A journey into Menger-type properties in locales

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Quasi-Menger locales

Weakly Menger locales

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where Menger defined the following basis covering property for metric spaces:

For each basis *B* for the topology of a metric space *X*, there is a sequence $(B_n)_{n \in \mathbb{N}}$ in *B* such that $\lim_{n \to \infty} diam(B_n)_{n \in \mathbb{N}} = 0$ and *X* is covered by $(B_n)_{n \in \mathbb{N}}$.

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Uber die Verallgemeinerung des Borelschen Theorems

Mathematische Zeitschrift 24 (1925), 401-425,

Let \mathcal{A} and \mathcal{B} be families of subsets of an infinite set X, for each sequence $(A_n)_{n \in \mathbb{N}}$ of elements of \mathcal{A} , there is a sequence $(B_n)_{n \in \mathbb{N}}$ of finite sets such that for each n, we have $B_n \subseteq A_n$, and $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{B}$.

Definition

A topological space X is Menger, if for every sequence (\mathscr{U}_n) of open covers of X, we can select, for each n, a finite $\mathscr{V}_n \subseteq \mathscr{U}_n$ such that $\bigcup_{n \in \mathbb{N}} \mathscr{V}_n$ is a cover of X.

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 - Games there is a natural connection between the Menger property and an infinitely long game for two players. (See W. Hurewicz: Uber die Verallgemeinerung des Borelschen Theorems: Mathematische Zeitschrift, 24 (1925), 401-425.)
- Ramsey theory Ramsey theoretical results can be derived from game-theoretic statements, and selection hypotheses can be derived from Ramseyan partition relations. (See – Lj.D.R. Kočinac & M. Scheepers; *Combinatorics of open covers (VII)*: Groupability, Fundamenta Mathematicae 179 :

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holds for all $a \in L$ and $S \subseteq L$.

- A frame homomorphism is a mapping h : L → M between frames which preserves arbitrary joins and finite meets.
- An element p of a frame L is called a point (or a prime) if it satisfies the property that p < 1 and (∀x, y ∈ L)(x ∧ y ≤ p ⇒ x ≤ p or y ≤ p).
- For any a ∈ L, the pseudocomplement of a is defined by a^{*} = a → 0 = \/{x ∈ L | x ∧ a = 0}.

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• A sublocale *S* of the locale *L* is a subset $S \subseteq L$ such that

- (i) for every $A \subseteq S$, $\bigwedge A$ is in *S* (in particular $1 = \bigwedge \emptyset \in S$), and
- (ii) for each $x \in L$ and $s \in S$, $x \to s \in S$.
- The lattice of sublocales of a frame L, ordered by inclusion, is a coframe denoted by S(L).

For any a in a locale L,

- o_L(a) = {a → x | x ∈ L} = {x ∈ L | a → x = x} is called an open sublocale of L.
- $c_L(a) = \uparrow a = \{x \in L \mid x \ge a\}$ is called a closed sublocale of *L*.
- A sublocate of L is called regular-closed (resp. regular-open) in case it is of the form c_L(a) (resp. σ_L(a)) with a regular, that is, a = a^{**}.

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- By a cover of *L* we mean a set $C \subseteq L$ such that $\bigvee C = 1$.
- There is a bijection between covers and open coverings given by $C \mapsto \mathscr{C}^{C} := \{o_{L}(c) \mid c \in C\} \text{ and } \mathscr{C} \mapsto C^{\mathscr{C}} := \{x \in L \mid o_{L}(x) \in \mathscr{C}\}.$

 A cover C of L is said to refine a cover D if for every c ∈ C there is a d ∈ D such that c ≤ d. In this case, C is called a refinement of D.

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 If every sublocale in a covering 𝒞 of L is open, then 𝒞 is an open covering of L.
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A frame *L* is Menger if for every sequence (\mathscr{C}_n) of open coverings of *L*, there exists, for each *n*, a finite $\mathscr{D}_n \subseteq \mathscr{C}_n$ such that $\bigcup_{n \in \mathbb{N}} \mathscr{D}_n$ is a covering of *L*. In this case, we say the sequence (\mathscr{D}_n) is a Menger witness for (\mathscr{C}_n) .

Examples: every compact frame (in fact, every σ -compact one – meaning one that is a join of countably many compact sublocales) is Menger.

Proposition

A frame L is Menger iff for every sequence (C_n) of covers of L, there exists, for each n, a finite $D_n \subseteq C_n$ such that $\bigcup_{n \in \mathbb{N}} D_n$ is a cover of L.

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We give an example of a non-spatial compact locale.

This is recorded in Stone Spaces by P.T. Johnstone.

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Let $\alpha \mathbb{Q}$ denote the one-point compactification of \mathbb{Q} , and let A be the locale $\Omega(\alpha \mathbb{Q}) \times \Omega(\alpha \mathbb{Q})$. A is compact. But, since \mathbb{Q} is an open subspace of $\alpha \mathbb{Q}$, we have that $\Omega \mathbb{Q} \times \Omega \mathbb{Q}$ is an open sublocale of $\Omega(\alpha \mathbb{Q}) \times \Omega(\alpha \mathbb{Q})$; and since $\Omega \mathbb{Q} \times \Omega \mathbb{Q}$ is not spatial, it follows that A is not spatial.

