Continuous convergence in $\mathcal{R}L$

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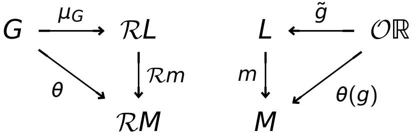
The pointfree Yosida Adjunction

W is the category of archimedean vector lattice with designate weak order unit. $\mathcal{R}L$ is the **W**-object of frame homomorphisms from the topology $\mathcal{O}\mathbb{R}$ of the real numbers into a completely regular frame L.

Theorem (Madden-Vermeer, 1983)

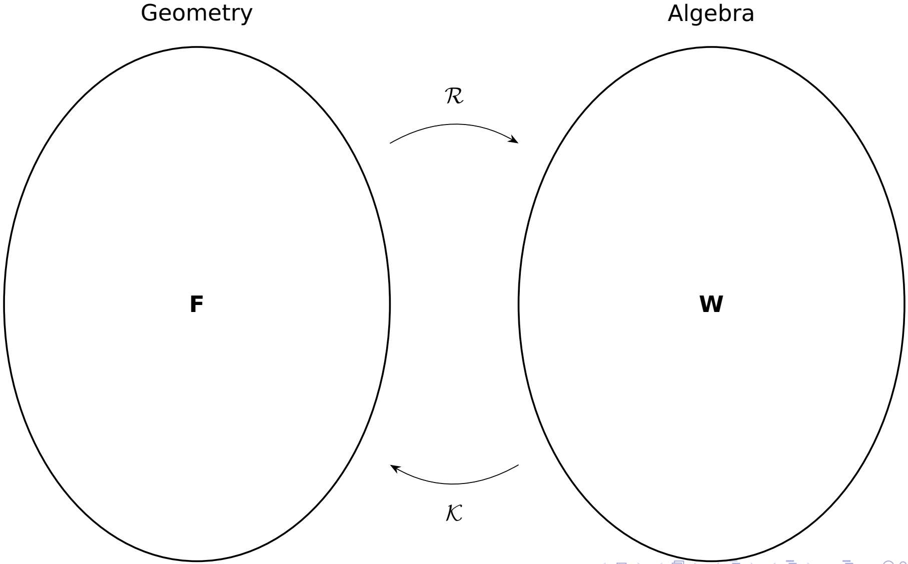
For each **W**-object G there is a completely regular frame L and a **W**-injection $\mu_G\colon G \to \mathcal{R}L$ such that for any completely regular frame M and **W**-homomorphism θ there is a unique frame homomorphism m such that $\theta(g) = m \circ \tilde{g}$ for all $g \in G$.



The frame L is called the *Madden frame of* G. It is the frame of **W**-kernels of G and is designated $\mathcal{K}G$.

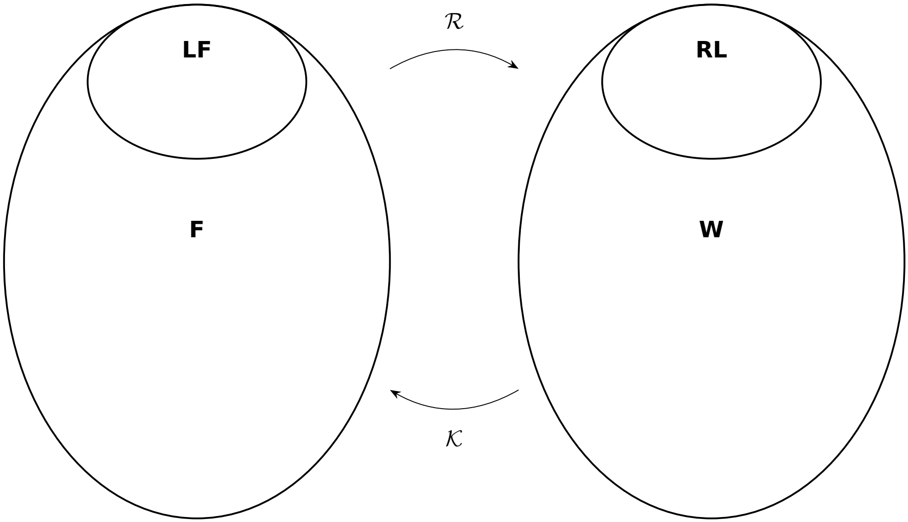
The pointfree Yosida adjunction

F is the category of completely regular frames and frame homomorphisms.



