A journey into Menger-type properties in locales

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where Menger defined the following basis covering property for metric spaces:

For each basis $\mathcal{B}$ for the topology of a metric space X, there is a sequence $(B_n)_{n\in\mathbb{N}}$ in $\mathcal{B}$ such that $lim_{n\to\infty} diam(B_n)_{n\in\mathbb{N}} = 0$ and X is covered by $(B_n)_{n\in\mathbb{N}}$.

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Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of an infinite set X, for each sequence $(A_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{A}$, there is a sequence $(B_n)_{n\in\mathbb{N}}$ of finite sets such that for each n, we have $B_n \subseteq A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

Definition

A topological space X is Menger, if for every sequence $(\mathscr{U}_n)$ of open covers of X, we can select, for each n, a finite $\mathscr{V}_n \subseteq \mathscr{U}_n$ such that $\bigcup_{n\in\mathbb{N}} \mathscr{V}_n$ is a cover of X.

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- Combinatorics and Forcing – Menger property characterizes filters whose Mathias forcing notion does not add dominating functions. (See – D. Chodounsky, D. Repovš, L. Zdomskyy: *Mathias forcing and combinatorial properties of filters*: J. Symbolic logic, 80 (2015), 1398 – 1410.)

- Games – there is a natural connection between the Menger property and an infinitely long game for two players. (See – W. Hurewicz: *Über die Verallgemeinerung des Borelschen Theorems*: Mathematische Zeitschrift, 24 (1925), 401–425.)

- Ramsey theory – Ramsey theoretical results can be derived from game-theoretic statements, and selection hypotheses can be derived from Ramseyan partition relations. (See – Lj.D.R. Kočinac & M. Scheepers: *Combinatorics of open covers (VII): Groupability*, Fundamenta Mathematicae 179 : 2 (2003), 131–155.)

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- A frame is a complete lattice L in which the distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge x \mid x \in S\}$$

holds for all $a \in L$ and $S \subseteq L$.

- A frame homomorphism is a mapping $h : L \longrightarrow M$ between frames which preserves arbitrary joins and finite meets.

- An element p of a frame L is called a point (or a prime) if it satisfies the property that $p < 1$ and $(\forall x, y \in L)(x \wedge y \leq p \implies x \leq p \text{ or } y \leq p)$.

- For any $a \in L$, the pseudocomplement of a is defined by $a^* = a \to 0 = \bigvee\{x \in L \mid x \wedge a = 0\}$.

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- A sublocale S of the locale L is a subset $S \subseteq L$ such that
  - (i) for every $A \subseteq S$, $\bigwedge A$ is in S (in particular $1 = \bigwedge \emptyset \in S$), and
  - (ii) for each $x \in L$ and $s \in S$, $x \to s \in S$.

- The lattice of sublocales of a frame L, ordered by inclusion, is a coframe denoted by $\mathcal{S}(L)$.

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  - A sublocale of L is called regular-closed (resp. regular-open) in case it is of the form $\mathfrak{c}_L(a)$ (resp. $\mathfrak{o}_L(a)$) with a regular, that is, $a = a^{**}$.

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- By a cover of L we mean a set $C \subseteq L$ such that $\bigvee C = 1$.

- A collection $\mathscr{C}$ of sublocales of L is a covering of L if $\bigvee\{T \mid T \in \mathscr{C}\} = L$, where the join is calculated in $\mathcal{S}(L)$.
  If every sublocale in a covering $\mathscr{C}$ of L is open, then $\mathscr{C}$ is an open covering of L.

- There is a bijection between covers and open coverings given by

$$C \mapsto \mathscr{C}^C := \{\mathfrak{o}_L(c) \mid c \in C\} \text{ and } \mathscr{C} \mapsto C^{\mathscr{C}} := \{x \in L \mid \mathfrak{o}_L(x) \in \mathscr{C}\}.$$

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Definition

A frame L is Menger if for every sequence $(\mathscr{C}_n)$ of open coverings of L, there exists, for each n, a finite $\mathscr{D}_n \subseteq \mathscr{C}_n$ such that $\bigcup_{n\in\mathbb{N}} \mathscr{D}_n$ is a covering of L. In this case, we say the sequence $(\mathscr{D}_n)$ is a Menger witness for $(\mathscr{C}_n)$.

Examples: every compact frame (in fact, every σ-compact one – meaning one that is a join of countably many compact sublocales) is Menger.

Proposition

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Every Menger frame is Lindelöf.

A Menger frame need not be spatial.

We give an example of a non-spatial compact locale.

