \( \mathcal{K} \) is CD.

**Theorem (Jónsson 1967)**

If \( \mathcal{V} = V(\mathcal{K}) \) is congruence distributive then \( \mathcal{V}_{SI} \subseteq HSP_u \mathcal{K} \).

**Theorem**

If \( \mathcal{K} \) is a finite class of finite algebras and \( \mathcal{V}(\mathcal{K}) \) is CD then \( \mathcal{V}_{SI} \subseteq HS \mathcal{K} \).

If \( \mathcal{A}, \mathcal{B} \in \mathcal{V}_{SI} \) are finite nonisomorphic and \( \mathcal{V} \) is CD then \( \mathcal{V}(\mathcal{A}) \neq \mathcal{V}(\mathcal{B}) \).

\( \mathcal{V} \) is finitely generated if \( \mathcal{V} = V(\mathcal{K}) \) for some finite class of finite algebras.

**Theorem**

A finitely generated CD variety has only finitely many subvarieties.

Lattices of subvarieties

\( \mathcal{A}' = (A, f_{i_1}, f_{i_2}, \ldots) \) is a reduct of \( \mathcal{A} = (A, f_1, f_2, \ldots) \) if \( i_1 < i_2 < \cdots \) in \( \mathbb{N} \).

In this case \( \mathcal{A} \) is called an expansion of \( \mathcal{A}' \).

Exercise: If \( \mathcal{A}' \) is a reduct of \( \mathcal{A} \) then \( \text{Con}(\mathcal{A}) \) is a sublattice of \( \text{Con}(\mathcal{A}') \).

The variety of lattices is CD, so any variety of lattice expansions is CD.

Recall that for a variety \( \mathcal{V} \) the lattice of subvarieties is denoted by \( \Lambda_{\mathcal{V}} \).

**Theorem**

\( HSP_u (\mathcal{K} \cup \mathcal{L}) = HSP_u \mathcal{K} \cup HSP_u \mathcal{L} \) for any classes \( \mathcal{K}, \mathcal{L} \).

If \( \mathcal{V} \) is CD then \( \Lambda_{\mathcal{V}} \) is distributive and the map \( \mathcal{V} \mapsto \mathcal{V}_{SI} \) is a lattice embedding of \( \Lambda_{\mathcal{V}} \) into \( \mathcal{P}(\mathcal{V}_{SI}) \) (unless \( \mathcal{V}_{SI} \) is a proper class).

Size of the lattice of subvarieties

A set \( S \) is countable if it there is a one-to-one function \( f : S \rightarrow \mathbb{N} \).

The subvariety lattices of \( \text{HA} \) (Heyting algebras) and \( \text{Br} \) (Brouwerian algebras) are uncountable, hence so are \( \Lambda_{\text{FL}} \) and \( \Lambda_{\text{RL}} \).

We will

- determine the size of the set of atoms in \( \Lambda_{\text{FL}} \).
- outline a method for finding axiomatizations of certain varieties.
- give a description of joins in \( \Lambda_{\text{FL}} \).
Boolean Algebras and 2

The variety BA of Boolean algebras is generated by the 2-element alg. 2
i.e. $BA = HSP(2) = V(2)$

Proof idea: In a distributive lattice, any maximal proper congruence has
two classes: a (prime) filter and a (prime) ideal, and each pair of distinct
elements can be separated by such a congruence (the prime ideal-filter
theorem for distributive lattices)

Then show that every BA has a subdirect decomposition with copies of 2
So $BA = SP(2) = HSP(2)$

Now, $HSP_U(2) = \{2, 1\}$, hence $\text{SI} = \{2\}$

Finitely generated atoms

Consider the residuated chain $T_n$ defined to the right

We also let $Tu = uT = u$, hence $T\backslash 1 = u$

Note that $T_n$ is strictly simple, i.e. has no non-trivial
subalgebras or homomorphic images

It follows that $V(T_n)$ is an atom of $\mathbb{A}_{RL}$

Moreover, all these atoms are distinct so $\mathbb{A}_{RL}$ has
at least countably many atoms

$\mathbb{Z}^- = \{n \in \mathbb{Z} \mid n \leq 0\}$ is a cancellative RL with min, max, + as operations

Claim: If for all $x < 1$, we have $1 < 1/x$, then $L$ is an $\ell$-group.

For $a \in L$ set $x = (1/a)a$. Note that $x \leq 1$, and if $x < 1$, then
$1/x = 1/(1/a)a = (1/a)/(1/a) = 1$, cancellativity; so $x = 1$.

