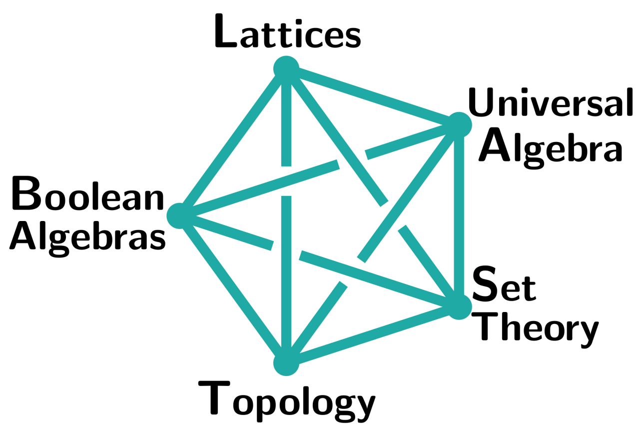
BLAST 2022

Chapman University
8–12 August, Orange, CA



— Book of Abstracts —

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Acknowledgements

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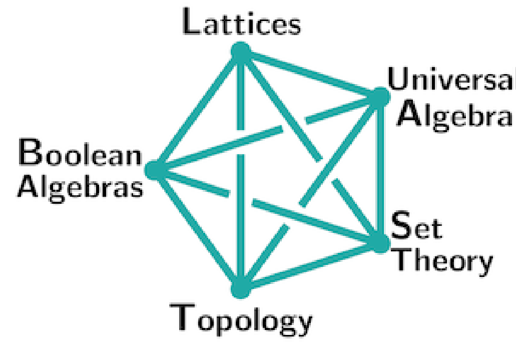
BLAST 2022



Schedule (check the webpage in case of minor changes)

The talks are held in the Beckman Hall **BK 104** and in **BK 102**.
Registration is in BK 104. Coffee and snacks are available there as well.
Invited and tutorial talks are 50-55 minutes with 5-10 minute turnaround.
Contributed talks are 25 minutes (including question time) with 5 minute turnaround.
For contributed talks, **Session A is in BK 104** and **Session B is in BK 102**.

All times in Pacific Daylight Time (UTC-7)	**Monday** August 8 Session A	Session B	**Tuesday** August 9 Session A	Session B	**Wednesday** August 10 Session A	Session B	**Thursday** August 11 Session A	Session B	**Friday** August 12 Session A	Session B
8^{30} - 9^{00}	Registration/Coffee		Coffee		Coffee		Coffee		Coffee	
9^{00} - 10^{00}	Lapenta		Mazari-Armida		Fussner		Ramanayake		Ighedo	
10^{00} - 10^{30}	Coffee		Coffee		Coffee		Coffee		Coffee	
10^{30} - 10^{55}	Yamamoto	Harding	Madden	Zhang	Meredith	Kurz	Romanowska	De La Pava Castro	de Ceglie	Y. Zhuchok
11^{00} - 11^{25}	Mayr	McDonald	Massas	Darnel	A. Zhuchok	Evans	Smith	B. Wynne	Hart	
11^{30} - 12^{00}	Hart		Panagiotopoulos		Hart		Panagiotopoulos			
12^{00} - 12^{30}									Sobocinski	
12^{30} - 2^{00}	Lunch time		Lunch time		Conference Picture Lunch time		Lunch time			
2^{00} - 2^{30}	Panagiotopoulos		Bergman				Bergman			
2^{30} - 3^{00}					Afternoon outing and conference dinner at Heisler Park **Laguna Beach** Beckman Hall pick-up 2:30 pm ALO hotel pick-up 3:00 pm					
3^{00} - 3^{25}	P. Wynne	Kornell	Jack	Galatos				Semrl		
3^{30} - 4^{00}	Coffee		Coffee				Coffee			
4^{00} - 4^{25}	Wires	Ball	Bloomfield	Jipsen			Mozo Carollo	Schmid		
4^{30} - 5^{30}	Baez		Chornomaz				Barriga-Acosta			
5^{30} - 8^{30}	Reception in the **Faculty Athenaeum**									

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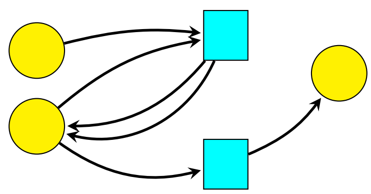
Contributed Talks **15**

Invited Lectures

Petri Nets as a Source of Mathematical Structures

John C. Baez
University of California, Riverside, USA
baez@math.ucr.edu

A Petri net has a set of **places**, drawn here as circles, and a set of **transitions**, drawn as squares:



Each transition goes from a finite formal sum of places to a finite formal sum of places. Petri nets are used in computer science, chemistry and other disciplines to describe processes where finite collections of entities of different kinds turn into other such collections [2]. Here we explore Petri nets as a source of interesting categories, monoids, posets and other structures.

Mathematically, a Petri net P can be seen as a way of presenting a free "commutative monoidal category" FP: that is, commutative monoid object in Cat, which can be seen as symmetric monoidal category of an especially strict sort. We can then extract other structures from FP. First, we can turn any category into a preorder, or by taking a skeleton, a poset. This process turns FP into partially ordered commutative monoid where $x \leq y$ iff there is a morphism in FP from the object x to the object y. The question of explicitly computing this partial order for a finite Petri net is a famous problem in computer science: the "reachability problem" [2, Sec. 25.1].

Next, given a partially ordered commutative monoid, its posets of downsets (ordered by inclusion) forms a commutative quantale. This can be seen as model of a fragment of linear logic where there are two notions of "and": the cartesian & and a noncartesian $\otimes$. The interpretation of these in terms of "the logic of resources" is especially vivid for commutative quantales arising from Petri nets.

Alternatively, we can take any partially ordered commutative monoid and construct a commutative monoid by modding out by the smallest congruence containing $\leq$. This amounts to promoting inequalities to equations. Thus, Petri nets give presentations of commutative monoids. Any presentation of commutative monoids arises this way.

When we label the transitions of a finite Petri net with numbers, we can obtain further interesting structures. If we use complex numbers we obtain commutative algebra of a special sort, which is the ring of regular functions on an "affine toric variety". Every affine toric variety arises this way. If we use positive real numbers we obtain a system of ordinary differential equations called the "rate equation", important in chemistry [2, Chap. 2]. In some cases the steady states of this equation have an interesting relation to the aforementioned toric variety [4]. Labeling the transitions of a Petri net with real numbers also gives an operator on the Fock space of the Hilbert space having the species as an orthonormal basis; this connects the category FP to Feynman diagrams in quantum field theory [2, Sec. 6.3].

In a bit more detail, let $\mathsf{CommMon}$ be the category of commutative monoids and monoid homomorphisms. Let $J\colon \mathsf{Set} \to \mathsf{CommMon}$ be the free commutative monoid functor, that is, the left adjoint of the forgetful functor $K\colon \mathsf{CommMon} \to \mathsf{Set}$. Let $\mathbb{N}\colon \mathsf{Set} \to \mathsf{Set}$ be the composite KJ. Concretely, elements of $\mathbb{N}[S]$ can be written as finite formal sums of elements of S.

While there are various definitions of "Petri net" in the literature, reviewed in [1], here we use this:

Definition. Define a **Petri net** to be a pair of functions of the following form:

$$T \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{N}[S].$$

We call T the set of **transitions**, S the set of **places**, s the **source** function and t the **target** function.

Definition. A **Petri net morphism** from the Petri net $s, t\colon T \to \mathbb{N}[S]$ to the Petri net $s', t'\colon T \to \mathbb{N}[S']$ is a pair of functions $(f\colon T \to T', g\colon S \to S')$ such that the following diagrams commute:

$$\begin{tikzcd} T \arrow[r, "s"] \arrow[d, "f"'] & \mathbb{N}[S] \arrow[d, "\mathbb{N}[g]"] \\ T' \arrow[r, "s'"] & \mathbb{N}[S'] \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "t"] \arrow[d, "f"'] & \mathbb{N}[S] \arrow[d, "\mathbb{N}[g]"] \\ T' \arrow[r, "t'"] & \mathbb{N}[S']. \end{tikzcd}$$

Let Petri be the category of Petri nets and Petri net morphisms.

Definition. A **commutative monoidal category** is a commutative monoid object internal to Cat, or equivalently, a strict symmetric monoidal category where all the symmetries $S_{x,y}\colon x \otimes y \to y \otimes x$ are identity morphisms. Let CMC be the category of commutative monoidal categories and symmetric strict monoidal functors.

Theorem. *There is a left adjoint functor*

$$F\colon \mathsf{Petri} \to \mathsf{CMC}$$

Proof. More details are provided in [3, Lem. 9], and this is a special case of [5, Thm. 5.1]. ☐

Starting from the commutative monoidal category FP we can construct a partially ordered commutative monoid, or a commutative quantale, or a commutative monoid, by applying further left adjoints.

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On the Weak Pseudoradiality of CSC Spaces

Hector Barriga-Acosta   Alan Dow

University of North Carolina at Charlotte, Charlotte

hbarriga@uncc.edu   adow@uncc.edu

Sequentially compact spaces are topological spaces in which the closure of any subset is generated by the limit points of convergent (countable) sequences within the subset. The collection of points generated by limit points of such sequences from a set A is called the *radial closure* of A. If we extend this notion (that is, by letting the sequences have any regular length) we call the spaces *pseudoradial*. A space is *weakly pseudoradial* if the radial closure of every countable set is closed. In this talk we'll see how elementary submodels and convergent sequences interact, and how this interaction behaves in some forcing extension. In this direction, we explore when compact sequentially compact spaces are weakly pseudoradial, and we'll show a model by a matrix forcing that gives an example of a compact sequentially compact space which is NOT weakly pseudoradial.

On SSP = RC Conjecture

Bogdan Chornomaz

Vanderbilt University, Nashville, TN, USA

markyz.karabas@gmail.com

A set family $\mathcal{F}$ over a base set X of size n *shatters* $Y \subseteq X$ iff for all $Z \subseteq Y$ there is $F \in \mathcal{F}$ such that $Y \cap F = Z$. The family of sets shattered by $\mathcal{F}$ is denoted by $\mathit{Str}(\mathcal{F})$. A famous Sauer–Shelah–Perles (SSP) lemma then says that $\mathcal{F}$ shatters at least as many elements as it has, that is, $|\mathcal{F}| \leq |\mathit{Str}(\mathcal{F})|$. It is perhaps more known in its uniform version: If $\mathcal{F}$ does not shatter any $(d+1)$-set, that is, if its VC-dimension is at most d, then

$$|\mathcal{F}| \leq \binom{n}{0} + \cdots + \binom{n}{d}.$$

The definition of shattering can be easily reformulated in terms of lattices, namely, for a lattice L and $F \subseteq L$, F *shatters* $y \in L$ iff for all $z \leq y$ there is $f \in F$ such that $y \wedge f = z$. Again, $\mathit{Str}(F) \subseteq L$ is the set of all elements shattered by F. It is trivial to show that $\mathit{Str}(F)$ is an order-ideal. We say that a lattice L satisfies the SSP property if the conclusion of SSP lemma holds for it; that is, if $|F| \leq |\mathit{Str}(F)|$ for every $F \subseteq L$.

The SSP lemma thus states that every boolean lattice is SSP. In [3], Babai and Frankl generalized this by proving that any lattice with a nonvanishing Möbius function (NMF) satisfies SSP. This includes, for example, all geometric (atomic and semimodular) lattices. Their proof is very short, elegant, and obviously non-sharp, that is, it is straightforward to construct a lattice satisfying SSP, but not NMF. It is then natural to ask for a characterization of SSP lattices, if, indeed, there is one.

A bridge to our conjecture can be made by observing that a most simple example of a non-NMF lattice is a three-element interval. Consecutively, any lattice containing a three-element interval is non-NMF as well, and, as it turns out, also will not satisfy SSP. On top of it, such lattices have a very nice structural local-to-global characterization by Björner [1]: They turn out to be precisely *relatively-complemented* (RC) lattices. We thus have SSP⇒RC, and, in [2], we conjectured that this is actually equality, that is, that every relatively complemented lattice satisfies SSP.

This talk is based on a preprint of the same name where we are trying to prove the SSP = RC conjecture. We are far from succeeding, and our main result in this description is as follows:

Theorem 1. *In an RC lattice L every set F, whose antichain of minimal non-shattered elements has at most three elements, satisfies SSP inequality, that is, $|F| \leq |\mathit{Str}(F)|$.*

This is a rather modest result; the statement itself is more telling about our approach than of anything else. But it is a modest result in the right direction: RC⇒SSP would follow if, as we hope, we manage to change three to infinity. It is noteworthy that there are several simple approaches to the proof of the original SSP lemma. Sauer used a simple inductive argument; Frankl, and then Bollobás and Radcliffe, used the shifting technique; finally, the aforementioned result of Babai and Frankl uses the dimensionality argument in linear algebra. All of these approaches found applications to different combinatorial problems, for example, to dealing with Erdős matching conjecture, and to Frankl–Wilson and Ray-Chaudhuri–Wilson theorems. All of these approaches turn out to be quite inapplicable to the conjecture at hand. Our motivation, apart from the minimalistic beauty of the conjecture itself, is thus that should we prove it, the methods we develop might prove useful outside of their immediate goal.

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Representations by Antichain Labelings

Wesley Fussner

Mathematical Institute, University of Bern, Switzerland

wesley.fussner@unibe.ch

It is well known that each Heyting algebra is isomorphic to an algebra of monotone functions from a poset $(X, \leq)$ into the two-element totally ordered set $\{0, 1\}$. This fact has deep consequences in intuitionistic logic, where it connects algebraic semantics (Heyting algebras) to relational semantics (Kripke frames). In that setting, X is thought of as a set of 'possible worlds' or 'alternatives' and the order $\leq$ is thought of as an 'accessibility relation' connecting these possible alternative states. The values 0 and 1 are thought of as truth values in the two-element Boolean algebra. By replacing the two-element Boolean algebra by a richer algebraic structure—such as a totally ordered MV-algebra—one obtains relational semantics that can interpret more complicated non-classical logics as well as a more powerful representation theory, capable of realizing many generalizations of Heyting algebras. However, here monotone functions are no longer sufficient to obtain natural representations and must be replaced by a different family of maps.

The correct family of functions for this purpose are *antichain labelings*, and were first identified in a line of papers by P. Jipsen and F. Montagna. An antichain labeling amounts to an assignment to each point x in a poset $(X, \leq)$ of a value in some ordered algebraic structure with least element 0 and greatest element 1. The defining characteristic of antichain labelings is that the points of $(X, \leq)$ assigned to values outside of $\{0, 1\}$ form an antichain, making this family of functions a concrete and easily-handled generalization of monotone functions.

In this talk, we discuss representations of ordered algebraic structures as algebras of antichain labelings, focusing in particular on (bounded, commutative, and integral) residuated lattices. We concentrate on representations by antichain labelings that are valued in simple residuated lattices, directly generalizing the classical representation of Heyting algebras using the two-element Boolean algebra. Our representation theorems generalize previous representations of Jipsen and Montagna that were confined to the divisible case. We also sketch some of the limitations of this kind of representation theory, and give some logical applications to relational semantics and translations in the spirit of the Gödel-McKinsey-Tarski translation. This talk discusses separate joint work with both P. Jipsen and W. Zuluaga Botero.