The objects fixed by the adjunction

LF is the full subcategory of Lindelöf frames. **RL** is the full subcategory of **W**-objects of the form $\mathcal{R}L$ for some L.



$\mathcal{KR}L = \mathcal{L}L$

- An interesting feature of **F** is that one has a Lindelöf coreflection $\mathcal{L}L \to L$.
- Since $\mathcal{O}\mathbb{R}$ is Lindelöf, every frame homomorphism $\mathcal{O}\mathbb{R} \to L$ factors through $\mathcal{L}L \to L$.

- This means that $\mathcal{R}L = \mathcal{RL}L$.
- In other words, $\mathcal{R}L$ does not "see" L very clearly.
- We address this issue in this talk.

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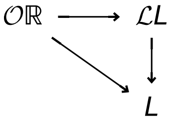

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Characterizing **RL**-objects

- The **W**-objects of the form $\mathcal{C}X$ are tricky to characterize algebraically. One of us has written several deep papers on the intricacies of this matter. Perhaps the most interesting results are negative.
- **W**-objects of the form $\mathcal{R}L$ are somewhat better described. Two of us have written several papers on this issue.
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Convergences on sets

A *convergence* on a set G is obtained by specifying which filters $\mathscr{F}$ converge to which points g, written $\mathscr{F} \longrightarrow g$, subject to three mild constraints.

- $\dot{g} \longrightarrow g$ for all $g \in G$. Here $\dot{g}$ is $\{ K \subseteq G : g \in K \}$, the principal ultrafilter of g.
- If $\mathscr{G} \supseteq \mathscr{F} \longrightarrow g$ then $\mathscr{G} \longrightarrow g$.
- If $\mathscr{F} \longrightarrow g$ and $\mathscr{G} \longrightarrow g$ then $\mathscr{F} \cap \mathscr{G} \longrightarrow g$. Here $\mathscr{F} \cap \mathscr{G}$ is the filter with base sets of the form $F \cup G$, $F \in \mathscr{F}$, $G \in \mathscr{G}$.
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A *convergence* on a **W**-object G is said to be a **W**-*convergence* if it is a convergence on G with the following properties.

- The **W** operations are continuous: if $\mathscr{F} \longrightarrow f$ and $\mathscr{G} \xrightarrow{g} g$ then $\mathscr{F} \diamond \mathscr{G} \longrightarrow f \diamond g$, where $\diamond$ can be taken to be $+$, $-$, $\vee$, or $\wedge$.
- The convergence is *convex*, i.e., if $\mathscr{F} \longrightarrow g$ then $\langle \mathscr{F} \rangle \longrightarrow g$. Here $\langle \mathscr{F} \rangle$ is the filter whose base sets are the convex sublattices generated by the sets of $\mathscr{F}$.
- $\mathscr{E}(g) \longrightarrow 0$ for all $g \in G^+$. Here $\mathscr{E}(g)$ is the filter with base sets of the form $\{ f : |f| \leq g/n \}$, $n \in \mathbb{N}$.
- The third item amounts to insisting that $g/n \to 0$ as $n \to \infty$ for all $g \in G^+$. This is equivalent to the archimedean property of G.
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A **W**-convergence is determined by the filters convergent to 0.

Theorem

Let $\xrightarrow{x}$ be a **W**-convergence on a **W**-object G. Then the family $\mathscr{W}_x \equiv \left\{ \mathscr{F} : \mathscr{F} \xrightarrow{x} 0 \right\}$ has the following properties.