The construction of $\mathbb{Z}^-$ from $\mathbb{Z}$ actually works in general

The negative cone of a RL $A = (A, \wedge, \vee, \cdot, \setminus, /, 1)$ is the RL
$A^- = (A^-, \wedge, \vee, \cdot, \setminus^{A^-}, /^{A^-}, 1)$, where $A^- = \{a \in A \mid a \leq 1\}$,
$a^{A^-}b = (a^1b)^1$ and $b^{A^-}a = (b/a)^1$.

Cancellative atoms

Left cancellativity ($ab = ac \Rightarrow b = c$) is equational: $x \setminus (xy) = y$

Right cancellativity is $(yx)/x = y$

CanRL denotes the variety of (left and right) cancellative RLs

Theorem

There are only 2 cancellative atoms: $V(\mathbb{Z})$ and $V(\mathbb{Z}^-)$

Let $L \in \text{CanRL}$. For $a \leq 1$, we have $1 \leq 1/a$.

Claim: If $a < 1$ and $1/a = 1$, then the subalgebra generated by $a$ is $\mathbb{Z}^-$

Since $a < 1$, we get $a_{n+1} < a^n$, for all $n \in \mathbb{N}$, by order preservation and
cancellativity

Moreover, $a^{k+m}/a^m = a^k$ and $a^m/a^{m+k} = 1$, for all $m, k \in \mathbb{N}$
Idempotent atoms generated by chains

For $S \subseteq \mathbb{Z}$, we define

$a_i b_i = a_i$, if $i \in S$ and

$a_i b_i = b_i$, if $i \notin S$.

Although we may have

- $S \neq T$, but $N_S \cong N_T$
- $N_S \not\cong N_T$, but $V(N_S) = V(N_T)$
- $V(N_S)$ is not an atom

N. Galatos [2004] proved that there are continuum many atoms $V(N_S)$

Representable RL’s

A resideduated lattice is called representable if it is a subdirect product of linearly ordered RL’s

RRL denotes the class of representable RL’s

A linearly ordered RL satisfies the first-order formula $\forall x, y (x \geq y \text{ or } y \geq x)$, i.e., $[(\forall x, y)(1 \leq x \cdot y\text{ or } 1 \leq y \cdot x)]$

Representable Heyting algebras form a variety axiomatized by $1 = (x \rightarrow y) \vee (y \rightarrow x)$ (= Gòdel algebras)

Representable commutative RL’s form a variety axiomatized by $1 \leq (x \rightarrow y) \vee (y \rightarrow x)$

RRL is a variety axiomatized by $1 = \gamma_1(x \cdot y) \vee \gamma_2(y \cdot x)$

Goal: Given a class $K$ of RL’s axiomatized by a set of positive universal first-order formulas (PUF’s), provide an axiomatization for $V(K)$

Joins of varieties

Recall that the meet of two varieties in $\mathcal{A}_{FL}$ is their intersection

In fact, if $\mathcal{V}_1 = \text{Mod}(E_1)$, $\mathcal{V}_2 = \text{Mod}(E_2)$, then $\mathcal{V}_1 \wedge \mathcal{V}_2 = \text{Mod}(E_1 \cup E_2)$

But the join of two varieties is the variety generated by their union

Also, if $\mathcal{V}_1$ is axiomatized by $E_1$ and $\mathcal{V}_2$ by $E_2$, then $\mathcal{V}_1 \vee \mathcal{V}_2$ is usually not axiomatized by $E_1 \cap E_2$.

Goals

- Find an axiomatization of $\mathcal{V}_1 \vee \mathcal{V}_2$ in terms of $E_1$ and $E_2$
- Find situations where: if $E_1$ and $E_2$ are finite, then $\mathcal{V}_1 \vee \mathcal{V}_2$ is finitely axiomatized

Finite basis

[K. Baker 1977, B. Jónsson 1979] If $\mathcal{V}$ is a congruence distributive variety of finite type and $\mathcal{V}_{FSI}$ is strictly elementary, then $\mathcal{V}$ is finitely axiomatized.