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A Journey into Menger-type Properties in Locales

Oghenetega Ighedo[1] Themba Dube[1] Tilahun E. Bayih[2]

[1]University of South Africa, Pretoria

[2]Addis Ababa University, Ethiopia

ighedo@unisa.ac.za dubeta@unisa.ac.za tilahun.esayiyas@aau.edu.et

Following terminology in spaces, which was introduced in [1], we say a frame L is *Menger* if for every sequence $(\mathcal{C}_n)$ of open coverings of L, there exists, for each n, a finite $\mathcal{D}_n \subseteq \mathcal{C}_n$ such that $\bigcup_{n\in\mathbb{N}}\mathcal{D}_n$ is a covering of L, by a *covering* of L we mean a collection of sublocales of L whose join (calculated in the sublocale lattice $\mathcal{S}(L)$) equals L. In the talk, I will characterize Menger locales and their weaker variants, and give some localic results with no topological counterparts. Transference of some of these properties via localic maps will also be discussed.

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Generalizing the Baker–Beynon Duality, from `Max` to `Spec`

Serafina Lapenta

University of Salerno, Italy

slapenta@unisa.it

Let G be a lattice ordered Abelian group (henceforth, just ℓ-group), the set of prime ideals (=prime convex ℓ-subgroups) of G, endowed with the hull-kernel topology is called the *spectrum* of G and is denoted $\mathtt{Spec}\,G$. Spectra of ℓ-groups have received much attention in the past. It is know that they are *generalised spectral spaces* —i.e. T_0, sober, and with a basis of compact open sets— and recently, in [4], it was proved that the above properties characterise the second-countable spectra of ℓ-groups; in the same paper it was proved that there cannot be any first order axiomatisation of the distributive lattices that are dual to spectra of ℓ-groups.

It was also observed that spectra of ℓ-groups do not retain enough information to characterise the ℓ-group they come form; e.g., $\mathbb{Z}$ and $\mathbb{R}$ have the same spectrum (a single point). In this talk we will see how it is possible to generalize the well known Baker–Beynon duality in such a way that allows to attach further information on `Spec` so to be able to reconstruct the original ℓ-group, up to isomorphism.

Baker–Beynon duality is a fundamental result in the theory of abelian lattice-ordered groups (ℓ-groups, for short) and vector lattices. In [1, 2], the category of finitely-presented vector lattices is proved to be equivalent to the one of *polyhedral cones* and piecewise (homogeneous) linear maps among them —the case of ℓ-groups being slightly more complicated, as it involves polyhedral cones with *rational vertices* and maps with *integer* coefficients.

In [3] the authors propose a general framework in which many dualities, including Baker–Beynon duality, can be set. One starts with picking an arbitrary object in a category with some mild properties. The object induces a contravariant adjunction with another category to be thought of as category of *spaces*. Then a number of results, which are parametric on the arbitrary choice of the object, help characterise the fixed points of the adjunction, i.e., the ones for which the adjunction restricts to a duality. Baker–Beynon duality can be obtained from this framework setting the fixed object to be $\mathbb{R}$ and restricting to finitely generated objects. In this case the adjunction will fix only the Archimedean vector lattices (or ℓ-groups), as they are exactly the subdirect products of $\mathbb{R}$. However, if one chooses a suitable ultrapower of $\mathbb{R}$, many more objects will be left fixed by the adjunction. The reason is to be found in the following result, which is an easy consequence of quantifier elimination for vector lattices and divisible ordered groups, respectively.

Theorem 1. *For every cardinal α there exists an ultrapower of $\mathbb{R}$ on an α-regular ultrafilter, in which all linearly ordered vector lattices of cardinality smaller than α embed. The same is true for linearly ordered groups.*

In greater detail, let V be taken to be either the variety of ℓ-groups or Riesz spaces, α is a fixed cardinal and $\mathcal{U}$ denotes invariably the α-regular ultrapower of $\mathbb{R}$ —in the appropriate language— given by Theorem 1. We denote by $\mathcal{F}_\kappa$ the free κ-generated algebra in V. Following the general framework of [3], for any $T \subseteq \mathcal{F}_\kappa$ and $S \subseteq \mathcal{U}^\kappa$, we define the following operators:

$$\mathbb{V}_{\mathcal{U}}(T) = \{x \in \mathcal{U}^\kappa \mid t(x) = 0 \text{ for all } t \in T\}, \qquad \mathbb{I}_{\mathcal{U}}(S) = \{t \in \mathcal{F}_\kappa \mid t(x) = 0 \text{ for all } x \in S\}.$$

The operators $\mathbb{V}_{\mathcal{U}}$ and $\mathbb{I}_{\mathcal{U}}$ form a Galois connection. Upon defining the appropriate notion of arrows between spaces, $\mathbb{V}_{\mathcal{U}}$ and $\mathbb{I}_{\mathcal{U}}$ can be lifted to a contravariant adjunction. A function $f : \mathcal{U}^n \to \mathcal{U}$ is called *definable* if there exists a term t in the language on V such that $f(p) = t(p)$ for all $p \in \mathcal{U}^n$. The definition easily generalises to functions from $S \subseteq \mathcal{U}^\mu$ into $S' \subseteq \mathcal{U}^\nu$, with μ and ν cardinals. Let $\mathsf{G}_{\mathsf{def}}$ be the category of subsets of $\mathcal{U}^\kappa$, with κ ranging among all cardinals, and definable maps among them.

The Galois connection induces functors between $\mathsf{G}_{\mathsf{def}}$ and V as follows. For any subset $S \subseteq \mathcal{U}^\kappa$ and for any algebra in V (assumed to be presented in the form $\mathcal{F}_\kappa/J$),

$$\mathcal{I}(S) = \mathcal{F}_\kappa/\mathbb{I}_{\mathcal{U}}(S), \quad \mathcal{V}(\mathcal{F}_\kappa/J) = \mathbb{V}_{\mathcal{U}}(J).$$

We omit the definition of $\mathcal{I}$ and $\mathcal{V}$ on arrows, as it is more technical and not necessary in this context. By [3, Corollary 4.8] the functors $\mathcal{V}$ and $\mathcal{I}$ form a contravariant adjunction.

The fixed points of the adjunction easily correspond to the fixed points of the compositions of the operators $\mathbb{V}_{\mathcal{U}}$ and $\mathbb{I}_{\mathcal{U}}$. Regarding the algebraic side, [3, Theorem 4.15] guarantees that the ideals J for which $\mathbb{I}_{\mathcal{U}} \circ \mathbb{V}_{\mathcal{U}}(J) = J$ holds are exactly the ones that can be obtained as intersections of ideals of the form $\mathbb{I}_{\mathcal{U}}(\{a\})$ for some $a \in \mathcal{U}$. Furthermore, [3, Theorem 4.15] implies that an ideal of $\mathcal{F}_\kappa$ has the form $\mathbb{I}_{\mathcal{U}}(\{a\})$ if and only if the quotient over it embeds in $\mathcal{U}$. Since both vector lattices and ℓ-groups are subdirect products of linearly ordered ones, an application of Theorem 1 proves that, up to the cardinality α, all the objects in the algebraic side of the adjunction are left fixed.

As per the fixed points in $\mathsf{G}_{\mathsf{def}}$, it is readily seen that they are the closed subspaces of $\mathcal{U}^\kappa$ under a Zariski-like topology whose closed sets are the ones of type $\mathbb{V}_{\mathcal{U}}(T)$, for T ranging among subsets of $\mathcal{F}_\kappa$.

Summing up, if V_α denotes the full subcategory of V whose objects have cardinality smaller than α, we obtain the following duality theorem.

Theorem 2. *There is a dual equivalence between V_α and the full subcategory K given by the closed objects in $\mathsf{G}_{\mathsf{def}}$.*

In addition to describing the dual categories to the classes of *all* Riesz spaces and ℓ-groups, Theorem 2 also enables the use of tools of non-standard analysis in the study of these structures from the dual, topological, point of view.

Indeed, it is easy to see that the topology on $\mathcal{U}^\kappa$ is not T_0, however it is sober and has a basis of compact open sets. Moreover it is compact only because the origin point O belongs to all closed sets. For any ultrapower $\mathcal{U}$ let $\mathcal{U}^k_o$ be the largest T_0 quotient of $\mathcal{U}^k \setminus \{O\}$. With these definitions, one of the main results is the following.

Theorem 3. *For any k, the space $\mathcal{U}^k_o$ is a* generalized spectral space*. Moreover for any ℓ-group or Riesz space A, there exists a cardinal k and an ultrapower $\mathcal{U}$ of $\mathbb{R}$, such that* $\mathtt{Spec}\,A$ *is dense in a closed subspace of $\mathcal{U}^k_o$. In particular, if $A \simeq \mathcal{F}_\kappa/J$, then* $\mathtt{Spec}\,A$ *is dense in $\mathbb{V}_{\mathcal{U}}(J)$.*

Finally, the following representation theorem holds.

Theorem 4. *Any ℓ-group or Riesz space $A \simeq \mathcal{F}_\kappa/J$ is isomorphic to the algebra of definable maps from* $\mathtt{Spec}(A)$ *to* $\mathcal{U}$*, with operations defined pointwise.*

Recall that definable maps are, in fact, piecewise linear functions in the case of ℓ-groups and piecewise linear functions with integer coefficients in the case of Riesz spaces. Hence, A is represented by restricting to (the homeomorphic image of) $\mathtt{Spec}\,A$ the elements of the free object.

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Universal Models for Classes of Abelian Groups for Purity

Marcos Mazari-Armida

Department of Mathematics, University of Colorado Boulder, USA

Marcos.MazariA@colorado.edu

The search for universal models began in the early twentieth century when Hausdorff showed that there is a universal linear order of cardinality $\aleph_{n+1}$ if $2^{\aleph_n} = \aleph_{n+1}$, i.e., a linear order U of cardinality $\aleph_{n+1}$ such that every linear order of cardinality $\aleph_{n+1}$ embeds in U. In this talk, we will study universal models in classes of abelian groups with respect to pure embeddings.

We will present a complete solution below $\aleph_\omega$ to Problem 5.1 in page 181 of *Abelian Groups* by László Fuchs, which asks to find the cardinals λ such that there is a universal abelian p-group for purity of cardinality λ. The solution presented will use both model-theoretic and set-theoretic ideas. The talk will be mostly based on [1], but we will mention a recent result which is joint work with Ivo Herzog.

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Algorithmic Complexity of Substructural Logics Through Bad Sequences

Revantha Ramanayake

Bernoulli Institute, University of Groningen, Netherlands

d.r.s.ramanayake@rug.nl

We enter the world of substructural logics by considering logical connectives that lack some familiar structural properties of classical propositional logic such as commutativity ($a \cdot b \to b \cdot a$), contraction ($a \to a \cdot a$), or weakening ($a \cdot b \to a$). In terms of their algebraic models, this is a move from boolean algebras to more general residuated structures. The absence of structural properties makes it possible for substructural logics to model resource-awareness (e.g. an ability to count) and this expressiveness means that the algorithmic complexity of deciding validity in certain substructural logics may be fantastically high, or even undecidable.

Lazić and Schmitz [2] recently presented a rather general methodology for upper bounding the complexity of decision problems in suitable transition systems through an analysis of the lengths of bad sequences of transitions, and they identified a sufficient condition on ideal decompositions that yields sharper bounds. This talk will examine such upper bounding arguments specialised to the case of substructural logics.

A sequence $(a_i)_{i\in\mathbb{N}}$ over a quasi-ordered set $(X, \preceq)$ is *good* if there exists $i < j$ such that $a_i \preceq a_j$ otherwise it is *bad*; it is a well-quasi-ordering (wqo) if every infinite sequence is good. For example, $(\mathbb{N}^d, \leq)$ with $\leq$ as the usual component-wise ordering is a wqo. Moreover if $(X, \preceq)$ is a wqo then so is $(\mathcal{P}_f(X), \preceq_{\mathbb{H}})$ where $\mathcal{P}_f(X)$ denotes the finite powerset and $U \preceq_{\mathbb{H}} V$ iff for every $u \in U$ there exists $v \in V$ such that $u \preceq v$ (this is known as the majoring ordering). An equivalent definition of a wqo $\preceq$ is that every descending chain (wrt $\subseteq$) of $\preceq$-downsets stabilises.

Our starting point is a formal proof system defined over the quasi-ordered set $(X, \preceq)$ of statements of substructural logic. A concrete illustration from a proof theoretic perspective is the set of *sequents* where the ordering is contractibility. The proof system $\mathcal{C}$ consists of a set of rules of the form $S \to x$ such that $S\cup\{x\} \subseteq_f X$. Here $\subseteq_f$ denotes the finite subset relation. Read $S \to x$ as the rule that obtains the conclusion x from the set S of premises. If $S = \emptyset$ then the rule is said to be axiom. We suppose that the proof system is *finitary* in the sense that $\{x|S \to x \in \mathcal{C}\}$ is finite for every $S \subseteq_f X$, and that its rules are *compatible* with $\preceq$. A *proof of x from assumptions* S (the existence of such a proof is denoted by $S \vdash x$) is a finite tree rooted at x with its leaves in S such that $S_y \to y \in \mathcal{C}$ for every vertex y with children S_y. We write $S \vdash^k x$ $(k > 0)$ to mean that there exists such a tree of height $\leq k$, and $S \vdash^0 x$ to mean $x \in S$.

We want to obtain an upper bound on the algorithmic complexity of deciding $\emptyset \vdash x$ (i.e. does x have a proof with no assumptions?). This is the derivation problem for the logic; in terms of its algebraic models it concerns the equational theory of the corresponding variety of algebras.

Consider $\mathcal{D}_0 \supseteq \mathcal{D}_1 \supseteq \ldots$ where each $\mathcal{D}_k = \{S|\forall y \succeq x.S \not\vdash^k y\} \subseteq \mathcal{P}_f(X)$ consists of those sets from which (no element above) x cannot be proved with height $\leq k$. From compatibility it follows that each $\mathcal{D}_k$ is a $\preceq_{\mathbb{H}}$-downset. Since $(\mathcal{P}_f(X), \preceq_{\mathbb{H}})$ is a wqo it follows that the sequence stabilises in the sense that there exists N such that $\mathcal{D}_N = \mathcal{D}_M$ for $M \geq N$. It is then possible to decide the problem by checking $\emptyset \notin \mathcal{D}_N$ since the latter holds iff $\emptyset \vdash x$. The dominant term in the algorithmic complexity of the derivation problem is the value of N. Yet even in the case that $(X, \preceq)$ is $(\mathbb{N}^d, \leq)$, it turns out that the value of N (as a function of the size of x) cannot be bounded by any primitive recursive function. Nevertheless, following [2] a much more modest double-exponential upper bound can be obtained in some cases by extracting a certain sequence $D_0 \supset \ldots \supset D_{N-2}$ with a further property on the ideals in their decompositions, namely that for every ideal $I \subseteq D_{k+1}$ with $I \not\subseteq D_{k+2}$ there exists an ideal $J \subseteq D_k$ with $J \not\subseteq D_{k+1}$ such that I is better behaved/less infinitary than J.