This is recorded in Stone Spaces by P.T. Johnstone.

Example

Let $\alpha\mathbb{Q}$ denote the one-point compactification of $\mathbb{Q}$, and let A be the locale $\Omega(\alpha\mathbb{Q}) \times \Omega(\alpha\mathbb{Q})$. A is compact. But, since $\mathbb{Q}$ is an open subspace of $\alpha\mathbb{Q}$, we have that $\Omega\mathbb{Q} \times \Omega\mathbb{Q}$ is an open sublocale of $\Omega(\alpha\mathbb{Q}) \times \Omega(\alpha\mathbb{Q})$; and since $\Omega\mathbb{Q} \times \Omega\mathbb{Q}$ is not spatial, it follows that A is not spatial.

Every subframe of a Menger frame is Menger. Thus, a localic image of a Menger frame is Menger.

Since every cover of a subframe is a cover of the ambient frame.

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Corollary

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In light of the dual adjunction

$$\mathbf{Top} \underset{\Sigma}{\overset{\Omega}{\rightleftarrows}} \mathbf{Frm}$$

We recall that L is spatial if and only if the frame homomorphism $\eta_L : L \to \Omega(\Sigma L)$ is one-one.

Example

Let L be a frame with no points, such as the smallest dense sublocale of $\Omega(\mathbb{R})$. Let $\tilde{L}$ be the frame obtained from L by adjoining a new top element $1_{\tilde{L}} > 1_L$. Then $\tilde{L}$ is not spatial and $\mathrm{Pt}(\tilde{L}) = \{1_L\}$. From the latter, it is not hard to see that $\eta_{\tilde{L}}$ is codense.

Proposition

A frame whose spatial reflection is a codense sublocale is Menger iff its spectrum is Menger.

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Let L be a frame with no points, such as the smallest dense sublocale of $\Omega(\mathbb{R})$. Let $\tilde{L}$ be the frame obtained from L by adjoining a new top element $1_{\tilde{L}} > 1_L$. Then $\tilde{L}$ is not spatial and $\mathrm{Pt}(\tilde{L}) = \{1_L\}$. From the latter, it is not hard to see that $\eta_{\tilde{L}}$ is codense.

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A frame whose spatial reflection is a codense sublocale is Menger iff its spectrum is Menger.

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The following are equivalent for a frame L.

(a) *L is Menger.*

(b) *For every sequence (C_n) of directed covers of L, there exists, for each n, an element $c_n \in C_n$ such that $\{c_n \mid n \in \mathbb{N}\}$ is a cover of L.*

(c) *For every sequence $(\mathscr{U}_n)$ of directed open coverings of L, there exists, for each n, a sublocale $U_n \in \mathscr{U}_n$ such that $\{U_n \mid n \in \mathbb{N}\}$ is a covering of L.*

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The join of finitely many Menger sublocales of a frame is Menger.

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Suppose we summarize “pictorially" using the acronyms below that a frame homomorphism satisfies:

- (DJ) if its right adjoint preserves directed joins;
- (DC) if its right adjoint sends directed covers to directed covers; and
- (CC) if its right adjoint sends covers to covers.

Then

$$(\mathrm{DJ}) \;\not\Longrightarrow\; (\mathrm{CC}) \;\not\Longrightarrow\; (\mathrm{DJ}); \quad (\mathrm{CC}) \;\Longrightarrow\; (\mathrm{DC}) \;\not\Longrightarrow\; (\mathrm{CC});$$

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A cover B of a frame L is called a strong refinement of a cover C if for every $b \in B$ there is a $c \in C$ such that $b \prec c$.

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Corollary

Let $h\colon L \to M$ be a frame homomorphism.

(a) *If h is weakly perfect and L is Menger, then M is Menger.*

(b) *If h is dense and weakly perfect, L is cover regular, and M is Menger, then L is Menger.*

Corollary

If L is compact and M is Menger, then $L \oplus M$ is Menger.

It is shown in

Wei He, Maokang Luo
Completely regular proper reflection of locales over a given locale
Proc. Amer. Math. Soc. 141 (2013), 403–408.

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Definition

A frame L is projectively Menger if every subframe of L with a countable base is Menger.

Rephrasing, we say, L is projectively Menger in case whenever $h\colon M \to L$ is a one-one frame homomorphism and M has a countable base, then M is Menger.

Lemma

Every frame with a countable base is Lindelöf.

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The following are equivalent for a frame L.