Strictly elementary: Axiomatized by a single first-order sentence

Finitely SI: $id_A$ is not the intersection of two non-trivial congruences

Cor. For every variety $\mathcal{V}$ of RL’s, if $\mathcal{V}_{FSI}$ is strictly elementary, then the finitely axiomatized subvarieties of $\mathcal{V}$ form a lattice

Pf. For finitely axiomatized subvarieties $\mathcal{V}_1$, $\mathcal{V}_2$, $(\mathcal{V}_1 \vee \mathcal{V}_2)_{FSI} = (\mathcal{V}_1 \cup \mathcal{V}_2)_{FSI}$ is strictly elementary.

Let $\mathcal{V}_1$, $\mathcal{V}_2$ be subvarieties of RL axiomatized by $E_1$, $E_2$, respectively, where $E_1$, $E_2$ have no variables in common.

The class $\mathcal{V}_1 \cup \mathcal{V}_2$ is axiomatized by the universal closure of $(\text{AND } E_1)$ or $(\text{AND } E_2)$, over infinitary logic, which is equivalent to the set $\{\forall \gamma (\varepsilon_1 \lor \varepsilon_2) : \varepsilon_1 \in E_1, \varepsilon_2 \in E_2\}$ of positive universal first-order formulas (PUFs).
Residuated Lattices

P. Jipsen (Chapman), N. Galatos (DU)

ICLA, January 2009 55 / 72

Residuated Lattices

\( \Psi \), by RL axiomatized, relative to \( \bar{\alpha} \) is weakly join irreducible

K \( \in \) RL

Let \( \Psi \) be a class of RLs axiomatized by a set \( \Gamma \) of equations of the form

\[ \bar{\alpha} \equiv \gamma_1 \lor \cdots \lor \gamma_k = 1 \]

for \( k \geq 1 \), where \( \gamma_i \in \Gamma \) is an equation of the form

\[ \bar{\alpha} \equiv \alpha_1 \lor \cdots \lor \alpha_n \]

for some fresh variables \( \alpha_1, \ldots, \alpha_n \). Set \( \bar{\alpha} = \bigcup_{n \in \omega} \bar{\alpha} \)

Here \( \Gamma^m = \{ \pi_1 \pi_2 \cdots \pi_{\gamma_0} \mid \pi_i \in Y, \pi_0 \in \{ \lambda_1, \rho_1 \} \} \)

Proof.

Let \( A \in RL_{\bar{\alpha}} \). By congruence distributivity and Jónsson’s Lemma, \( A \in V(\Gamma) \) iff \( A \in HSP_U(\Gamma) \). Since PUFs are preserved under H, S and \( P_U \), \( A \in HSP_U(\Gamma) \) iff \( A \in K \). Finally, \( A \in K \) iff \( A \models \psi \) iff \( A \models \psi \).

Let \( \mathcal{V}_1, \mathcal{V}_2 \) be subvarieties of RL axiomatized by \( E_1, E_2 \), respectively, where \( E_1, E_2 \) have no variables in common.

The class \( \mathcal{V}_1 \cup \mathcal{V}_2 \) is axiomatized by the set of PUFs

\[ \Psi = \{ \forall \forall (1 \leq r_1 \lor 1 \leq r_2) \mid (1 \leq r_1) \in E_1, (1 \leq r_2) \in E_2 \} \]

Theorem

\[ \mathcal{V}_1 \lor \mathcal{V}_2 \text{ is axiomatized by} \]

\[ \bar{\Psi} = \{ \gamma_1(r_1) \lor \gamma_2(r_2) = 1 \mid (1 \leq r_1) \in E_1, (1 \leq r_2) \in E_2, \gamma_i \in \Gamma \} \]
Residuated Lattices

**Thm.** The variety RRL generated by all linearly ordered residuated lattices is axiomatized by the identity \( \lambda_2((x \lor y)\land x) \lor \rho_w((x \lor y)\land y) = 1 \)

**Pf.** A RL is a chain iff it satisfies \( \forall x, y (x \leq y \lor y \leq x) \), or
\[
\forall x, y (1 \leq (x \lor y) \land x) \lor 1 \leq (x \lor y) \land y).
\]
Thus, RRL is axiomatized by the identities
\[
1 = \gamma_1((x \lor y)\land x) \lor \gamma_2((x \lor y)\land y); \quad \gamma_1, \gamma_2 \in \Gamma \quad (\Gamma)
\]
So, RRL satisfies the identity
\[
\lambda_2((x \lor y)\land x) \lor \rho_w((x \lor y)\land y) = 1. \quad (\lambda, \rho)
\]
Conversely, the variety axiomatized by this identity satisfies
\[
x \lor y = 1 \Rightarrow \lambda_2(x) \lor y = 1 \quad \forall x \lor y = 1 \Rightarrow x \lor \rho_w(y) = 1. \quad (imp)
\]
By repeated applications of (imp) on (\(\lambda, \rho\)), we get (\(\Gamma\)).