The sufficient condition for the modest upper bound essentially translates to the proof system being closed under the replacement of any premise y of a rule with any $y' \succeq y$. This property holds for multiplicative ('context-insensitive') rules and fails for additive ('context-sensitive') rules. This is in keeping with the non-primitive recursive complexity of the substructural logic $\mathrm{FL_{ec}}$ [3] (the commutative Full Lambek calculus with contraction) and 2EXPTIME-completeness [1] of its multiplicative fragment.

This talk is based on a collaboration with Adrián Puerto Aubel and Amir Akbar Tabatabai.

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Graphical Relational Algebras

Paweł Sobociński

Tallinn University of Technology, Estonia

pawel.sobocinski@taltech.ee

The talk will start with an exposition of Graphical Linear Algebra [2], a diagrammatic calculus for linear (a.k.a. additive) relations – those relations between vector spaces that are linear subspaces. The primitives of the calculus are closely related to the algebraic structure present in abelian bicategories, in the sense of Carboni and Walters [1].

The calculus is visually close to classical diagrammatic circuit notations in various application domains, for example signal flow graphs in engineering and control theory [3, 4, 5, 6], allowing for sound and complete calculi for diagrammatic reasoning. Further work [7, 8, 10] explored various extensions of the calculus in order to increase expressivity in terms of the class of relations that can be denoted, including affine relations, additive relations on the rig of natural numbers, and polyhedral relations. We will go through some of these extensions and showcase some applications, including reasoning about non-passive electrical circuits [9].

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Varieties, Quasivarieties, and Maltsev Products

Clifford Bergman

Iowa State University, USA

cbergman@iastate.edu

There is considerable overlap between universal algebra and model theory. In universal algebra, we typically focus on theories that are universal and either atomic or Horn. The classes defined by the former are called *varieties,* those of the latter are *quasivarieties.* These are the classes that are closed under the most important constructions in algebra: subalgebras, products, homomorphic images, direct limits, and free algebras.

In this series of lectures I will present some of the classical machinery related to these concepts. Following that, I will introduce a topic of current interest, namely the Maltsev product, that allows one to combine two well-understood varieties (or quasivarieties) and produce one that is far more general.

My plan is to provide lots of examples and few proofs, although I can provide proofs in some supplementary material if there is interest.

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Three Different Papers

KLAAS PIETER HART
Faculty EEMCS, TU Delft
k.p.hart@tudelft.nl

In these lectures I will discuss on three papers that I read and commented on in the last few years. Each paper has something to teach us, and gives rise to a warning of sorts.

1 Brouwer and Cardinalities

On L. E. J. Brouwer's first presentation of his foundational ideas to the rest of the world and the difficulties we may have reading it.

2 Machine Learning and the Continuum Hypothesis

I will discuss a recent article about an independence result in Machine Learning and its connection with some results of Kuratowski's that characterize the cardinals $\aleph_n$ for $n \in \omega$.

The warning here is about the way some people represent results in the media and that those representations are not always faithful.

3 When Linguists do Set Theory

About a claim made by linguists in the 1980's: "Natural languages form proper classes." This sounds very exciting but an analysis of a proof of this by two linguists reveals a serious misunderstanding of the way mathematics (and in particular Set Theory) works.

The implicit warning: "Do not try this at home."

The Definable Content of (Co)homological Invariants

ARISTOTELIS PANAGIOTOPOULOS
Department of Mathematical Sciences, Carnegie Mellon University, PA, USA
apanagio@andrew.cmu.edu

Some of the best-known and most versatile invariants in mathematics arise as the (co)homology groups of (co)chain complexes. Such invariants are computed by first associating to each object X a chain complex

$$C_\bullet(X) := \quad 0 \xleftarrow{\partial_0} C_0(X) \xleftarrow{\partial_1} C_1(X) \xleftarrow{\partial_2} C_2(X) \xleftarrow{\partial_3} \cdots$$

of abelian groups that encode the relevant data regarding the object X. The n^{th} *homology group of* X is then defined to be the quotient group of all "n-cycles" modulo all "n-boundaries":

$$\mathrm{H}_n(X) := \mathrm{Z}_n(X)/\,\mathrm{B}_n(X) := \ker(\partial_n)/\,\mathrm{ran}(\partial_{n+1}).$$

Overwhelmingly, the tendency is to regard the homology groups H_n as discrete objects. This is despite the fact that the chain groups $C_n(X)$ often carry a natural topology, and even one encoding data about X that is not captured by the abstract group structure alone. Historically, there have been attempts to incorporate this additional information in the groups $\mathrm{H}_n(X)$ by treating them as topological groups endowed with the induced quotient topology; see [4, p. 67]. However this approach was later abandoned since in most interesting cases $\mathrm{B}_n(X)$ is dense in $\mathrm{Z}_n(X)$, rendering the quotient topology on $\mathrm{H}_n(X)$ trivial and hence rather unavailing.

On an entirely separate trajectory, interactions between logic and analysis in the later 20th century gave birth to *invariant descriptive set theory*: a framework for measuring the Borel complexity of any classification problem which can be formulated as a quotient X/E, where X is a Polish space and E is an analytic equivalence relation on X. Of course, the original motivation was the problem of classifying all unitary representation of any given group Γ up to unitary equivalence—a problem which we now know (using techniques from invariant descriptive set theory) that it cannot be classified using any reasonably simple type of invariants, at least when Γ is not close to being abelian; see [5].

In this three part series we will provide an overview of a currently ongoing research program that is being developed in joint work with J. Bergfalk and M. Lupini; see [1, 2]. Its goal is to employ various concepts and tools from invariant descriptive set theory in order to enrich classical (co)homological invariants from algebraic topology and homological algebra with a finer structure than the quotient topology. In particular, we will show that many of these classical invariants can be naturally regarded as functors to the category of *groups with a Polish cover*. The resulting definable invariants provide far stronger means of classification.

Part 1. Definable (co)homological invariants: background, motivation, and main results. In this talk we will review some standard constructions from homological algebra and algebraic topology and we will provide an overview of the Borel reduction hierarchy of classification problems. We will then introduce the category of *definable homomorphisms* between *groups with a Polish cover* and illustrate how various (co)homology theories give rise to functors that naturally take values in this category. The resulting *definable invariants* provide far stronger means of classification. For example, we will provide an uncountable family of topological spaces whose homology groups are isomorphic but not definably isomorphic. This additional rigidity of the definable invariants is a consequence of certain "Ulam stability" phenomena for groups with a Polish cover. A detailed discussion of such phenomena will be the topic of our next talk.

Part 2. An Ulam stability result for quotients of abelian, non-archimedean Polish groups. Based on an earlier work of Shelah concerning the relationship of the continuum hypothesis to the cardinality of the set of automorphisms of the Boolean algebra $\mathcal{P}(\omega)/\mathrm{fin}$, Veličkovíc showed that if such an automorphism admits a Borel lift $\mathcal{P}(\omega) \to \mathcal{P}(\omega)$, then it is of a certain "trivial" form; [8]. Similarly, Kanovei and Reeken showed that if N, M are countable dense subgroups of the reals R, then every homomorphism $\mathrm{R}/N \to \mathrm{R}/M$ with a Borel lift $\mathrm{R} \to \mathrm{R}$, is of a certain "trivial" form; [6]. Kanovei and Reeken asked in [7] whether quotients of the p-adic groups satisfy similar "Ulam stability" phenomena. In this talk, we will settle this question by providing Ulam-stability phenomena for definable homomorphisms $G/N \to H/M$ when G, H are arbitrary abelian non-archimedean Polish groups and N, M are Polishable subgroups. While this result is in the heart of the definable (co)homology, this talk will be kept entirely independent of Part 1.

Part 3. Applications: definable Čech cohomology and homotopy classification. In this talk we will illustrate how the framework we developed in Part 1 for enriching classical (co)homological invariants with "definable content" applies to the classical Čech cohomology theory. As a consequence we will introducing a new invariant for locally compact metrizable spaces up to homotopy equivalence which we call "definable cohomology." Using the rigidity results from Part 2 we will show that definable cohomology provides a much finer invariant for classifying such spaces. For example, in strong contrast to its classical counterpart, this definable cohomology theory provides complete classification to homotopy classes of mapping telescopes of d-tori, and for homotopy classes of maps from mapping telescopes of d-tori to spheres. The latter problem was raised in the $d = 1$ case by Borsuk and Eilenberg in 1936; [3].

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Contributed Talks

Continuous Convergence in $\mathcal{R}L$

R. N. Ball[1] A. W. Hager[2] J. Walters-Wayland[3]

[1]Department of Mathematics, University of Denver, Denver CO, USA

[2]Department of Mathematics, Wesleyan University, Middletown CT, USA

[3]CECAT, Chapman University, Orange CA, USA

richardn.ball@gmail.com ahager@wesleyan.edu walterswayland@chapman.edu

This is a talk about the classical notion of continuous convergence viewed from a pointfree perspective. The context is the category $\mathbf{W}$ of archimedean lattice-ordered groups with distinguished weak (order) unit. A filter $\mathcal{F}$ on $\mathcal{C}X$, the $\mathbf{W}$-object of continuous functions on a Tychonoff space X, is said to *converge continuously to* $0 \in \mathcal{C}X$ if for each $x \in X$ and $\varepsilon > 0$ there exists a neighborhood U of x and a set $F \in \mathcal{F}$ such that $|f(x')| < \varepsilon$ for all $x' \in U$ and $f \in F$. That is, $\mathcal{F}$ converges continuously to 0 if

$$\forall \varepsilon > 0 \ \Big(\bigcup \big\{ U \in \mathcal{O}X : \exists F \in \mathcal{F} \ \forall f \in F \ \big(f^{-1}(-\varepsilon, \varepsilon) \supseteq U \big) \big\} = X \Big).$$

In the latter form, continuous convergence translates directly into terms of $\mathcal{R}L$, the $\mathbf{W}$-object of frame homomorphisms from the topology $\mathcal{O}\mathbb{R}$ of the real numbers into a (completely regular) frame L.

Definition (c-convergence $\xrightarrow{c}$)**.** A filter $\mathcal{F}$ on $\mathcal{R}L$ is said to *c-converges to* 0, written $\mathcal{F} \xrightarrow{c} 0$, if

$$\forall \varepsilon > 0 \ \left(\bigvee_{\mathcal{F}} \bigwedge_{F} f(-\varepsilon, \varepsilon) = \top \right).$$

A filter $\mathcal{F}$ *c-converges to an element* $g \in G$, written $\mathcal{F} \xrightarrow{c} g$, if $(\mathcal{F} - \dot{g}) \xrightarrow{c} 0$.

Continuous convergence has very nice properties: it is Hausdorff, i.e., if $\mathcal{F} \xrightarrow{c} f$ and $\mathcal{F} \xrightarrow{c} g$ then $f = g$, it is a $\mathbf{W}$-convergence, i.e., all $\mathbf{W}$-operations and $\mathbf{W}$-homomorphisms are c-continuous, and every c-Cauchy filter on $\mathcal{R}L$ is convergent. It is also canonical in the following sense.

Definition ($\mathbf{cW}$, $\mathcal{R}_c$)**.** The category $\mathbf{CW}$ has objects of the form $(G, \xrightarrow{x})$, where G is a $\mathbf{W}$-object and $\xrightarrow{x}$ is an admissible Hausdorff $\mathbf{W}$-convergence. The $\mathbf{CW}$-morphisms are the continuous $\mathbf{W}$-homomorphisms. The functor $\mathcal{R}_c : \mathbf{Frm} \to \mathbf{CW}$ assigns to a frame L the $\mathbf{CW}$-object $(\mathcal{R}L, \xrightarrow{c})$.

Theorem. *$\mathcal{R}_c$ is adjoint, and its (left) adjoint is the functor $\mathcal{K}_c : \mathbf{CW} \to \mathbf{Frm}$, which assigns to a $\mathbf{CW}$-object $(G, \xrightarrow{x})$ its frame of x-closed $\mathbf{W}$-kernels. This restricts to an equivalence between $\mathbf{Frm}$ and the full subcategory of Cauchy complete $\mathbf{CW}$-objects.*

And this is the point. The classical (pointfree) Yosida adjunction of Madden and Vermmer is a blunt instrument for studying the $\mathbf{W}$-manifestations of frame properties; after all, the Madden frame of a $\mathbf{W}$-object is always ω_1-Lindelöf. By enriching $\mathbf{W}$ with convergences, we have enough algebraic structure to fully articulate these manifestations.

Fibered Universal Algebra for First-Order Logics

Colin Bloomfield[1] Yoshihiro Maruyama[2]
[1]Vanderbilt University, USA
[2]Austral ian National University
colinbloomfield1@gmail.com yoshihiro.maruyama@anu.edu.au

We extend Lawvere-Pitts prop-categories (aka. hyperdoctrines) to develop a general framework for providing "algebraic" semantics for nonclassical first-order logics. This framework includes a natural notion of substitution, which allows first-order logics to be considered as structural closure operators just as propositional logics are in abstract algebraic logic. We then establish an extension of the homomorphism theorem from universal algebra for generalized prop-categories and characterize two natural closure operators on the prop-categorical semantics. The first closes a class of structures (which are interpreted as morphisms of prop-categories) under the satisfaction of their common first-order theory and the second closes a class of prop-categories under their associated first-order consequence. It turns out these closure operators have characterizations that closely mirror Birkhoff's characterization of the closure of a class of algebras under the satisfaction of their common equational theory and Blok and Jónsson's characterization of closure under equational consequence, respectively. These "algebraic" characterizations of the first-order closure operators are unique to the prop-categorical semantics and do not have analogs, for example, in the Tarskian semantics for classical first-order logic. The prop-categories we consider are much more general than traditional intuitionistic prop-categories or triposes (i.e., topos representing indexed partially ordered sets). Nonetheless, to our knowledge, our results are still new even when restricted to these special classes of prop-categories.

Quasi-Engel Varieties of Lattice-ordered Groups

Michael Darnel
Indiana University South Bend, USA
mrdarnel@gmail.com

We show that any ordered group satisfying the identity $[x_1^{k_1}, \dots, x_n^{k_n}] = e$ must be weakly abelian and that when $x_i \neq x_1$ for $2 \leq i \leq n$, ℓ-groups satisfying the identity $[x_1^n, \dots, x_k^n] = e$ also satisfy the identity $(x \vee e)^{y^n} \leq (x \vee e)^2$. These results are used to study the structure of ℓ-groups satisfying identities of the form $[x_1^{k_1}, x_2^{k_2}, x_3^{k_3}] = e$.

MV-topologies and MV-uniformities

Luz Victoria De La Pava C. Marby Zuley Bolaños O.
Universidad del Valle Cali, Colombia
victoria.delapava@correounivalle.edu.co marby.bolanos@correounivalle.edu.co

MV-topological spaces are fuzzy topological spaces in which Łukasiewicz t-norm and t-conorm play the role of strong intersection and union of fuzzy sets. They were introduced by C. Russo in [7] with the aim of extending Stone duality to semisimple MV-algebras. Many basic notions and results of general topology have been succefully extended to MV-topologies in [7] and [1]. These and others results obtained so far, as [2], indicate that MV-topological spaces constitute a pretty well-behaved fuzzy generalization of classical topological spaces.