(a) *L is projectively Menger.*

(b) *Every Lindelöf subframe of L is Menger.*

(c) *For every sequence (C_n) of countable covers of L, there exists, for each n, a finite $D_n \subseteq C_n$ such that $\bigcup_{n\in\mathbb{N}} D_n$ is a cover of L.*

(d) *For every sequence (C_n) of increasing countable covers of L, there exists, for each n, an element $c_n \in C_n$ such that $\{c_n \mid n \in \mathbb{N}\}$ is a cover of L.*

(e) *For every sequence $(\mathscr{C}_n)$ of countable open coverings of L, there exists, for each n, a finite $\mathscr{D}_n \subseteq \mathscr{C}_n$ such that $\bigcup_{n\in\mathbb{N}} \mathscr{D}_n$ is a covering of L.*

Corollary

A frame is Menger if and only if it is Lindelöf and projectively Menger.

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Corollary

A frame is Menger if and only if it is Lindelöf and projectively Menger.

A space X is called almost Menger if for every sequence $(\mathscr{C}_n)$ of open covers of X, there exists, for each n, a finite $\mathscr{D}_n \subseteq \mathscr{C}_n$ such that $\bigcup\{\overline{D} \mid D \in \bigcup_{n\in\mathbb{N}} \mathscr{D}_n\} = X$.

Lj. D. R. Kočinac
Star-Menger and related spaces II
Filomat 13 (1999), 129–140.

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Theorem

Let X be a topological space.

(a) *If X is almost Menger, then $\Omega(X)$ is almost Menger.*

(b) *If X is sober T_D-space, then $\Omega(X)$ is almost Menger iff X is almost Menger.*

Proposition

- *A frame L is almost Menger iff for every sequence* $(\mathscr{C}_n)$ *of directed open coverings of L, there exists, for each n, a sublocale* $C_n \in \mathscr{C}_n$ *such that* $\bigvee\{\overline{C_n} \mid n \in \mathbb{N}\} = L$.
- *A localic image of an almost Menger frame is almost Menger.*
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A frame L is almost Menger iff for every sequence (C_n) *of directed covers of L, we can select, for each n, an element* $c_n \in C_n$ *such that any* $a \in L$ *is expressible as* $a = \bigwedge_{n\in\mathbb{N}} t_n$ *for some elements* $t_n \in L$ *with each* $t_n \geq c_n^*$.

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In topological spaces,

- Regular-closed subspaces of almost Menger spaces need not be almost Menger.
- Clopen subspaces inherit the almost Menger property.

Definition

An element a of a frame L is co-linear in case $a \vee \bigwedge_{i\in I} x_i = \bigwedge_{i\in I}(a \vee x_i)$ for all families $\{x_i\}_{i\in I}$ of elements of L.

Example

Let a be a co-linear element of L and let $\kappa_a : L \to \uparrow a$ be the map given by $\kappa_a(x) = a \vee x$. Then κ_a is an onto, weakly perfect (actually, perfect) frame homomorphism preserving meets (since a is co-linear). Note though that κ_a is not an isomorphism if $a \neq 0$.

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J. Gutiérrez García, I. Mozo Carollo, J. Picado
Normal semicontinuity and the Dedekind completion of pointfree function rings
Algebra Universalis **75** (2016), 301–330

If $h\colon L \to M$ is a perfect frame homomorphism, then $h_*(a^*) \leq h_*(a)^*$ for every $a \in M$.

Proposition

Let $h\colon L \to M$ be a meet-preserving perfect onto frame homomorphism. If L is an almost Menger frame, then so is M.

Corollary

If L is almost Menger, then $c_L(a)$ is almost Menger for every co-linear $a \in L$. In particular, every closed sublocale of an almost Menger frame which is also a coframe is almost Menger.

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In

G. Di Maio, Lj.D.R. Kočinac
A note on quasi-Menger and similar spaces
Topology. Appl. **179** (2015) 148–155a

A topological space X is called quasi-Menger if for every closed set $F \subseteq X$ and every sequence $(\mathscr{V}_n)$ of covers of F by sets open in X, there exists, for each n, a finite $\mathscr{U}_n \subseteq \mathscr{V}_n$ such that $F \subseteq \overline{\bigcup_{n \in \mathbb{N}} \bigcup \mathscr{U}_n}$.

Definition

A frame L is quasi-Menger (resp. regularly quasi-Menger) if for every closed (resp. regular-closed) sublocale F of L and each sequence $(\mathscr{V}_n)$ with $\mathscr{V}_n$ consisting of open sublocales of L which cover F, there exists, for each n, a finite $\mathscr{U}_n \subseteq \mathscr{V}_n$ such that $F \subseteq \overline{\bigvee_{n \in \mathbb{N}} \bigvee \mathscr{U}_n}$.