- $\mathscr{E}(g) \in \mathscr{W}_x$ for all $g \in G^+$.
- $\mathscr{G} \supseteq \mathscr{F} \in \mathscr{W}_x$ implies $\mathscr{G} \in \mathscr{W}_x$.
- Whenever $\mathscr{W}_x$ contains filters $\mathscr{F}$ and $\mathscr{G}$, it also contains filters $-\mathscr{F}$, $2\mathscr{F}$, $\langle \mathscr{F} \rangle$, and $\mathscr{F} \cap \mathscr{G}$.

Conversely, if $\mathscr{W}$ is a family of filters on G with these properties, then the unique **W**-convergence $\xrightarrow{x}$ for which $\mathscr{W} = \mathscr{W}_x$ is defined by declaring $\mathscr{F} \xrightarrow{x} g$ if $\mathscr{F} - \dot{g} \in \mathscr{W}$. And the convergence is topological if and only if $\mathscr{W}_x$ contains a smallest filter.

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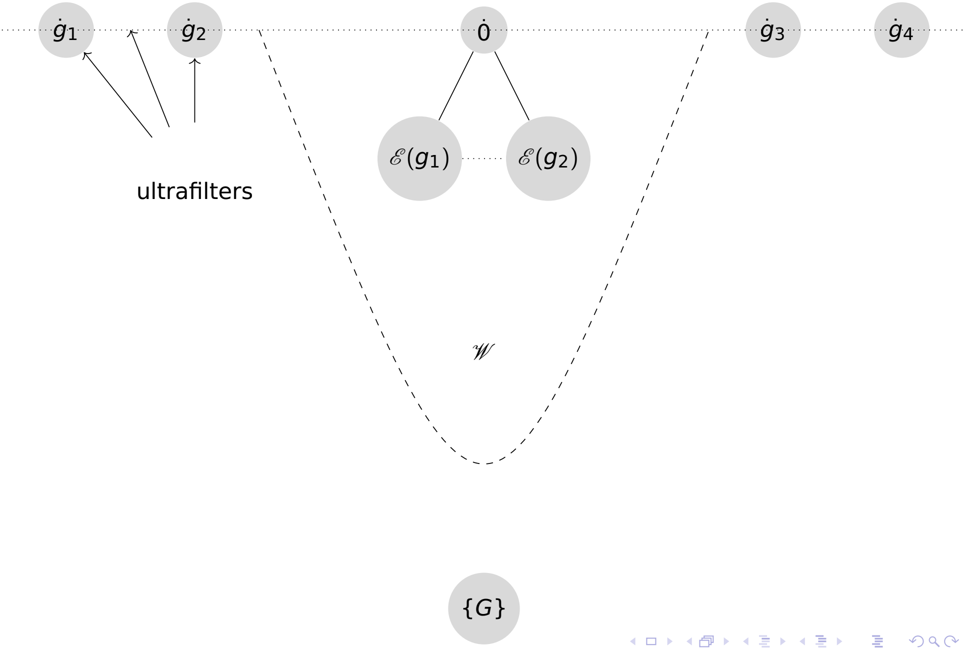
The poset $\mathscr{W}$ of filters convergent to 0



Examples of **W**-convergences

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- Classical uniform convergence is generated by the single filter $\mathscr{E}(1)$.
- The convergence generated by all of the $\mathscr{E}(g)$'s, $g \in G^+$, is called *archimedean convergence*, and is designated $\xrightarrow{a}$.
- A filter $\mathscr{F}$ on G *order converges to* 0, designated $\mathscr{F} \xrightarrow{o} 0$, if

$$\bigwedge \left\{ g \in G^+ : \exists F \in \mathscr{F} \forall f \in F \; (|f| \leq g) \right\} = 0.$$

  The meet is reckoned in G.
- A filter $\mathscr{F}$ on a **W**-object G is said to *α-converge to* 0, written $\mathscr{F} \xrightarrow{\alpha} 0$, provided that

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Admissible **W**-convergences

Let G be a **W**-object.