Now, we define MV-uniformities and we show that each MV-uniformity generates an MV-topology. We define an MV-uniformity for the fuzzy unite interval defined by B. Hutton in [3]. Besides, we define complete regularity for MV-spaces and we show that each MV-topological space that is generated from an MV-uniformity is completely regular. Definitions and results are inspired by [4, 5, 6].

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Axiomatising Set Theoretic Multiverses

Matteo de Ceglie

Fachberei Philosophie, Kultur- und Gesellschaftwissenschaft Fakultät, Universität Salzburg, Austria

decegliematteo@gmail.com

Ten years ago, [6] changed the landscape of the foundations of mathematics, by introducing a novel conception that tried to clarify some ambiguous notions in current set theoretic practice. In particular, he provided a revolutionary interpretation for the practice of forcing: a multiverse of different set theoretic universes. Such an idea immediately sparked an intense debate in the philosophy of set theory and the foundations of mathematics. In the following years, several crucial contributions were made by [1], [2], [4], [5], [7], [9], [11], [10], [13], [15], [17], [18], and [19], just to name a few. These contributions can be roughly divided in two broad categories:

- the general debate between *universism* (the position that there is a single, determined universe of sets) and *pluralism* (the position that there are several universes of sets, all of them equally interesting, i.e. the multiverse);[1]
- the introduction of novel mathematical characterisations of the set theoretic multiverse.[2]

Indeed, while the general idea behind pluralism in the philosophy of mathematics is more or less the same every time, the actual mathematical details can vary enormously from one characterisation to the other. We have multiverses based upon different kinds of forcing[3] , multiverses with different background logics[4],

[1]Examples of such papers are [8], [10] and [9].

[2]For example, [5], [13] and [16].

[3]For example, [13] is based upon set-generic forcing, while [18] on Robinson infinite forcing.

[4][17] and [4] are the prime examples.

multiverses that try to accommodate the highest number of different universes[5], etc. Even though all these different set theoretic multiverses share the same, general, philosophical idea, they differ wildly from the mathematical perspective. There are some proposal of assessing all these differences[6], but this research field is still in its infancy. In this paper, I propose a novel, more general, framework for the set theoretic multiverse: the *Universal Multiverse*.

Drawing from the ideas of Universal Logic[7], this new framework is *not* a new set theoretic multiverse. Instead, it is a *general theory of the multiverse*, that investigates the various set theoretic multiverses from a common and abstract point of view. This means highlighting the common features of all multiverses, studying them as mathematical structures without any other metaphysical and ontological connotation.

The first step to carry out this program is to define a method of comparing and categorising all the different multiverses. A multiverse can be characterised in several ways: the most common one is to describe it as a set of universes. For example, Steel's set generic multiverse is the set of all set-generic extensions of a core universe, Friedman's Hyperuniverse is the set of all countable transitive models of ZFC, etc..

Generalising on this idea, a general theory of the set theoretic multiverse is the theory of the *Multiverse Operator*, Mlt_i. This theory is analogous to Tarski's theory of the consequence operator as introduced in [14]. A multiverse operator is a function defined on the powerset of a set S of sentences. This set S is the set of all possible set theoretic axioms, so each set theory T is a subset of $\mathcal{P}(S)$. A multiverse operator maps each theory T with the set of all universes, M, that are part of the multiverse generated from T:

$$Mlt_i : T \mapsto M.$$

For example, the operator $Mlt_{generic}$ applied to the set of sentences MV (the multiverse theory from [13]), written $Mlt_{generic}(MV)$, will map to Steel's set generic multiverse. Another example is the operator Mlt_{vlogic}, that maps ZFC to the set of all outer models of V (not only the set-generic ones).[8] In this way we can define in very general terms what a set theoretic multiverse is: it is an ordered couple (T, Mlt_i), where T is a set of axioms and Mlt_i is a multiverse operator.

The general theory of the multiverse that I propose is the study of the multiverse operator in *general*, without considering the special feature of each of them (e.g. the difference between $Mlt_{generic}$ and Mlt_{vlogic}). From this very abstract and general perspective, I claim that the general structure $\langle S, Mlt\rangle$ forms a *Tarski structure*. That is, it obeys the following axioms:

1. $T \subseteq Mlt(T)$;
2. $T \subseteq U \implies Mlt(T) \subseteq Mlt(U)$;
3. $Mlt(Mlt(T)) \subseteq Mlt(T)$.

In this paper, I will expand on the meaning of these axioms for the multiverse operator, with particular attention on their philosophical justification and why they were chosen.

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[5] [6] is the maximal multiverse conception, encompassing all possible universes.

[6] See for example [12].

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Satisfiability Degrees in BCK-algebras

Matt Evans

Oberlin College, USA

Matt.Evans@oberlin.edu

It is a fascinating result of Gustafson that the probability two elements commute in a finite non-Abelian group is no greater than 5/8. Inspired by this, we investigate the satisfiability degree of some equations in finite BCK-algebras; that is, given a finite BCK-algebra and an equation in the language of BCK-algebras, what is the probability that elements chosen uniformly randomly with replacement satisfy that equation?

Specifically we consider the equations for commutativity, double negation, and positive implicativity. For each of these equations we give an upper bound on the probability and show it is sharp. As corollaries, we will see that none of these three equations has a satisfiability gap. For commutativity and double negation we give sharp lower bounds as well, and for positive implicativity we give sharp lower bounds for the classes of linear BCK-algebras and BCK-algebras with unique atom.

Idempotent Residuated Chains

Nick Galatos[1] Wesley Fussner[2]

[1]University of Denver, USA

[2]University of Bern, Switzerland

ngalatos@du.edu wesley.fussner@unibe.ch

Idempotent residuated structures lie on the other side of the spectrum compared to lattice-ordered groups and cancellative residuated lattices and thus are of independent interest. Moreover, idempotent elements in residuated lattices play an important role in understanding the overall structure of the algebra (for example in the structure of BL-algebras, or in creating localizations via double-division conuclei). We focus our attention on totally-ordered residuated lattices that are also idempotent, and on the variety that they generate.

A central and recurring theme in the talk will be the importance of the two inverse operations, $x\backslash 1$ and $1/x$, in the study of idempotent residuated chains. For example, they are crucial in proving the congruence extension property (CEP) by controlling the behavior of conjugation, which is present in congruence generation in non-commutative residuated lattices. We show that the elements of an idempotent residuated chain that are inverses form a skeleton, which can be characterized abstractly by the quasi-involutive identity. (In the commutative case, the inverses form an involutive structure that is an odd Sugihara monoid). We define a closure operator whose image is precisely these inverse elements and show that it is a nucleus, thus guaranteeing that the quasi-involutive skeleton is itself a residuated lattice, but also transparently (via the induced partition) breaking up the whole chain into an ordinal sum of topped chains over the skeleton. (In addition to this, we also present a second construction of idempotent residuated chains: nested sums).

We prove that if we retain the total-lattice order, the inverses and the constant 1, then the divisions and multiplication are definable in the resulting reduct, and we further characterise abstractly these reducts. This observation, together with the flow diagrams that describe the precise action of the inverse operations, paves the way for showing that idempotent residuated chains are equivalent to enhanced monoidal preorders. The latter are enhancements of the preorder defined by Gil-Férez, Jipsen and Metcalfe, by including the sign of each element and also the appropriate inverses that need to exist in order to obtain the equivalence. We show that we can move back and forth between the order information in the preorders and the action of the inverses in the flow diagram and this yields a transparent understanding of the subalgebras (and subalgebra generation) that is crucial in investigating the amalgamation property. This understanding of subalgebras allows us to locate a failure of the amalgamation for idempotent residuated chains (even though amalgamation holds in the commutative case), which exhibits a forbidden behavior of the inverses, and also to carve out precisely the natural class where this behavior does not occur. We further obtain an equational characterization of the avoidance of the bad behavior, which we call rigidity. Finally, we prove the strong amalgamation property for rigid idempotent residuated chains and, by making use of the previously established CEP, we also obtain the strong amalgamation for the variety of rigid semilinear residuated lattices that they generate. This entails the epimorphism surjectivity of this variety and also interpolation and Beth definability properties for the associated substructural logic.

Quantum Monadic Algebras

John Harding

New Mexico State University, Las Cruces NM, USA

jharding@nmsu.edu

We introduce quantum monadic and quantum cylindric algebras. These are adaptations to the quantum setting of the monadic algebras of Halmos, and cylindric algebras of Henkin, Monk and Tarski, that are used in algebraic treatments of classical and intuitionistic predicate logic. Primary examples in the quantum setting come from von Neumann algebras, subfactors, commuting squares of subfactors, and a version of a quantum cylindric set algebra related to Weaver's quantum predicate logic. Here we develop the basic properties of these quantum monadic and cylindric algebras and a rudimentary type of Kripke frames for them.

Computational Complexity of Checking Semigroup Properties in Partial Bijection Semigroups and Inverse Semigroups

Trevor Jack

Illinois Wesleyan University, USA

trevor.jack@colorado.edu

We will compare and contrast the computational complexities of checking partial bijection semigroups and inverse semigroups for various properties. We prove that the decision problems for checking the following properties are NL-complete for partial bijection semigroups and AC^0 for inverse semigroups: (1) nilpotence, (2) $\mathcal{R}$- and $\mathcal{L}$-triviality, and (3) idempotent centrality. We describe algorithms that require at most logarithmic space to compute the following: (1) the left and right identities in a transformation semigroup and (2) whether a partial bijection semigroup has a zero element. We finally prove that checking membership of an idempotent in an inverse semigroup is a PSPACE-complete problem.

On Partially Ordered Algebras and Preclones

Peter Jipsen

Chapman University, USA

jipsen@chapman.edu

This talk reports on joint work with Reinhard Pöschel and Erkko Lehtonen. Partially ordered algebras are relational structures with a partial order and fundamental operations that are order-preserving or order-reversing in each argument. These two attributes are usually denoted by the signum + and − respectively for each argument. In general, the clone of operations of a partially ordered algebra may contain operations that are neither order-preserving nor order-reversing in some argument. We define S-preclones, based on a monoid S of signa, suitable for studying partially ordered algebras and more general relational structures. (Abstract) preclones are also known as operads, and we show that the notion of S-preclone admits a Pol-Inv Galois connection that extends the standard one from clones. If S is a group then the Galois connection gives a bijective correspondence between all S-preclones and S-relational clones on a finite set. We prove that the lattice of S-preclones on a finite set is atomic and coatomic, and for the case when $S = \{+, -\}$ we obtain a full description of the maximal S-preclones and minimal S-preclones on a two-element set.

Natural Deduction in Quantum Logic

Andre Kornell

Tulane University, United States

akornell@tulane.edu

The study of quantum logic was initiated by Birkhoff and von Neumann [1], who gave a semantics for the set $\mathcal{P}$ of propositional formulas in the connectives $\wedge$, $\vee$, and $\neg$. A large variety of sound deductive systems have been proposed for this semantics [9], and this contribution proposes another [8]. Whether a sound and complete deductive system exists appears to be an open problem [4].

The proposed deductive system is a sequent calculus called *NOM*. Its primitive connectives are $\wedge$, $\rightarrow$, and $\neg$, which satisfy the following familiar logical rules:

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \tag{1}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \qquad \frac{\Gamma \vdash A \rightarrow B}{\Gamma, A \vdash B} \qquad \frac{\Gamma, A \vdash B \quad \Gamma, \neg A \vdash B}{\Gamma \vdash B} \qquad \frac{\Gamma \vdash \neg A}{\Gamma, A \vdash B} \tag{2}$$

The antecedent of each sequent is a finite sequence of formulas, and the succedent of each sequent consists of exactly one formula.

The structural rules of *NOM* are weaker than those of classical logic, and in this sense, *NOM* is a substructural logic. However, *NOM* is not formally a substructural logic because the cut rule is not admissible in its standard formulation [11]. Instead, *NOM* includes the following basic structural rules:

$$\frac{}{\Gamma, A \vdash A} \qquad \frac{\Gamma \vdash A \quad \Gamma, A \vdash B}{\Gamma \vdash B} \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma, A \vdash B} \tag{3}$$

The second of these rules is the *cut rule*, and the third of these is its inverse, which we call the *paste rule*. The exchange rule is also not admissible in its standard formulation. Instead, *NOM* includes a *conditional exchange rule*, which is in some sense characteristic of the system:

$$\frac{\Gamma, A, B \vdash A \quad \Gamma, A, B \vdash C \quad \Gamma, B, A \vdash B}{\Gamma, B, A \vdash C} \tag{4}$$

The rules of *NOM* are the seven given logical rules and the four given structural rules.

This deductive system is sound for the semantics of Birkhoff and von Neumann in the sense that

$$\vdash A \quad \text{implies} \quad [\![A]\!] = \top$$

for each $A \in \mathcal{P}$ and each interpretation $[\![\cdot]\!]\colon \mathcal{P} \rightarrow C$. Here, C is the ortholattice of closed subspaces of Hilbert space, and $\top$ is it maximum element, i.e., Hilbert space itself. An expression of the form $A \vee B$ is treated as an abbreviation of $\neg(\neg A \wedge \neg B)$. The deductive system *NOM* satisfies the deduction theorem by fiat, and this addresses a prominent critique of previous deductive systems [6].

The basic soundness result of the previous paragraph extends to arbitrary sequents. The most natural formulation of this extended soundness result is in terms of physical notions. The closed subspaces of Hilbert space, which interpret our formulas, model observables that take values in the set $\{\texttt{true}, \texttt{false}\}$.

Theorem 1. *Let $A_1, \ldots, A_n \vdash B$ be a sequent of formulas in $\mathcal{P}$ that is derivable in NOM. Let $[\![\cdot]\!]\colon \mathcal{P} \rightarrow C$ be an interpretation. In any physical system that is modelled by Hilbert space, if $[\![A_1]\!], \ldots, [\![A_n]\!]$ are measured to be* `true` *in that order, then a measurement of $[\![B]\!]$ is certain to find that it is* `true`*.*

In principle, an observable might be measurable by multiple procedures that can be distinguished by how they transform the physical system. Theorem 1 refers only to those procedures that produce minimal decoherence in the physical system. These are the procedures that transform the physical system by $\rho \mapsto [\![A]\!]\rho[\![A]\!] + [\![\neg A]\!]\rho[\![\neg A]\!]$. Here, ρ is a density matrix, and $[\![A]\!]$ and $[\![\neg A]\!]$ are projection operators.

The notation *NOM* refers to orthomodularity. This system is sound and complete for orthomodular lattices in the sense that for each formula $A \in \mathcal{P}$, the sequent $\vdash A$ is derivable in *NOM* if and only if $[\![A]\!] = \top$ for every interpretation $[\![\cdot]\!]\colon \mathcal{P} \rightarrow L$, where L is any orthomodular lattice. Indeed, the familiar adjunction between the Sasaki projection and the Sasaki arrow is at the heart of the logical rules for $\rightarrow$ [2]. However, it must be emphasized that neither orthomodularity nor the Sasaki connectives appear in any way in the statement of Theorem 1.