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Theorem

A frame L is qM *(resp.* rqM*) iff for every $a \in L$ (resp. regular $a \in L$) and every sequence (V_n) of subsets of L with $a \vee \bigvee V_n = 1$ for each n, there is a finite $U_n \subseteq V_n$ such that $\left(\bigvee_{n\in\mathbb{N}} u_n\right)^* \leq a$, where $u_n = \bigvee U_n$.*

Corollary

A subframe of a qM *frame is* qM*. Hence, a localic image of a* qM *frame is* qM*.*

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A space X is qM *iff $\Omega(X)$ is* qM*.*

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A subframe of a qM *frame is* qM*. Hence, a localic image of a* qM *frame is* qM*.*

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A space X is qM *iff $\Omega(X)$ is* qM*.*

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A topological space X is called weakly Menger if for every sequence $(\mathscr{V}_n)$ of open covers of X, there exists, for each n, a finite $\mathscr{U}_n \subseteq \mathscr{V}_n$ such that $\bigcup_{n\in\mathbb{N}}\bigcup\mathscr{U}_n$ is dense in X.

P. Daniels
Pixley-Roy spaces over subsets of the reals
Topology Appl. 29 (1988) 93–106.

these spaces are called "weakly Hurewicz".

Definition

A frame L is weakly Menger (abbreviated wM) if for every sequence $(\mathscr{C}_n)$ of open coverings of L, there exists, for each n, a finite $\mathscr{D}_n \subseteq \mathscr{C}_n$ such that $\bigvee\{T \mid T \in \bigcup_{n\in\mathbb{N}}\mathscr{D}_n\}$ is a dense sublocale of L. We shall say the sequence $(\mathscr{D}_n)$ is a *weakly Menger witness* (abbreviated wM-witness) for the sequence $(\mathscr{C}_n)$.

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Proposition

The following are equivalent for a frame L.

1. L *is* wM.
2. *For every sequence* $(\mathscr{C}_n)$ *of directed open coverings of* L*, there exists, for each* n*, some* $C_n \in \mathscr{C}_n$ *such that* $\bigvee_{n\in\mathbb{N}} C_n$ *is a dense sublocale of* L*.*
3. *For every sequence* (C_n) *of covers of* L*, there exists, for each* n*, a finite* $D_n \subseteq C_n$ *such that* $\bigvee D$ *is a dense element in* L*, where* $D = \bigcup_{n\in\mathbb{N}} D_n$*.*
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Proposition

A space X is wM *iff* $\Omega(X)$ *is* wM.

Remark

Abbreviate by M the Menger property. We then have the non-reversible implications:

$$\text{M} \implies \text{aM} \implies \text{wM} \qquad \text{and} \qquad \text{M} \implies \text{qM} \implies \text{wM}.$$

In the case of Boolean frames all these implications are equivalences.

To see this, observe that if L is Boolean and wM, then it is Menger because the only dense element in a Boolean frame is the top.

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Definition

We say an open covering $\mathscr{U}$ of a frame L is a κ-covering if $L \notin \mathscr{U}$ and for every compact sublocale K of L, there exists some $U \in \mathscr{U}$ such that $K \subseteq U$.

Let L_s denote the *semiregularization* of L.

Theorem

The following are equivalent for a frame L.

1. *L is wM.*
2. *L_s is wM.*
3. *Every sequence of κ-coverings of L has a wM-witness.*
4. *Whenever $h: M \to L$ is a dense homomorphism, then M is wM.*

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Corollary

Let L be a frame.

(a) *The closure of any* wM *sublocale of L is* wM.

(b) *If the smallest dense sublocale of L is* wM, *then L is* wM.

Corollary

Let $(L_i \mid i \in I)$ be a family of frames, and $(X_j \mid j \in J)$ a family of topological spaces.

(a) *If the coproduct $\bigoplus_{i\in I} L_i$ is* wM, *then each L_i is* wM.

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Remark

Even though the localic product $\prod_{j \in J} \Omega(X_j)$ is not necessarily isomorphic to the locale $\Omega\left(\prod_{j \in J} X_j\right)$, we are able to deduce as a corollary of the localic result that if the product of topological spaces is weakly Menger, then so is each factor.

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An infinitely extremally disconnected frame is wM iff its smallest dense sublocale is wM.

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THANK YOU FOR YOUR ATTENTION.