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Oghenetega Ighedo (Unisa)

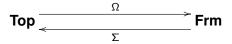
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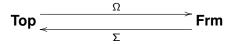
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Let L be a frame with no points, such as the smallest dense sublocale of $\Omega(\mathbb{R})$. Let \tilde{L} be the frame obtained from L by adjoining a new top element $\mathbf{1}_{\tilde{L}} > \mathbf{1}_{L}$. Then \tilde{L} is not spatial and $Pt(\tilde{L}) = \{\mathbf{1}_{L}\}$. From the latter, it is not hard to see that $\eta_{\tilde{I}}$ is codense.

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A frame whose spatial reflection is a codense sublocale is Menger iff its spectrum is Menger.



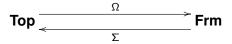
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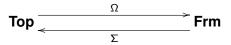
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(DJ) if its right adjoint preserves directed joins;
 (DC) if its right adjoint sends directed covers to directed covers; and

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Then

 $(DJ) \Rightarrow (CC) \Rightarrow (DJ); (CC) \Rightarrow (DC) \Rightarrow (CC);$

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A cover *B* of a frame *L* is called a strong refinement of a cover *C* if for every $b \in B$ there is a $c \in C$ such that $b \prec c$.

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- Let $h: L \to M$ be a frame homomorphism.
- (a) If h is weakly perfect and L is Menger, then M is Menger.
- (b) If h is dense and weakly perfect, L is cover regular, and M is Menger, then L is Menger.

Corollary

If L is compact and M is Menger, then $L \oplus M$ is Menger.

It is shown in



Completely regular proper reflection of locales over a given locale Proc. Amer. Math. Soc. **141** (2013), 403–408,

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Definition

A frame *L* is projectively Menger if every subframe of *L* with a countable base is Menger.

Rephrasing, we say, L is projectively Menger in case whenever $h: M \rightarrow L$ is a one-one frame homomorphism and M has a countable base, then M is Menger.

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Every frame with a countable base is Lindelöf.

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The following are equivalent for a frame L.

- (a) L is projectively Menger.
- (b) Every Lindelöf subframe of L is Menger.
- (c) For every sequence (C_n) of countable covers of L, there exists, for each n, a finite $D_n \subseteq C_n$ such that $\bigcup_{n \in \mathbb{N}} D_n$ is a cover of L.
- (d) For every sequence (C_n) of increasing countable covers of L, there exists, for each n, an element $c_n \in C_n$ such that $\{c_n \mid n \in \mathbb{N}\}$ is a cover of L.
- (e) For every sequence (𝔅_n) of countable open coverings of L, there exists, for each n, a finite 𝔅_n ⊆ 𝔅_n such that ⋃_{n∈ℕ}𝔅_n is a covering of L.

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A space *X* is called almost Menger if for every sequence (\mathscr{C}_n) of open covers of *X*, there exists, for each *n*, a finite $\mathscr{D}_n \subseteq \mathscr{C}_n$ such that $\bigcup \{\overline{D} \mid D \in \bigcup_{n \in \mathbb{N}} \mathscr{D}_n\} = X.$



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Star-Menger and related spaces II

Filomat 13 (1999), 129-140.

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Theorem

- Let X be a topological space.
 - (a) If X is almost Menger, then $\Omega(X)$ is almost Menger.
 - (b) If X is sober T_D -space, then $\Omega(X)$ is almost Menger iff X is almost Menger.

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- A frame L is almost Menger iff for every sequence (𝒞_n) of directed open coverings of L, there exists, for each n, a sublocale C_n ∈ 𝒞_n such that \\{ (C_n | n ∈ ℕ} = L.
- A localic image of an almost Menger frame is almost Menger.
- A frame L is almost Menger iff for every sequence (C_n) of covers of L, there exists, for each n, a finite D_n ⊆ C_n such that every element a of L is expressible as a = Λ_αt_α where each t_α ≥ d^{*}_α for some d_α ∈ U_{n∈N}D_n.

Corollary

A frame L is almost Menger iff for every sequence (C_n) of directed covers of L, we can select, for each n, an element $c_n \in G_n$ such that any $a \in L$ is expressible as $a = \bigwedge_{n \in \mathbb{N}} t_n$ for some elements $t_n \in L$ with each $t_n \ge c_n^*$.

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A frame L is almost Menger iff for every sequence (C_n) of directed covers of L, we can select, for each n, an element $c_n \in G_n$ such that any $a \in L$ is expressible as $a = \bigwedge_{n \in \mathbb{N}} t_n$ for some elements $t_n \in L$ with each $t_n \ge c_n^*$.

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 Regular-closed subspaces of almost Menger spaces need not be almost Menger.

Clopen subspaces inherit the almost Menger property.