Theorem 1 provides an additional argument for the position that the Sasaki arrow is the canonical implication connective of quantum logic [5]. This argument makes two assumptions. First, we assume that a sequent of physical propositions may be naturally interpreted as indicated in Theorem 1. The soundness of the structural rules of *NOM* is then experimentally falsifiable. Second, we assume that physical propositions admit the connectives $\wedge$, $\rightarrow$, and $\neg$ in a way that preserves their essential meaning as expressed in the logical rules of NOM. It follows that the implication connective is the Sasaki arrow:

Theorem 2. *Let A and B be formulas of NOM, and let Γ be a finite sequence of such formulas. Then, the sequents $\Gamma \vdash A \rightarrow B$ and $\Gamma \vdash \neg(A \wedge \neg(A \wedge B))$ are interderivable in NOM.*

The notation NOM also refers to natural deduction. This system is defined as a calculus of sequents, but its rules make it suitable for natural deduction [10, sec. 4]. It is routine to show that NOM is equivalent to a natural deduction system in the style of Fitch [3].

The preprint [8] also includes two extensions of NOM to a system of quantum predicate logic: the first is sound for quantum sets in the sense of [12], and the second is sound for quantum sets in the sense of [7].

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Dualities from Double Categories of Relations

Alexander Kurz

Chapman University

akurz@chapman.edu

The purpose of this note is to show how various duality theorems for so-called subordinations can be obtained from a "master theorem" that extends Priestley duality to double categories of relations. We write $\mathsf{DL}, \mathsf{BA}, \mathsf{Pri}, \mathsf{Stone}$ for the categories of bounded distributive lattices, Boolean algebras, Priestley and Stone spaces, respectively. The notion of a **$\mathbb{B}$-relation** for a concrete order-enriched category $\mathbb{B}$ (satisfying mild conditions) induces a **double category [6, 8] of $\mathbb{B}$-relations** $\mathbb{R}$ as follows [7].

- $\mathbb{R}$ has as objects $\mathbb{B}$-relations $R : A \looparrowright B$ with $A, B \in \mathbb{B}$. A **$\mathbb{B}$-relation** R can be tabulated in $\mathbb{B}$, that is, represented as a 'span of projections' ($\mathbf{R} \to A, \mathbf{R} \to B$). The span must be jointly mono. Moreover, relations are **weakening**, that is $x' \leq xRy \leq y' \ \Rightarrow\ x'Ry'$.
- A pair of maps $(f : X \to X', g : Y \to Y')$ in $\mathbb{B}$ is an arrow $R \to R'$ in $\mathbb{R}$ if $xRy \ \Rightarrow\ f(x)R'g(y)$. An arrow $(f, g) : R \to R'$ is also called a **square** and we draw f, g vertically and R, R' horizontally. Relations are ordered since $(id, id) : R \to R'$ is a square iff $R \subseteq R'$.
- **Vertical composition** in $\mathbb{R}$ is inherited from $\mathbb{B}$. **Horizontal composition** is composition of relations.

Master Theorem [7]. The dual equivalence between Pri and DL extends to Pri-relations and DL-relations. The dual of a DL-relation $\prec : A \nrightarrow B$ is $R : X \nrightarrow Y$ where X, Y are the Priestley spaces of prime filters of A, B, respectively, and

$$xRy \;\Leftrightarrow\; \prec[x] \subseteq y,$$

that is, xRy iff $a \in x$ and $a \prec b$ implies $b \in y$. The dual of a Pristley-relation $R : X \nrightarrow Y$ is $\prec : A \nrightarrow B$, where A, B are the DLs of upper clopens of X, Y, respectively, and

$$a \prec b \;\Leftrightarrow\; R[a] \subseteq b,$$

that is, $a \prec b$ iff $x \in a$ and xRy implies $y \in b$.

Remark. This duality does not flip relations but the order between relations. From the double theoretic point view relations are better understood as objects in $\mathbb{R}$ rather than arrows.

We are interested in restricting this duality to endo-relations, that is, to categories where, on the one hand, objects are of the kind (X, R) with X a Priestley space and R a relation on X and where, on the other hand, objects are of the kind $(B, \prec)$ with B a distributive lattice and $\prec$ is a relation on B. The latter ones are known as proximities or subordinations in the literature, see eg [1].

A **subordination** $\prec$ on a distributive lattice (or a Boolean algebra) B is a DL-relation $B \nrightarrow B$. The (order-enriched) category SubDL of **subordination algebras** is the (vertical) subcategory of the double category of DL-relations with subordination algebras $(B, \prec)$ as objects and arrows $f : B \to B'$ with $a \prec b \;\Rightarrow\; f(a) \prec' f(b)$. The category of **BA-based subordinations** SubBA further restricts to Boolean algebras B.

We now go through a list of corollaries to the master theorem.

Corollary. *The category of Priestley spaces with closed weakening relations is dually equivalent to* SubDL.

The next corollary is a variation of [3, Thm.9] and [5, Thm.3], see [1, Remark 2.23].

Corollary. *The category* SubBA *is dually equivalent to the category of Stone spaces with closed relations.*

The next examples exploit that duality of relations reverses the order between relations. For example, *reflexive* is dual to *below identity*, *transitive* is dual to *interpolative*.

Corollary. *The category of Stone spaces with reflexive closed relations is dually equivalent to the category of BA-based subordinations below identity.*

Corollary. *The category of Stone spaces with transitive closed relations is dually equivalent to the category of BA-based interpolative subordinations.*

Corollary. *The category of Stone spaces with reflexive and transitive closed relations (aka preorders, quasi-orders) is dually equivalent to the category of BA-based interpolative subordinations below identity.*

A **descriptive general frame** (X, R) is a Stone space with a closed relation R such that R^{-1} preserves clopens. It is easy to see that the inverse image of R preserves clopens iff, for all clopens b, the downsets $\{a \mid a \prec b\}$ of the dual relation have a maximal element (given by $R^{-1}[b]$). Such a relation is called $\prec$ **modally definable**, see [1, Def.2.9,Lem.4.3]. We now obtain [1, Thm.4.5(1)] as another corollary.

Corollary. *The category of BA-based modally definable subordination algebras is dually equivalent to the category of DGF's with relational morphisms.*

The constraints stemming from the order of relations and the constraint of preservation of clopens/modal definability can be combined in a modular fashion:

Corollary. *The category of reflexive DGF's with relational morphisms is dually equivalent to the category of modally definable BA-based subordinations below identity.*

Corollary. *The category of transitive DGF's with relational morphisms is dually equivalent to the category of modally definable BA-based interpolative subordinations.*

Corollary. *The category of reflexive and transitive DGF's with relational morphisms is dually equivalent to the category of modally definable BA-based interpolative subordinations below identity.*

We phrased the results in terms of the more widely considered BA-based subordinations but all of our proofs are obtained by restricting more general results about (DL-based) subordination algebras.

Acknowledgements I am grateful to Laurent De Rudder for introducing me to subordination algebras.

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Recent Work on Conjunctive Join-Semilattices

James J. Madden[1] Charles Delzell[1] Oghenetega Ighedo[2]

[1]Louisiana University, Baton Rouge, LA 70803, USA

[2]UNISA, Johannesburg SA

`madden@math.lsu.edu` `delzell@math.lsu.edu` `Ighedo@unisa.ac.za`

A join-semilattice is said to be *conjunctive* if it has a top element 1 and it satisfies the following first-order condition: for any two distinct elements a, b, there is an element c such that either $a \vee c \neq 1 = b \vee c$ or $a \vee c = 1 \neq b \vee c$. It is known that this is equivalent to the condition that every principal ideal is an intersection of maximal ideals. Conjunctivity is closely related to the property of subfitness in frames.

In the rest of this abstract, L will always denote a conjunctive join semilattice. L may fail to be distributive and may even fail to have any prime ideals whatsoever. Nonetheless, there is a satisfactory representation theory.

Theorem 1. *Let* $\operatorname{Max} L$ *denote the set of maximal ideals of* L *in the topology generated by the sets* $\widehat{a} := \{\, \mathbf{m} \in \operatorname{Max} L \mid a \notin \mathbf{m} \,\}$, $a \in L$. *Then* $\operatorname{Max} L$ *is a compact* T_1 *space and* $a \mapsto \widehat{a}$ *is an isomorphism of* L *with a join-closed subbase for* X. *Moreover, if* X *is a compact* T_1*-space and* L *is a join-closed subbase for* X, *then* $\operatorname{Max} L$ *is naturally homeomorphic with* X.

With some additional assumptions, the representation is functorial.

Theorem 2. *The following are equivalent:*

- *All the maximal ideals of* L *are prime;*
- L *is* 1-$\vee$*-distributive, i.e., for all* $a \in L$, $\{\, x \in L \mid x \vee a = 1 \,\}$ *is a filter;*
- $\{\, \widehat{a} \mid a \in L \,\}$ *is a base (not merely a subbase) for* $\operatorname{Max} L$.

There are examples of L *that satisfy these conditions and yet fail to be distributive.*

Our paper [1] presents the results above and much more, including a simplification and extension of the theory of Yosida frames, due to Martinez and Zenk. Recently, we have been investigating colimits in the category of join-semilattices with special attention on understanding what properties of join-semilattices (e.g., distributive, conjunctive) are preserved by colimits. In particular, when is the pushout of a pair of conjunctive join-semilattices over a shared ideal conjunctive? Some of the problems that arise seem to be very difficult.

In our BLAST 2022 talk, we shall provide an overview of the findings of [1] and describe our work in progress on colimits.

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The Stone Space of an Orthomodular Lattice

Joseph McDonald Katalin Bimbó

Department of Philosophy, University of Alberta, Canada

jsmcdon1@ualberta.ca bimbo@ualberta.ca

An *ortholattice* is an algebraic structure $A = \langle A; \wedge, \vee, -, 0, 1\rangle$ of similarity type $\langle 2, 2, 1, 0, 0\rangle$ such that $\langle A; \wedge, \vee, 0, 1\rangle$ is a bounded lattice and $-\colon A \to A$ is an orthocomplementation, i.e., a complementation operator that is an order-reversing involution. An *orthoframe* is a relational structure $\langle X; \perp\rangle$ such that X is a set and $\perp \subseteq X^2$ is irreflexive and symmetric, i.e., an *orthogonality relation.* For any $U \subseteq X$, defining $U^\perp = \{x \in X : \forall y[y \in U \Rightarrow x \perp y]\} = \{x \in X : x \perp U\}$ and then composing $^\perp$ with itself induces a closure operator $^{\perp\perp}\colon \wp(X) \to \wp(X)$ in which the $\perp$-*stable* sets are elements of $\{U \in \wp(X) : U = U^{\perp\perp}\}$. Goldblatt in [3] proved that every ortholattice is isomorphic to the algebra of clopen $\perp$-stable subsets of a Stone space X endowed with an orthogonality relation. Bimbó in [1] added a topology to the class of orthoframes used by Goldblatt and extended his topological representation of ortholattices to a full duality between homomorphisms and continuous frame morphisms.

An *orthomodular lattice* is an ortholattice satisfying $a \leq b \Rightarrow b = a \vee (b \wedge -a)$. Orthomodular lattices are well-known to arise via the lattice $\mathbb{C}(H)$ of closed linear subspaces of an infinite-dimensional separable Hilbert space H over the complex numbers, and provide an algebraic foundation for orthomodular quantum logic. Goldblatt in [4] proved that the condition of orthomodularity in $\mathbb{C}(H)$ cannot be characterized by any first-order properties of the relation of orthogonality between vectors in a pre-Hilbert space H. This result raised the natural question as to whether there exists an elementary (first-order definable) class of relational structures that can capture the condition of orthomodularity? This question was answered in the affirmative by Hartonas in [5] who introduced an elementary subclass of partially ordered quasi-orthoframes, known as *orthomodular frames*, and proved them to characterize orthomodular quantum logic.

In this talk, we describe a class of ordered relational topological spaces, which we call *orthomodular spaces*, whose construction involves adding a topology to the class of orthomodular frames along the lines of [1]. We then use these spaces to study the duality theory of orthomodular lattices. Time permitting, we will explore the possibility of casting this duality in a choice-free setting using spectral spaces along the general lines of McDonald and Yamamoto in [6].

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Choice-free De Vries Duality

Guillaume Massas
UC Berkeley, USA
gmassas@berkeley.edu

De Vries [2] generalized Stone duality between Boolean algebras and Stone spaces to a duality between compact Hausdorff spaces and complete compingent algebras, which are complete Boolean algebras enriched with a subordination relation. Just like Stone duality or Isbell's pointfree duality between compact regular frames and compact Hausdorff frames [3], de Vries duality relies on the Boolean Prime Ideal Theorem, a nonconstructive principle that follows from the Axiom of Choice.

In this talk based on [4], I will present a choice-free approach to de Vries Duality that generalizes the recent choice-free Stone Duality developed by Bezhanishvili and Holliday [1]. I will define a category of ordered topological spaces called *de Vries spaces* and show that it is dual to the category of complete compingent algebras. I will also discuss connections with Isbell Duality and with the Vietoris functor on compact Hausdorff spaces before concluding with some consequences of this duality in a choice-free setting.

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Vaughan–Lee's Loop is Finitely Based

Peter Mayr
University of Colorado, Boulder, USA
peter.mayr@colorado.edu

We show that every finite nilpotent Mal'cev algebra has a supernilpotent reduct, that is, a reduct that is a direct product of Mal'cev algebras of prime power order.

Recall that every finite supernilpotent Mal'cev algebra is finitely based by work of Vaughan–Lee (1983) and Freese and McKenzie (1987). In his paper Vaughan–Lee also points out a particular nilpotent loop of size 12, which is not supernilpotent and hence not covered by their techniques. Using its supernilpotent reduct, we can now show that this loop is still finitely based and also that its subpower membership problem can be solved in polynomial time.

This is joint work with Michael Kompatscher and Patrick Wynne.

Towards a Big New Obstacle Theorem: Challenges in Characterizing Dualizability of Finite Algebras in Congruence Modular Varieties

Connor Meredith

Department of Mathematics, University of Colorado, Boulder CO, USA

Connor.Meredith@Colorado.EDU

The Big NU Obstacle Theorem of Davey, Heindorf, and McKenzie from [1] characterizes dualizability of algebras that generate congruence distributive varieties and implies the following theorem.

Theorem. *Let* $\mathbb{A}$ *be a finite algebra such that* $\operatorname{typ}(\mathbb{A}) \subseteq \{3, 4\}$. *Then* $\mathbb{A}$ *is dualizable if and only if it has a near-unanimity term.*

The existence of an extension of the Big NU Obstacle Theorem to the setting of congruence modular varieties is currently unknown, but some progress has been made. M. Moore has shown in [5] that in the setting that $\mathbb{A}$ is finite and $\operatorname{typ}(\mathbb{A}) \subseteq \{2, 3, 4\}$, if $\mathbb{A}$ is dualizable, then $\mathbb{A}$ has a parallelogram term. However, the unmodified converse is known not to hold. Instead, in [3], Kearnes and Szendrei have shown in this setting that if $\mathbb{A}$ has a parallelogram term and satisfies the split centralizer condition, then $\mathbb{A}$ is dualizable. It is currently unknown if every finite dualizable algebra that generates a residually small variety omitting types 1 and 5 satisfies the split centralizer condition. If this were the case, then the following extension of the Big NU Obstacle Theorem would hold:

Conjecture. *Let* $\mathbb{A}$ *be a finite algebra such that* $\mathcal{V}(\mathbb{A})$ *is residually small and* $\operatorname{typ}(\mathbb{A}) \subseteq \{2, 3, 4\}$. *Then* $\mathbb{A}$ *is dualizable if and only if it satisfies the split centralizer condition and has a parallelogram term.*

In this talk, we will describe the split centralizer condition and a connection to neutrabelian algebras that was proven by Kearnes, Szendrei, and myself in [4]. We will also give a brief overview of a method, inspired by the work of Idziak in [2], that may leverage neutrabelian algebras to finally verify the above conjecture.