**Lemma**

Let \( A \) be an \( ib\ell \)-groupoid and let \( c \in B(A) \). Then \( x \land c = xc = cx \) for all \( x \in A \), and the Boolean center is a Boolean sublattice of central idempotent elements.

**Proof.**

Suppose \( A \) is an \( ib\ell \)-groupoid and \( c \land d = 0 \), \( c \lor d = 1 \). By integrality
\[
xc \leq x \land c = (x \land c)(c \lor d) = (x \land c)c \lor (x \land c)d \leq xc \lor 0 = xc,
\]
and similarly \( cx \leq c \land x \leq cx \).

Suppose we also have \( a \land b = 0 \), \( a \lor b = 1 \). To see that \( B(A) \) is a sublattice of \( A \), it suffices to show that \( a \land c \) and \( b \land d \) are complements:
\[
(a \lor c) \land (b \land d) = (a \lor c)bd = abd \lor cbd = 0 \quad \text{and} \quad (a \lor c) \lor (b \land d) = a \lor c \lor bd = a \lor c \lor b(c \lor d) = a \lor c \lor b = 1.
\]
Now \( B(A) \) is complemented by definition, and it is a distributive lattice since \( \lor \) distributes over \( \land \), hence it is a Boolean lattice.

**The boolean center is closed under residuals**

**Lemma**

If \( A \) is a residuated \( ib\ell \)-groupoid then \( B(A) \) is also closed under the residuals, the complement of \( c \) is \( \neg c = 0/c = c\lor 0 \) and \( c\lor x = x/c = \neg c \lor x \) for all \( c \in B(A) \) and \( x \in A \).

**Proof.**

For complements \( c, d \) and any \( x \in A \) we have
\[
c\lor x = (c \lor d)(c\lor x) = c(c\lor x) \lor d(c\lor x) \leq x \lor d
\]
On the other hand \( c(x \lor d) = cx \lor cd \leq x \) implies \( x \lor d \leq c\lor x \).

Hence \( c\lor x = d \lor x \), and for \( x = 0 \) we obtain \( \neg c = c\lor 0 = d \).

Therefore \( c\lor x = \neg c \lor x \) for all \( x \in A \)

The results for \( / \) follow similarly.
For an \(ib(r)\)-\(\ell\)-groupoid \(A\) and an element \(c \in B(A)\), define

the \textit{relativized subalgebra} \(A_c = \downarrow c\) with unit \(1^{A_c} = c\), operations \(\land, \lor, \cdot\) restricted from \(A\),

and \(a \land b = (a \land A b) \land c\), \(a / b = (a / A b) \land c\) for all \(a, b \in \downarrow c\).

\textbf{Lemma}

For any \(ib(r)\)-\(\ell\)-groupoid \(A\) and any \(c \in B(A)\), the relativized subalgebra \(A_c\) is an \(ib(r)\)-\(\ell\)-groupoid.

\textbf{Proof.}

By the first lemma, \(x \land c = xc = cx\), so \(A_c\) has \(c\) as a unit and is closed under \(\land, \lor, \cdot\), hence it is an \(ib\)-\(\ell\)-groupoid.

If \(A\) has residuals then for all \(a, b, x \in A_c\) we have

\[ax \leq b \iff x \leq^A a \land A b \text{ and } x \leq^A c,\]

whence \(a \land b = (a \land A b) \land c\), and similarly \(a / b = (a / A b) \land c\).

\textit{Complements in FL\textsubscript{w} give direct decompositions}

\textbf{Theorem}

If \(A\) is an FL\textsubscript{w}-algebra and if \(c, d \in B(A)\) are complements then

\[A \cong A_c \times A_d\]

\textbf{Proof.}

Consider the map \(h : A \to A_c \times A_d\) defined by \(h(a) = (ac, ad)\).

The preceding lemma shows that \(h\) is a homomorphism, and

\(h\) has an inverse given by \((x, y) \mapsto x \lor y\)

since \(ac \lor ad = a(c \lor d) = a1 = a\) and

for \(x \leq c, y \leq d\) we have

\[(x \lor y)c, (x \lor y)d = (xc \lor yc, xd \lor yd) = (x, y)\]

\textit{Direct decompositions imply complements}

Conversely, any direct decomposition of an \(ib(r)\)-\(\ell\)-groupoid is obtained in this way, since the elements \((0, 1), (1, 0)\) are complements.