Definition

An element *a* of a frame *L* is co-linear in case $a \vee \bigwedge_{i \in I} x_i = \bigwedge_{i \in I} (a \vee x_i)$ for all families $\{x_i\}_{i \in I}$ of elements of *L*.

Example

Let *a* be a co-linear element of *L* and let $\kappa_a: L \to \uparrow a$ be the map given by $\kappa_a(x) = a \lor x$. Then κ_a is an onto, weakly perfect (actually, perfect) frame homomorphism preserving meets (since *a* is co-linear). Note though that κ_a is not an isomorphism if $a \neq 0$.

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if h: $L \to M$ is a perfect frame homomorphism, then $h_*(a^*) \le h_*(a)^*$ for every $a \in M$.

Proposition

Let $h: L \rightarrow M$ be a meet-preserving perfect onto frame homomorphism. If L is an almost Menger frame, then so is M.

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If L is almost Menger, then $\epsilon_L(a)$ is almost Menger for every co-linear $a \in L$. In particular, every closed sublocale of an almost Menger frame which is also a coframe is almost Menger. We recall the following remark from

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G. Di Maio, Lj.D.R. Kočinac A note on quasi-Menger and similar spaces Topology. Appl. **179** (2015) 148–155a

A topological space X is called quasi-Menger if for every closed set $F \subseteq X$ and every sequence (\mathscr{V}_n) of covers of F by sets open in X, there exists, for each n, a finite $\mathscr{U}_n \subseteq \mathscr{V}_n$ such that $F \subseteq \overline{\bigcup_{n \in \mathcal{W}}} \bigcup \mathscr{U}_n$.

Definition

A frame *L* is quasi-Menger (resp. regularly quasi-Menger) if for every closed (resp. regular-closed) sublocale *F* of *L* and each sequence (\mathscr{V}_n) with \mathscr{V}_n consisting of open sublocales of *L* which cover *F*, there exists, for each *n*, a finite $\mathscr{U}_n \subseteq \mathscr{V}_n$ such that $F \subseteq \sqrt{v_n} \sqrt{\mathscr{U}_n}$.

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A frame L is qM (resp. rqM) iff for every $a \in L$ (resp. regular $a \in L$) and every sequence (V_n) of subsets of L with $a \lor \bigvee V_n = 1$ for each n, there is a finite $U_n \subseteq V_n$ such that $(\bigvee_{n \in \mathbb{N}} u_n)^* \leq a$, where $u_n = \bigvee U_n$.

Corollary

A subframe of a qM frame is qM. Hence, a localic image of a qM frame is qM.

Theorem A space X is qM iff $\Omega(X)$ is qM.

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A normal frame is gM iff it is rgM.

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Topology Appl. 29 (1988) 93-106,

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A frame *L* is weakly Menger (abbreviated wM) if for every sequence (\mathscr{C}_n) of open coverings of *L*, there exists, for each *n*, a finite $\mathscr{D}_n \subseteq \mathscr{C}_n$ such that $\bigvee \{T \mid T \in \bigcup_{n \in \mathbb{N}} \mathscr{D}_n\}$ is a dense sublocale of *L*. We shall say the sequence (\mathscr{D}_n) is a *weakly Menger witness* (abbreviated wM-witness) for the sequence (\mathscr{C}_n) .

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Oghenetega Ighedo (Unisa)

A space X is wM iff $\Omega(X)$ is wM.

Abbreviate by M the Menger property. We then have the non-reversible implications:

$M \implies aM \implies wM$ and $M \implies qM \implies wM$.

In the case of Boolean frames all these implications are equivalences.

To see this, observe that if *L* is Boolean and wM, then it is Menger because the only dense element in a Boolean frame is the top.

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We say an open covering \mathscr{U} of a frame *L* is a κ -covering if $L \notin \mathscr{U}$ and for every compact sublocale *K* of *L*, there exists some $U \in \mathscr{U}$ such that $K \subseteq U$.

Let L_s denote the *semiregularization* of *L*.

Theorem

The following are equivalent for a frame L.

L is wM.

■ L_s is wM.

Every sequence of κ-coverings of L has a wM-witness.

Whenever h: M → L is a dense homomorphism, then M is wM.

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- (a) The closure of any wM sublocale of L is wM.
- (b) If the smallest dense sublocale of L is wM, then L is wM.

Corollary

Let $(L_i \mid i \in I)$ be a family of frames, and $(X_j \mid j \in J)$ a family of topological spaces.

- (a) If the coproduct $\bigoplus_{i \in I} L_i$ is wM, then each L_i is wM.
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Remark

Even though the localic product $\prod_{j \in J} \Omega(X_j)$ is not necessarily isomorphic to the locale $\Omega(\prod_{j \in J} X_j)$, we are able to deduce as a corollary of the localic result that if the product of topological spaces is weakly Menger, then so is each factor.

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On infinite variants of De Morgan law in locale theory

J. Pure Appl. Algebra 225 (2021) article 106460,

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Menger-type properties in biframes.



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THANK YOU FOR YOUR ATTENTION.

Oghenetega Ighedo (Unisa)

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