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Putting Cats Together: Completions of Mereological Models

Imanol Mozo Carollo

Department of Applied Economics, University of the Basque Country UPV/EHU, Spain

imanol.mozo@ehu.eus

In logic and philosophy, *mereology* (from the Greek term μερος) is the theory of parthood relations, that is, the study of parts, the wholes that they form and their relation. The idea of *mereological composition* is a commonplace in philosophical literature. Intuitively, we think of a composition of a collection of parts as "what you get when you put those parts together". When a more substantial and precise definition is needed, literature offers several alternatives, namely, the notions of fusion and sum (using the non-standard but useful in our case terminology in [1]). The very existence of compositions is a disputed topic in philosophical research. *Universalism* is the thesis that claims that mereological compositions of any nonempty collection of parts of a whole exist. As many other mereological principles, universalism, although widely accepted, is not completely unquestioned. One of the main objections to closure principles like universalism is that they may enlarge the collection of objects that are allowed in the domain of a theory. However there are reasons to believe that universalism does not actually require a further ontological commitment. After all, a composition may not be more than the parts that it is composed from. In words of David Lewis in ([4, 81–82]),

> To be sure, if we accept mereology, we are committed to the existence of all manner of mereological fusions. But given a prior commitment to cats, say, a commitment to cat-fusions is not a further commitment. The fusion is nothing over and above the cats that compose it. It just is them. They just are it. Take them together or take them separately, the cats are the same portion of Reality either way. Commit yourself to their existence all together or one at a time, it's the same commitment either way.

The main goal of this talk is to analyze this argument from a mathematical point of view. We will discuss how models of mereological theories can be extended into models that satisfy universalism.

Mereological theories are usually formalized in first order predicate logic with identity and a distinguished binary predicate constant, $\leq$, meant to represent the parthood relation. Even though there are interesting exceptions, the usual starting point is to consider that parthood relations are reflexive, antisymmetric and transitive. In other words, parthood relations $\leq$ are partial orders. We call this theory Core Mereology and its models are posets.

Let P be a poset and $A \subseteq P$. We say that an element $x \in P$ is a *fusion* of A if, for any $y \in P$, $x \circ y$ if and only if there exists $z \in A$ such that $z \circ y$, with $x \circ y$ denoting the fact that x and y have a common lower bound, in other words, that x and y *overlap*. We say that an element $x \in P$ is a *sum* of A if it is an upper bound of A and for any $y \leq x$ there exists $z \in A$ such that $z \circ y$. Fusion and sums are the two most common definitions of *composition of parts* considered in formal mereology (see Hovda's work in [3] for a profound study of these formal definitions, some variants, and the consequences of their use in the statement of mereological formal theories). For an arbitrary poset, neither existence nor uniqueness of sums and fusions are guaranteed. We will say that a poset is *fusion-complete* (resp. *sum-complete*) if every subset has a fusion (resp. sum) in P.

For the aim of this talk, we first need to discuss what is the convenient definition of a mereological completion of a poset P. We will argue that it should be an extension of P into a fusion- or sum-complete poset M which preserves the main features of P as a model of Core Mereology: the extension should be an order-embedding that preserves and reflects the overlap relation and either fusions or sums. Further, in order to be as economical as possible, every element of M should be a composition of a subset of P. We will show that the following definitions are sufficient to satisfy the requirements above: a *sum-completion* of a poset P a pair (M, e) where M is a sum-complete poset and $e\colon P \to M$ is a sum-dense order-embedding; a *fusion-completion* of a poset P is a pair (M, e) where M is a fusion-complete poset and $e\colon P \to M$ is an overlap-reflecting, fusion-preserving and fusion-dense order-embedding.

Strong Supplementation is a commonly assumed mereological principle:

$$y \not\leq x \to \exists z(z \leq y \wedge x \wr z).$$

Models of Core Mereology + Strong Supplementation are known in lattice theory as *separative* posets. We will use a construction used in the context of forcing techniques and Boolean-valued models in set theory [2] to show that, assuming strong supplementation, there is a unique (up to isomorphism) fusion-completion and a unique sum-completion. Furthermore, both coincide. This result is a strong argument in favor of Lewis's idea.

In the general case of an arbitrary poset P, we will show that there always exist fusion- and sum-completions, but, unfortunately, they need no to be unique. However, assuming a weaker supplementation principle, there is a sum-completion $\sigma\colon P \to S(P)$ of P such that for any other sum-completion $e\colon P \to M$

there exists an onto monotone map $q\colon M \to S(P)$ such that $q \circ e = \sigma$. In other words, $S(P)$ is a quotient of any other sum-completion of P.

Finally, we will present a construction of a fusion-completion $\alpha\colon P \to H(P)$ valid for atomic posets P. Further, we will show that if $e\colon P \to M$ is a fusion-completion of P such that e is a dense map, that is, for each element of M there an element in P mapped below it, then there is an injective fusion-preserving monotone map $g\colon H(P) \to M$ such that $g \circ \alpha = e$.

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Mal'tsev Products of Varieties

Tomasz Penza   Anna Romanowska

Faculty of Mathematics and Information Science, Warsaw University of Technology, Warsaw, Poland

T.Penza@mini.pw.edu.pl   Anna.Romanowska@pw.edu.pl

The Mal'tsev product of two classes of algebras consists of those algebras that have a homomorphic image in the second class where, for each idempotent in the image, the preimage lies in the first class. We present conditions for the Mal'tsev product of two varieties to be a variety.

1 Background

Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of the same similarity type $\tau\colon \Omega \to \mathbb{N}$. The *Mal'tsev product* $\mathcal{V} \circ \mathcal{W}$ of $\mathcal{V}$ and $\mathcal{W}$ consists of all algebras A of type τ with a congruence θ, such that A/θ belongs to $\mathcal{W}$ and every congruence class of θ that is a subalgebra of A belongs to $\mathcal{V}$. Each algebra A of type τ has a smallest congruence ϱ (called the $\mathcal{W}$-*replica congruence*) such that the corresponding quotient algebra belongs to $\mathcal{W}$. The congruence θ in the definition of the Mal'tsev product may be taken to be the $\mathcal{W}$-replica congruence ϱ of A. (See [3].) Thus the definition of the Mal'tsev product of varieties becomes

$$\mathcal{V} \circ \mathcal{W} = \{A \mid (\forall a \in A)\ (a/\varrho \leq A \ \Rightarrow\ a/\varrho \in \mathcal{V})\}. \tag{1}$$

By results of Mal'tsev [3, Ths. 1, 2], it is known that the Mal'tsev product $\mathcal{V}\circ\mathcal{W}$ is closed under the formation of subalgebras and of direct products. However in general, it is not closed under homomorphic images. We are interested in sufficient conditions for the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ to be a variety.

2 The Results

Definition 1. [4, Def. 2.1] Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of type τ, and let Σ be an equational base for $\mathcal{V}$. We define the following set $\Sigma^{\mathcal{W}}$ of identities:

$$\begin{aligned}\Sigma^{\mathcal{W}} := \{u(r_1,\dots,r_n) = v(r_1,\dots,r_n) \mid\ & \\ (u = v) \in \Sigma,\ & \\ \forall\, i = 1,\dots,n-1,\quad & \mathcal{W} \models r_i = r_{i+1}, \\ \forall\, \omega \in \Omega,\quad & \mathcal{W} \models \omega(r_i,\dots,r_i) = r_i\}.\end{aligned}$$

Theorem 2. *[4, Lem. 2.2] Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of type τ, and let Σ be an equational base for $\mathcal{V}$. Then the variety $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$ generated by the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is defined by the identities $\Sigma^{\mathcal{W}}$.*

Theorem 3. *[4, Th. 3.3] Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of type τ, and let $\mathcal{W}$ be idempotent. If there exist terms $f(x, y)$ and $g(x, y)$ such that*

(a) $\mathcal{V} \models f(x, y) = x$ and $\mathcal{V} \models g(x, y) = y$,

(b) $\mathcal{W} \models f(x, y) = g(x, y)$,

then the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.

Corollary 4. *[2, Th. 6.3] If $\mathcal{V}$ is a strongly irregular variety with no nullary operations and $\mathcal{S}_\tau$ is the variety of the same type as $\mathcal{V}$, equvalent to the variety of semilattices, then $\mathcal{V} \circ \mathcal{S}_\tau$ is a variety.*

If t is a term of type τ and $\mathcal{W}$ satisfies the identities

$$\omega(t, \dots, t) = t, \tag{2}$$

for all $\omega \in \Omega$ (as in Definition 1), then we say that t is a *term idempotent* of $\mathcal{W}$.

Theorem 5. *[5, Thm. 4.1] Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of the same type, and let $\mathcal{W}$ be term idempotent. If there exist terms $f(x, y, z)$ and $g(x, y, z)$ such that*

(a) $\mathcal{V} \models f(x, y, y) = x$ and $\mathcal{V} \models g(x, x, y) = y$,

(b) $\mathcal{W} \models f(x, x, y) = g(x, x, y)$,

(c) $f(x, x, y)$ is a term idempotent of $\mathcal{W}$,

then the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.

Theorem 5 has a number of interesting consequences and applications. It is a common generalization of Theorem 3 and of the following theorem of Bergman.

Theorem 6. *[1, Cor. 2.3] If $\mathcal{V}$ and $\mathcal{W}$ are idempotent subvarieties of a congruence permutable variety, then $\mathcal{V} \circ \mathcal{W}$ is a variety.*

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Finite Base Property for Algebras of Binary Relations

Jaš Šemrl

Department of Computer Science, University College London, UK

j.semrl@cs.ucl.ac.uk

Not every finite representable Tarskian Relation Algebra (RA) is representable over a finite base. However, this Finite Base Property (FBP) does hold for some of its reduct languages. As conjectured for some time, we have recently shown that the FBP is in fact neither weaker nor stronger than the finite axiomatisability (FA) of the representation class. Furthermore, while the FA is known to hold or fail for the majority of the RA reducts and many well known tools are available to prove or disprove it, the FBP remains largely unknown with results especially sparse on the positive side. We conjecture that the FBP holds if and only if the language contains neither composition and negation nor composition and meet. We then prove the left to right implication of the conjecture. We conclude by showing some cases on the right to left implication and outline a possible proof for the general case.

Presenting Quantitative Inequational Theories

TODD SCHMID
University College London, UK
toddwayneschmid@gmail.com

It came to the attention of myself and the coauthors of [1] that a number of process calculi can be obtained by algebraically presenting the branching structure of the transition systems they specify. Labelled transition systems, for example, branch into sets of transitions, terms in the free semilattice generated by the transitions. Interpreting equational theories in the category of sets has undesirable limitations, and we would like to have more examples of presentations in other categories.

In this brief article, I discuss monad presentations in the category of partially ordered sets and monotone maps. I focus on quantitative monads, namely free modules over ordered semirings, and give sufficient conditions for one of these to lift a monad on the category of sets. I also give a description of ordered semirings that are useful for specifying unguarded recursive calls. Examples include ordered probability theory and ordered semilattices.

Monad presentations Intuitively, a monad is a free-object construction that takes some desirable properties as input and outputs the initial object satisfying those properties. Classically, the objects of interest are algebras, the properties of interest are equations, and the free-object is the initial algebra satisfying said equations.

Fix an endofunctor $S : \mathsf{C} \to \mathsf{C}$ on a category C. An (S-)*algebra* is a pair (X, α) consisting of an object X of C and an arrow $\alpha : SX \to X$. A *homomorphism* $h : (X, \alpha) \to (Y, \beta)$ is an arrow $h : X \to Y$ in C such that $S(h) \circ \alpha = \beta \circ h$. $\mathrm{Alg}_{\mathsf{C}}(S)$ denotes the category of S-algebras and homomorphisms.

For the most unrestricted view of properties that an algebra can satisfy, we simply consider full subcategories of $\mathrm{Alg}(S)$. The following notion of monad presentation is inspired by [6].

Definition 1. Let T be a full subcategory of $\mathrm{Alg}_{\mathsf{C}}(S)$. A T-*presented monad* is a triple (M, η, ρ) consisting of the following ingredients: (i) An endofunctor $M : \mathsf{C} \to \mathsf{C}$, (ii) a natural transformation $\eta : \mathrm{Id} \Rightarrow M$, and (iii) a natural transformation $\rho : SM \Rightarrow M$ such that

(a) $(MX, \rho_X) \in \mathsf{T}$ for any object X of C, and

(b) for any S-algebra (Y, β) and any arrow $f : X \to Y$ in C, there is a unique S-algebra homomorphism $f^\beta : (MX, \rho_X) \to (Y, \beta)$ such that $f^\beta \circ \eta = f$.

Quantitative theories Discrete processes that branch with observable quantities are popular in the automata theory and process algebra literature [5, 2, 7]. In the cited works, formal calculi are used for specifying and reasoning about behaviours of such processes. Algebraically reasoning about behaviour inevitably relies on an algebraic characterisation of the branching structures of processes.

Quantitative branching is often depicted by decorating transitions with numbers, vectors, or other quantities. Generally, quantities appear as elements of a (*positive*) *semiring* $\mathbf{P} = (P, 0, 1, +, \cdot)$, a set P equipped with constants $0, 1$ and binary operations $+, \cdot$ such that $(P, 0, +)$ is a commutative monoid, $(P, 1, \cdot)$ is a monoid, $0r = 0$, $r(s + t) = rs + rt$, and $(s + t)r = sr + tr$ for all $r, s, t \in P$ [8].

Definition 2. A $\mathbf{P}$-*module* is a monoid $(X, 0, +)$ equipped with an action $P \times X \to X$, written $(r, x) \mapsto rx$, such that $0x = 0$, $1x = x$, $(rs)x = r(sx)$, and $(r + s)x = rx + sx$ for $x \in X$ and $r, s \in P$.

Abstractly, $\mathbf{P}$-modules are algebras for $S = P \times \mathrm{Id} + \{+\} \times \mathrm{Id}^2$ that satisfy a handful of equations, where Id is the identity on Sets. They form a full subcategory $\mathbf{P}\mathsf{Mod}$ of $\mathrm{Alg}_{\mathsf{Sets}}(S)$. The free $\mathbf{P}$-module on X consists of finitely supported $\theta : X \to P$, where the *support* of θ is $\mathrm{supp}(\theta) = \{x \in X \mid \theta(x) \neq 0\}$.