\textbf{Corollary}

A FL\textsubscript{w}-algebra is directly indecomposable iff its Boolean center contains only the elements \(\{0, 1\}\).

The structure of directly indecomposables can be further analyzed by using \textit{subdirect decompositions}

However, we now consider a \textit{poset product} that can even be used to decompose subdirectly irreducible algebras...
The **poset product** uses a partial order on the index set to define a subset of the direct product. Specifically, let \( X = (X, \leq) \) be a poset, and assume \( \{ A_i \mid i \in X \} \) is a family of algebras that have two constant operations denoted 0, 1.

The poset product of \( \{ A_i \mid i \in X \} \) is

\[
\prod_X A_i = \{ f \in \prod_{i \in X} A_i \mid f(i) = 0 \text{ or } f(j) = 1 \text{ for all } i < j \text{ in } X \}
\]

If \( X \) is an **antichain** then the poset product is the same as the direct product.

If \( X \) is a **chain** and the \( A_i \) are ordered, then the poset product is the (amalgamated) ordinal sum of the \( A_i \).

We now generalize the poset sum decomposition result of [J., Montagna 2006] from finite GBL-algebras to certain FL\(_w\)-algebras.

**Theorem**

Consider a FL\(_w\)-algebra \( A \) with a finite subalgebra \( C \) such that \( C \subseteq I_A \), and let \( X \) be the dual of the partially ordered set of completely join irreducible elements of \( C \).

For \( c \in X \), let \( c_* \) denote the unique lower cover of \( c \) in \( C \).

If \( A_c = \downarrow c_* \oplus [c_*, c] \) for all \( c \in X \) then \( A \cong \prod_X [c_*, c] \).

The condition \( A_c = \downarrow c_* \oplus [c_*, c] \) is actually satisfied for many GBL-algebras.

For an \( \ell\)-groupoid \( A \) define \( I_A = \{ c \in A \mid c \wedge x = cx = xc \text{ for all } x \in A \} \).

Note that \( \wedge \) distributes over \( \vee \) in \( I_A \), but \( I_A \) need not be a subalgebra of \( A \).

A **GBL-algebra** is a residuated lattice that satisfies

\[
x \leq y \Rightarrow x = (x/y)y = y(y \setminus x)
\]

[J., Montagna 2006] prove that for bounded GBL-algebras, \( I_A \) is a subalgebra, hence a Heyting algebra contained in \( A \), and \( B(A) \) is the subalgebra of complemented elements of \( I_A \).

For MV-algebras \( I_A = B(A) \).

**Lemma**

Let \( A \) be a FL\(_w\)-algebra and let \( a, b \in I_A \) with \( a \leq b \). Then the interval \( [a, b] = \{ x \in A \mid a \leq x \leq b \} \) is a FL\(_w\)-algebra, with

\[
0 = a, 1 = b, \wedge, \vee, \cdot \text{ inherited from } A, \text{ and } x \setminus y = (x \setminus A y) \wedge b, x / y = (x \setminus A y) \land b.
\]

If \( A \) is a GBL-algebra, then so is \( [a, b] \).

A GBL-algebra is **normal** if every filter is a normal filter.

**Theorem (J., Montagna)**

A Blok-Ferreirim decomposition for GBL-algebras: Every subdirectly irreducible normal integral GBL-algebra decomposes as the ordinal sum of an integral GBL-algebra and a linearly ordered integral GMV-algebra.

A residuated lattice is **\( n \)-potent** if it satisfies \( x^{n+1} = x^n \).

[J., Montagna] prove that any \( n \)-potent GBL-algebra is commutative, hence normal, so e.g. any finite GBL-algebra is commutative.

**Corollary**

Suppose \( A \) is an integral normal GBL-algebra such that \( I_A \) is finite. Then \( A \) is isomorphic to a poset product of GMV-algebras.
Residuated lattices have **many motivating examples** from algebra and logic

Residuation has powerful consequences for the generation of congruences

The structure theory of residuated lattices is still developing

Positive universal formulas can be translated to equations

For certain varieties (e.g. GBL) the structure theory is even **better behaved**

There are many further interesting results to be discovered

---

**Bibliography I**


---

**Bibliography II**


N. Galatos and H. Ono. *Cut elimination and strong separation for non-associative substructural logics*, manuscript.


---

**Bibliography III**