Definition 3. The *free* $\mathbf{P}$*-module* functor $\mathcal{O}_\omega : \mathsf{Sets} \to \mathsf{Sets}$ is given by

$$\mathcal{O}_\omega X = \{\theta : X \to P \mid \mathrm{supp}(\theta) \text{ is finite}\} \qquad \mathcal{O}_\omega(h)(\theta)(y) = h^\bullet\theta(y) := \sum_{h(x)=y} \theta(x)$$

for any set X and any $h : X \to Y$. The $\mathbf{P}\mathsf{Mod}$-presented monad is the triple $(\mathcal{O}_\omega, \delta, \rho)$, defined

$$\delta_X(x) = \delta_x := \lambda y. \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases} \qquad \rho(0) = 0 \qquad \rho(r, \theta) = r\theta \qquad \rho(+, \theta_1, \theta_2) = \theta_1 + \theta_2$$

where $0(x) = 0$, and $r\theta$ and $\theta_1 + \theta_2$ are evaluated pointwise.

Example 4. Consider $\mathbf{2} = (\{0, 1\}, 0, 1, \max, \min)$. $\mathbf{2}\mathsf{Mod}$ is equivalent to the category of join semilattices with bottom. In particular, $\mathcal{O}_\omega$ is naturally isomorphic to the finitary powerset functor $\mathcal{P}_\omega$.

Example 5. Consider $\mathbf{R}^+ = (\mathbb{R}_{\geq 0}, 0, 1, +, \times)$. $\mathbf{R}\mathsf{Mod}$ is equivalent to the category of positive cones of finite dimensional real-valued metric spaces and linear maps with nonnegative entries.

Ordered quantitative theories A notable example of the power of algebraic presentation for the purposes of reasoning about behaviour is Stark and Smolka's algebra of probabilistic actions [5], which is used to study processes that branch probabilistically, with weights from the semiring $\mathbf{R}^+$. An important feature of their calculus is its interpretation of recursive specifications as least fixed-points.

Recently [9], I made that the observation that the existence of these least fixed-points is a property of the *inequational theory* obtained from the theory of $\mathbf{R}^+$-modules. Where Pos is the category of partially ordered sets (posets) and monotone maps, an inequational theory is a set of inequalities that describes a category of S-algebras for a functor $S : \mathsf{Pos} \to \mathsf{Pos}$, or *ordered algebras*.

Definition 6. An *ordered semiring (with bottom)* $\mathbf{P} = (P, \leq, 0, 1, +, \cdot)$ is a semiring $(P, 0, 1, +, \cdot)$ where $(P, \leq)$ is a poset, $(\forall r \in R)\ 0 \leq r$, and $+, \cdot$ are monotone. An *ordered* $\mathbf{P}$*-module* is a $\mathbf{P}$-module $(X, 0, +)$ such that $r \leq s$ and $x \leq y$ implies $rx + z \leq sy + z$ for any $r, s \in P$ and $x, y, z \in X$.

Abstractly, an ordered $\mathbf{P}$-module is an algebra for $S = (P, \leq) \times \mathrm{Id} + (\{+\}, =) \times \mathrm{Id}^2$ satisfying a set of inequations, where Id is the identity functor on Pos now. We also write $\mathbf{P}\mathsf{Mod}$ for the full subcategory of $\mathrm{Alg}_{\mathsf{Pos}}(S)$ consisting of ordered $\mathbf{P}$-modules.

If the reader is anything like myself, they might expect me to say that $\mathcal{O}_\omega$ constructs the free ordered $\mathbf{P}$-module on a poset $(X, \leq)$. This is not always the case. For example, turn the semiring $\mathbf{2}$ from Example 4 into an ordered semiring by taking $0 < 1$. Then an ordered $\mathbf{2}$-module consists of the same data as a semilattice with bottom, in the sense of order theory [10]. The free ordered semilattice on a poset $(X, \leq)$ is carried by the finitely generated downwards-closed subsets of $(X, \leq)$, which is not equal to $\mathcal{P}_\omega X$ in general!

In many cases, quotienting the set $\mathcal{O}_\omega X$ by a certain preorder suffices. Call a subset $U \subseteq X$ *upwards closed* if $x \in U$ and $x \leq y$ means $y \in U$. The *heavier-higher order* is the preorder $\sqsubseteq$ on $\mathcal{O}_\omega X$ defined so that $\theta_1 \sqsubseteq \theta_2$ if and only if for any upwards-closed $U \subseteq X$, $\sum_{x \in U} \theta_1(x) \leq \sum_{x \in U} \theta_2(x)$ (see for e.g. [11]). We write $\theta_1 \equiv \theta_2$ if $\theta_1 \sqsubseteq \theta_2 \sqsubseteq \theta_1$, $[\theta_1] = \{\theta_2 \mid \theta_1 \equiv \theta_2\}$, and $\mathcal{O}_\omega^\wedge(X, \leq) = (\mathcal{O}_\omega X/{\equiv}, \sqsubseteq)$.

Say that a semiring is *cancellative* if $(\forall r, s, t)\ r + s \leq r + t$ implies $s \leq t$, *difference ordered* if $r \leq s$ implies $(\exists t)\ t + t = s$, and *idempotent* if $r + r = r$. Our main result states that the free ordered $\mathbf{P}$-module is $(\mathcal{O}_\omega^\wedge, [\rho])$ when $\mathbf{P}$ is made of cancellative difference ordered and idempotent semirings.

Theorem 7. *Suppose* $\mathbf{P} = \prod_{i \in I} \mathbf{P}_i$*, where* $\mathbf{P}_i$ *is either cancellative and difference ordered, or idempotent, for each* $i \in I$*. Then* $(\mathcal{O}_\omega^\wedge, [\delta], [\rho])$ *is the* $\mathbf{P}\mathsf{Mod}$*-presented monad.*

In fact, if $\mathbf{P}$ is cancellative, like in Example 5, then $\sqsubseteq$ is a partial order. Consequently, if $\mathbf{P}$ is cancellative and difference ordered, $(\mathcal{O}_\omega^\wedge, [\delta], [\rho])$ lifts the monad $(\mathcal{O}_\omega, \delta, \rho)$ on Sets.

A negative answer to this question would consist of an example of an ordered semiring $\mathbf{P}$ and a monotone map $f : (X, \leq) \to (Y, \leq)$ into a $\mathbf{P}$-module $(Y, \leq, \beta)$ such that the inductively defined homomorphism $f^\beta : \mathcal{O}_\omega X \to Y$ is not monotone with respect to the heavier-higher order.

Distribution theories Finally, I would like to mention sufficient conditions for ensuring that recursive specifications of probabilistic branching have least fixed-points, like in [5, 2]. Call an ordered semiring $\mathbf{P}$ *division* if $(\forall r > 0)(\exists s)\ rs = sr = 1$. Call a map $A : (\mathcal{O}_\omega X)^n \to (\mathcal{O}_\omega X)^n$ *stochastic affine* if $A = [a_{ij}] + \vec{\theta}$ for some square $\mathbf{P}$-matrix $[a_{ij}]$, $\vec{\theta} \in (\mathcal{O}_\omega X)^n$, and $\theta_i(X) + \sum_j a_{ij} = 1$.

Theorem 8. *If* $\mathbf{P}$ *is cancellative, difference ordered, and divisive, then every stochastic affine map has a least fixed-point with respect to the heavier-higher order.*

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Augmented Quasigroups and Heyting Algebras

Jonathan D. H. Smith

Department of Mathematics, Iowa State University, USA

jdhsmith@iastate.edu

Lambek and others have long remarked on the analogies between quasigroup divisions and residuations. Now, as a result of a recent study [7] of fusion algebras in algebraic combinatorics and quantum theory (e.g., anyons), the concept of an **augmented quasigroup** in a compact closed category emerges as a precise unifying framework for quasigroups, Heyting algebras, and related structures.

1 Compact Closed Categories

Recall that a symmetric monoidal or tensor category $(\mathbf{V}, \otimes, \mathbf{1})$ is said to be *compact closed* [1, 4] if it has:

(a) a contravariant *duality functor* ${}^* \colon \mathbf{V} \to \mathbf{V}$;

(b) a natural transformation $\mathrm{ev}_A \colon A \otimes A^* \to \mathbf{1}$ of *evaluation*, and

(c) a natural transformation $\mathrm{coev}_A \colon \mathbf{1} \to A^* \otimes A$ of *coevaluation*

such that $[A \xrightarrow{1_A \otimes \mathrm{coev}} A \otimes A^* \otimes A \xrightarrow{\mathrm{ev} \otimes 1_A} A]$ and $[A^* \xrightarrow{\mathrm{coev} \otimes 1_A} A^* \otimes A \otimes A^* \xrightarrow{1_A \otimes \mathrm{ev}} A^*]$ reduce respectively to the identities 1_A and 1_{A^*} for objects A of $\mathbf{V}$. Typical examples of compact closed categories are provided by finite-dimensional vector spaces, finite multisets, and relations on sets.For objects A_1, A_2, A_3 in a compact closed category $(\mathbf{V}, \otimes, \mathbf{1})$, we have the natural isomorphisms

$$\begin{aligned} \phi_{A_1,A_2,A_3} \colon \mathbf{V}(A_2 \otimes A_1, A_3) &\to \mathbf{V}(A_2, A_3 \otimes A_1^*), \\ \tau_{13} \colon A_3 \otimes A_2 \otimes A_1 &\to A_1 \otimes A_2 \otimes A_3; a_3 \otimes a_2 \otimes a_1 \mapsto a_1 \otimes a_2 \otimes a_3, \\ \tau_{23} \colon A_1 \otimes A_3 \otimes A_2 &\to A_1 \otimes A_2 \otimes A_3; a_1 \otimes a_3 \otimes a_2 \mapsto a_1 \otimes a_2 \otimes a_3, \end{aligned}$$

and then define

$$\begin{aligned} \tau_{13}^* \colon \mathbf{V}(A_1 \otimes A_2 \otimes A_3, \mathbf{1}) &\to \mathbf{V}(A_3 \otimes A_2 \otimes A_1, \mathbf{1}); \theta \mapsto \tau_{13}\theta \text{ and} \\ \tau_{23}^* \colon \mathbf{V}(A_1 \otimes A_2 \otimes A_3, \mathbf{1}) &\to \mathbf{V}(A_1 \otimes A_3 \otimes A_2, \mathbf{1}); \theta \mapsto \tau_{23}\theta \end{aligned}$$

for morphisms

$$\theta \colon A_1 \otimes A_2 \otimes A_3 \to \mathbf{1} \text{ in } \mathbf{V} \text{ (using “algebraic” or “diagrammatic” notation).}$$

2 Augmented Magmas and Augmented Quasigroups

An *augmented magma* [7, Def'n. 3.6] has a *multiplication* (*structure*) $\mu \colon A \otimes A \to A^*$; a *comultiplication* $\Delta \colon A \to A \otimes A$, and an *augmentation* $\varepsilon \colon A \to \mathbf{1}$, such that the diagram

$$\begin{tikzcd} A \otimes A \arrow[r, "\mathrm{coev}_A \otimes \mu"] \arrow[d, "\varepsilon \otimes \varepsilon"'] & A^* \otimes A \otimes A^* \arrow[r, "1_{A^*} \otimes \Delta \otimes 1_{A^*}"] & A^* \otimes A \otimes A \otimes A^* \arrow[d, "\tau \otimes \mathrm{ev}_A"] \\ \mathbf{1} & & A \otimes A^* . \arrow[ll, "\mathrm{ev}_A"] \end{tikzcd}$$

commutes. Here, $\tau \colon A^* \otimes A \to A \otimes A^*$ swaps factors. Let $(A, \mu, \Delta, \varepsilon)$ be an augmented magma in $(\mathbf{V}, \otimes, \mathbf{1})$.

(a) The $\mathbf{V}$-morphism $\rho = \mu\phi^{-1}_{A,A\otimes A,\mathbf{1}}\tau^*_{13}\phi_{A,A\otimes A,\mathbf{1}} \colon A \otimes A \to A^*$ is called the *right division* (*structure*) on the augmented magma $(A, \mu, \Delta, \varepsilon)$.

(b) The $\mathbf{V}$-morphism $\lambda = \mu\phi^{-1}_{A,A\otimes A,\mathbf{1}}\tau^*_{23}\phi_{A,A\otimes A,\mathbf{1}} \colon A \otimes A \to A^*$ is called the *left division* (*structure*) on the augmented magma $(A, \mu, \Delta, \varepsilon)$.

(c) The structure $(A, \mu, \rho, \lambda, \Delta, \varepsilon)$ is an (*augmented*) *quasigroup* [7, Def'n. 4.1] if $(A, \rho, \Delta, \varepsilon)$ and $(A, \lambda, \Delta, \varepsilon)$ are augmented magmas.

3 Set-valued Functions and Structures in the Relation Category

We focus on the compact closed category $(\mathbf{Rel}, \otimes, \top)$ of relations on sets. Recall that the tensor product $\otimes$ here is the Cartesian product, with a singleton $\top$ (say $\{0\}$) as the tensor unit. Composition is the relation product. Let A be a set with a “set-valued” function $A \times A \to 2^A; (x, y) \mapsto x \diamond y$.

(a) The set $\{(x \otimes y, z) \mid x, y, z \in A,\ z \in x \diamond y\}$ is the *multiplication relation* $\mu \colon A \otimes A \to A^*$.

(b) The *comultiplication* $\Delta \colon A \to A \otimes A$ is the *diagonal* $\{(x, x \otimes x) \mid x \in A\}$.

(c) Define augmentation $\varepsilon \colon A \to \top$ as the relation $\{(x, 0) \mid x \in A\}$.

This defines $(A, \mu, \Delta, \varepsilon)$ in $(\mathbf{Rel}, \otimes, \top)$. A *hypermagma* [5, Defn. 6.1(a)] $(A, \diamond)$ is a set A equipped with a function $A \times A \to 2^A \smallsetminus \{\ \}; (x, y) \mapsto x \diamond y$.

Proposition. *A set A with $A \times A \to 2^A; (x, y) \mapsto x \diamond y$ is a hypermagma iff the structure $(A, \mu, \Delta, \varepsilon)$ is an augmented magma in $(\mathbf{Rel}, \otimes, \top)$.*

The proposition shows that the categorical augmented magma concept captures the set-theoretical non-emptiness condition for the set-valued product of a hypermagma.

Now, on a set A, take three hypermagma structures $(A, \diamond)$, $(A, \swarrow)$, and $(A, \searrow)$. Then $(A, \diamond, \swarrow, \searrow)$ is a *Marty quasigroup* if and only if the conditions

$$z \in x \diamond y \quad \Leftrightarrow \quad x \in z \swarrow y \quad \Leftrightarrow \quad y \in x \searrow z$$

hold for elements x, y, z of A. (Compare [6] for the associative case.)

A *residuated magma* is a partially ordered algebra $(A, \leq, \cdot, /, \backslash)$ such that $(A, \leq)$ is a poset, and $\cdot, /, \backslash$ satisfy the *residuation property*

$$x \cdot y \leq z \quad \Leftrightarrow \quad x \leq z/y \quad \Leftrightarrow \quad y \leq x\backslash z \tag{1}$$

[2]. For example, in a *Heyting algebra* $(A, \wedge, \rightarrow)$, consider

$$x \wedge y \leq z \quad \Leftrightarrow \quad x \leq y \rightarrow z \quad \Leftrightarrow \quad y \leq x \rightarrow z \tag{2}$$

as a residuation [3, §I.1.10]. Define the *up-sets* $\uparrow a = \{x \in A \mid a \leq x\}$ and the *down-sets* $\downarrow a = \{x \in A \mid x \leq a\}$ for elements a of A. Then the specifications

$$x \diamond y = \uparrow(x \cdot y)\,, \quad z \swarrow y = \downarrow(z/y) \quad \text{and} \quad x \searrow z = \downarrow(x\backslash z)$$

yield a Marty quasigroup $(A, \diamond, \swarrow, \searrow)$.

Theorem ([7, Th. 4.6])**.** *Marty quasigroups are equivalent to augmented quasigroups in $(\mathbf{Rel}, \otimes, \top)$.*

Corollary ([7, Cor. 4.7])**.** *Quasigroups are equivalent to augmented quasigroups in $(\mathbf{Rel}, \otimes, \top)$ for which all the structure actually lies in $(\mathbf{Set}, \times, \top)$.*

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Central Extensions, Presentations and Hochschild–Serre in Varieties with a Difference Term

Alexander Wires

Southwestern University of Finance and Economics, China

slawkenbergius@hotmail.com

There is a cohomology theory for arbitrary varieties characterizing extensions realizing affine datum. The theory is replete with semidirect products, derivations, stabilizing automorphism groups and admits an additional parameter controlling the equational theories of extensions. In varieties with a difference term, the second-cohomology with trivial actions characterize central extensions. In this setting, we establish a low-dimensional Hochschild–Serre exact sequence and discuss applications to lifting homomorphisms and characterizations of the second-cohomology group by datum derived from the commutator in free presentations. Varieties with a difference term cover many classical and non-classical algebras; by specialization, we recover the classical and motivating results from group theory and somewhat more recent results for subvarieties of modules expanded by multilinear maps such as Loday-type algebras.

Minimum Proper Extensions in Lattices of Subalgebras

Brian Wynne[1] A. W. Hager[2]

[1]Lehman College, CUNY, USA

[2]Wesleyan University, USA

brian.wynne@lehman.cuny.edu ahager@wesleyan.edu

An element b in a lattice L is called *strictly meet-irreducible* (smi) if there is $c > b$ in L such that $c < y$ or $c = y$ for every $y > b$ in L. Let I and A be algebras (in the sense of Universal Algebra) with I a subalgebra of A, and let $S(I, A)$ be the lattice of all subalgebras of A that contain I. If B is smi in $S(I, A)$ with unique cover C, then we call $B < C$ a *minimum proper extension* (mpe) in $S(I, A)$. We consider the problem of identifying all the mpe's in lattices of the form $S(I, A)$. After giving a few general results, we solve the problem for $S(\mathbb{Z}, \mathbb{R})$, where $\mathbb{Z}$ and $\mathbb{R}$ represent the additive groups of integers and real numbers, respectively. If time allows, we will also discuss how this work relates to the problem of identifying atoms in the lattice of hull operators on various categories of Abelian lattice-ordered groups.

On the Number of Clonoids Between Finite Modules

Patrick Wynne Peter Mayr

University of Colorado Boulder, USA

pawy3273@colorado.edu peter.mayr@colorado.edu

A clonoid from an algebra A to an algebra B is a set of functions from powers of A into B that is closed with respect to precomposition with term functions of A and closed with respect to postcomposition with term functions of B. We investigate clonoids between finite modules. These structures arise in the description of nilpotent Mal'cev algebras. We show that if the two modules have orders that are not coprime then there are infinitely many clonoids between them. On the other hand, if the modules are of coprime order then the number of clonoids from A to B depends on the structure of A. We classify modules over commutative rings for which the clonoids are generated by their unary functions.

Generalized Ultraproducts for Positive Logic

Kentaro Yamamoto[1] Tommaso Moraschini[2] Johann J. Wannenburg[1]

[1]Institute of Computer Science of the Czech Academy of Sciences, Prague, Czech Republic

[2]Universitat de Barcelona, Catalonia, Spain

yamamoto@cs.cas.cz tommaso.moraschini@ub.edu wannenburg@cs.cas.cz

We present a generalization of the ultraproduct construction suitable for positive logic in the sense of [1]. While the classical notion of ultraproducts involves ultrafilters in the powerset Boolean algebra of some index set, the generalized notion of (prime) *filter products* is dependent on (prime) filters, respectively, in the *Heyting* algebra of up-sets of an index set, which is now equipped with a partial order. In the present abstract, we first define this new notion, and then proceed to present counterparts of classical theorems on ultraproducts, such as Łoś's Theorem, the Łoś-Suszko Theorem, and the Keisler–Shellac Theorem.

Definition 1.

1. A *wellfounded forest* is a poset in which every principal down-set is a wellorder.
2. Let $(I, \leq)$ be a wellfounded forest, F a filter in $\mathrm{Up}(I, \leq)$, and $(h_{ij} : M_i \to M_j \mid i \leq j \in I)$ be a direct system of homomorphisms between structures of a common signature. Define the set $\prod_F A_i$ to be

$$\{a \in \prod_{i \in I} M_i : \exists I' \in F\, \forall i \leq j \in I'\, h_{ij}(a(i)) = a(j)\}.$$

The relation defined by ($a \equiv_F b$ iff $[\![a = b]\!] \in F$) is a congruence of $\prod_F M_i$. We call $\prod_F M_i / \equiv_F$ a *filter product* of $\{M_i : i \in I\}$ and a *prime (filter) product*, when F is a prime filter of $\mathrm{Up}(I, \leq)$.

Remark 2.

1. Ultraproducts can be regarded as a special kind of prime filter products whose index sets are equipped with the diagonal binary relation.
2. This definition of prime filter products can be motivated in a sheaf-theoretic manner. Sheaf-theoretically, an ultraproduct $\prod_D M_i$ is the stalk at D of a sheaf obtained by composing the topological embedding of I into the Stone space of $\mathcal{P}(I)$ with the sheaf associating each finite subset $I' \subseteq I$ with the direct product $\prod_{i \in I'} M_i$. Naturally generalizing this, one can think of the stalk at D of a sheaf obtained by composing the topological embedding of a poset I into the Priestley space of the $\mathrm{Up}(I)$, where F is a prime filter of $\mathrm{Up}(I)$, with the sheaf

$$I' \mapsto \{a \in \prod_{i \in I'} M_i \mid \forall i \leq j \in I'\, h_{ij}(a(i)) = a(j)\}.$$

One can easily check that this stalk is nothing but the prime filter product $\prod_F M_i$.

To state our results, we need to define the following fragment of first-order logic studied in positive logic.

Definition 3 ([1])**.**

1. A *positive existential* (or $\exists^+$) formula is a first-order formula obtained by existentially quantifying, finitely many times, disjunctions of conjuctions of atomic formulae. We assume that a first-order language always contains as a 0-ary predicate $\perp$, the contradiction.
2. A basic *h-inductive* formula is a first-order formula obtained by universally quantifying, finitely many times, a conditional between $\exists^+$ formulae. An *h-inductive* (or $\forall_2^+$) formula is a conjunction of basic h-inductive formulae.

Then we have the following analogue of Łoś's Theorem.

Theorem 4. *Given a direct system $(h_{ij} : M_i \to M_j)$ of homomorphisms indexed by a wellfounded forest I and a filter F in $\mathrm{Up}(I)$, for every positive primitive formula $\phi(\overline{x})$ (that is, a $\exists^+$ formula with no nontrivial disjunctions) and a tuple $\overline{a}$ of elements of the filter product $\prod_F M_i$,*

$$\prod_F M_i \models \phi(\overline{a}) \iff [\![\phi(\overline{a})]\!] \in F.$$

If F is prime, the displayed biconditional is true of all $\exists^+$ formulae.

One can also characterize the definability of a class of structures by $\forall_2^+$ sentences à la Łoś-Suszko.

Theorem 5. *A class K of algebras is axiomatized by $\forall_2^+$ sentences if and only if K is closed under ultraroots and prime products.*

We conclude this abstract by stating a counterpart of the Keisler-Shelah Theorem. First, we can show the following:

Theorem 6. *Let K a quasivariety of finite type, with a finite nontrivial member. Then the following conditions are equivalent:*

1. *The nontrivial members of K satisfy the same existential positive sentences.*
2. *Any two nontrivial members of K have isomorphic prime powers.*

In view of this, we conjecture the following. Here, a *positively existentially closed model* of T is a structure $M \models T$ such that for every homomorphism $h : M \to M'$ into another model M' of T, every $\exists^+$ formula $\phi(\overline{x})$, and every tuple $\overline{a}$ in M, $M' \models \phi(h(\overline{a}))$ implies $M \models \phi(\overline{a})$.

Conjecture 1. *Let T be a $\forall_2^+$ theory and κ be a large enough cardinal such that $2^\kappa = \kappa^+$ with additional set-theoretic assumptions. The following are equivalent.*

1. *Positively existentially closed models of T have the same $\exists^+$ theory.*
2. *Positively existentially closed models of T of cardinality less than κ have isomorphic prime powers.*

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Directed Partial Orders and Birkhoff-Pierce Problem

Yuehui Zhang

School of Mathematical Sciences, Shanghai Jiao Tong University, China

zyh@sjtu.edu.cn

The so-called Birkhoff–Pierce problem ([1]) was raised by G. Birkhoff and R. Pierce in 1956: "Can the field $\mathbb{C}$ of complex numbers be made into a lattice-ordered field?" This problem has a history more than 65 years and is still open today. M. Henriksen listed Birkhoff–Pierce problem in [3] as the top 2 problems of lattice-ordered rings and the reviewer of [3] commented that Birkhoff–Pierce problem is the most famous old question about lattice-ordered rings.

Birkhoff and Pierce prove in [1] that $\mathcal{C}$ does not admit any such orderings as an algebra over the real number field $\mathbb{R}$. L. Fuchs generalized in 1963 ([2, problem 31, p.212]) Birkhoff–Pierce problem as to describe the directed partial orders of $\mathbb{C}$, $\mathbb{R}$ and the division ring $\mathbb{H}$ of the quaternions. In 1976, R. Wilson surprisingly proved in [10] that there are infinitely many lattice orderings on $\mathbb{R}$, partly answered the Fuchs' problem over $\mathbb{R}$. In 1986, N. Schwartz proved in [8] that algebraic number fields admitting no total order do not allow a lattice order. The most important progress is due to N. Schwartz and Y. Yang, who in 2011 proved in [9] that $\mathbb{C}$ admits directed partial orders. However, Birkhoff–Pierce problem is still open since Schwartz and Yang further proved that none of those orders in [9] are lattice ones. Since 2017, J. Ma, L. Wu and the third author of the present article classified in a series papers all directed partial orders over an imaginary quadratic extension of a non-archimedean o-field in [4], [5], [6]. In [4], they use the notion *admissible semigroup* to classify directed partial orders with $1 > 0$, while they use the notion *special convex set* to classify directed partial orders with $1 \not> 0$ [5, 6]. In 2020, Z. Xu and the third author of the present article use the new concept *doubly convex set* in [11] to uniformly classify all directed partial orders over $F(i)$ without dividing them into two cases as $1 > 0$ and $1 \not> 0$.

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The Least Rectangular Dimonoid Congruences on the Free Trioid

Anatolii V. Zhuchok

Luhansk Taras Shevchenko National University, Poltava, Ukraine

zhuchok.av@gmail.com

Following [2], a nonempty set T equipped with three binary associative operations $\dashv$, $\vdash$, and $\perp$ satisfying the following axioms:

$$(x \dashv y) \dashv z = x \dashv (y \vdash z), \tag{T1}$$

$$(x \vdash y) \dashv z = x \vdash (y \dashv z), \tag{T2}$$

$$(x \dashv y) \vdash z = x \vdash (y \vdash z), \tag{T3}$$

$$(x \dashv y) \dashv z = x \dashv (y \perp z), \tag{T4}$$

$$(x \perp y) \dashv z = x \perp (y \dashv z), \tag{T5}$$

$$(x \dashv y) \perp z = x \perp (y \vdash z), \tag{T6}$$

$$(x \vdash y) \perp z = x \vdash (y \perp z), \tag{T7}$$

$$(x \perp y) \vdash z = x \vdash (y \vdash z) \tag{T8}$$

for all $x, y, z \in T$, is called a *trioid*. In [2], Loday and Ronco described the free trioid of rank 1. The free trioid of an arbitrary rank was given in [4]. Recall this construction.

Let X be an arbitrary nonempty set, and let $F[X]$ be the free semigroup on X. The length of an arbitrary word $w \in F[X]$ is denoted by ℓ_w. For any $n, k \in \mathbb{N}$ and $L \subseteq \{1, 2, \ldots, n\}$, $L \neq \varnothing$, we let $L + k = \{m + k \mid m \in L\}$. Define operations $\dashv$, $\vdash$, and $\perp$ on the set

$$F = \{(w, L) \mid w \in F[X],\ L \subseteq \{1, 2, \ldots, \ell_w\},\ L \neq \varnothing\}$$

by

$$(w, L) \dashv (u, R) = (wu, L), \quad (w, L) \vdash (u, R) = (wu, R + \ell_w),$$
$$(w, L) \perp (u, R) = (wu, L \cup (R + \ell_w))$$

for all $(w, L), (u, R) \in F$. By [4, Lemma 7.1 and Theorem 7.1], $(F, \dashv, \vdash, \perp)$ is the free trioid on X. It is denoted by $FT(X)$.

Recall that a *dimonoid* [1] is a nonempty set T equipped with two binary associative operations $\dashv$ and $\vdash$ satisfying the axioms $(T1)$–$(T3)$. A dimonoid $(T, \dashv, \vdash)$ is called a *rectangular dimonoid* [3] if both semigroups $(T, \dashv)$ and $(T, \vdash)$ are rectangular bands. If ρ is a congruence on a trioid $(T, \dashv, \vdash, \perp)$ such that two operations of $(T, \dashv, \vdash, \perp)/\rho$ coincide and $(T, \dashv, \vdash, \perp)/\rho$ is a rectangular dimonoid, we say that ρ is a *rectangular dimonoid congruence*.

We characterize the least rectangular dimonoid congruences on the free trioid $FT(X)$.

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On Automorphisms of the Category of Free g-Dimonoids

Yurii V. Zhuchok

Department of Algebra and System Analysis, Luhansk Taras Shevchenko National University, Ukraine

zhuchok.yu@gmail.com

The problem of studying the automorphism group of the category of free algebras in a certain variety was raised by Plotkin in his papers on universal algebraic geometry (see, e.g., [1]). Now there are many papers devoted to describing automorphisms of the category of free finitely generated universal algebras in some varieties: groups, (inverse) semigroups, associative algebras, modules and semimodules, Lie algebras, commutative dimonoids and other algebras.

A nonempty set D with two binary associative operations $\dashv$ and $\vdash$ is called a *g-dimonoid* [2] if

$$(x \dashv y) \dashv z = x \dashv (y \vdash z),\ (x \dashv y) \vdash z = x \vdash (y \vdash z)$$

for all $x, y, z \in D$. In [3] the Plotkin's problem was considered for the variety of commutative g-dimonoids. Here we study the similar problem for the category of free g-dimonoids.

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