Definition

For a filter $\mathscr{F}$ on G and element $0 < g \in G$, we denote by $\mathscr{F}(g)$ the filter with base sets of the form $[0, g']$ for elements $0 \leq g' < g$ such that $g \downarrow g' \in \mathscr{F}$. A **W**-convergence $\xrightarrow{x}$ is said to be *admissible* $\mathscr{F}(g) \xrightarrow{x} 0$ whenever $\mathscr{F} \xrightarrow{x} 0$ and $0 < g \in G$.

Proposition

$\xrightarrow{a}$ is the finest admissible convergence on G and $\xrightarrow{\alpha}$ is the coarsest admissible convergence on G. And for every **W** convergence $\xrightarrow{x}$ between $\xrightarrow{a}$ and $\xrightarrow{\alpha}$ there is a finest admissible **W**-convergence coarser than $\xrightarrow{x}$.

The category **cW**

Definition

The category **cW** has objects of the form $(G, \xrightarrow{x})$, where G is a **W**-object and $\xrightarrow{x}$ is an admissible **W**-convergence on G. A **cW**-morphism is a continuous **W**-homomorphism $(G, \xrightarrow{x}) \to (H, \xrightarrow{y})$.

Propostion

Every **W**-homomorphism is a-continuous. A complete **W**-homomorphism is α-continuous.

The frame of **cW**-kernels of a **cW**-object $(G, \xrightarrow{x})$

Lemma

- A convex ℓ-subgroup of a **cW**-object $(G, \xrightarrow{x})$ is the kernel of a **cW**-homomorphism iff it is an x-closed **W**-kernel.
- The map $K \mapsto [K]_x$ which takes a **W**-kernel to its x-closure functions as a nucleus on $\mathcal{K}G$.
- The family $\{ K \in \mathcal{K}G : K \text{ is } x\text{-closed} \}$ is a sublocale of $\mathcal{K}G$, and so a frame in the containment order.
- We denote this sublocale by $\mathcal{K}_c(G, \xrightarrow{x})$ or simply $\mathcal{K}_c G$. We denote the (restriction of its) nucleus by $q_G^c : \mathcal{K}G \to \mathcal{K}_c G$.

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$\mathcal{K}_c$ is functorial

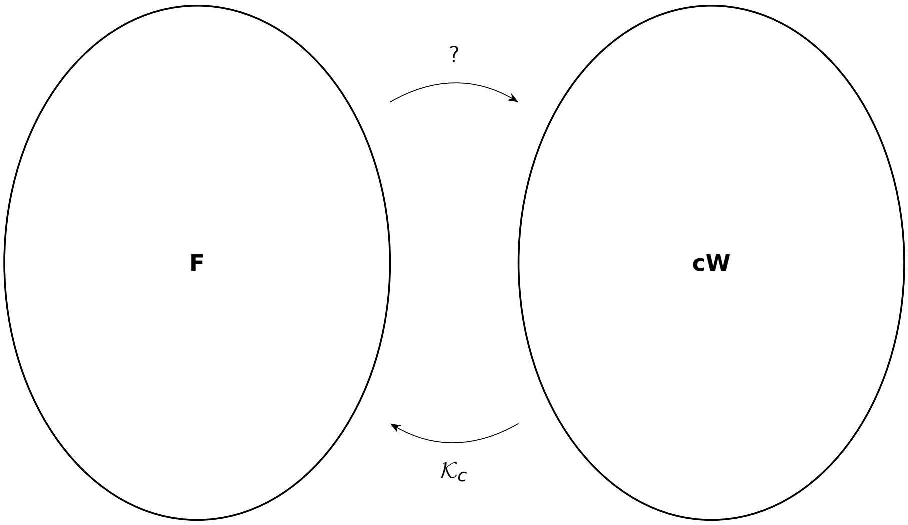
Proposition

For any **cW**-homomorphism $\theta\colon (G, \xrightarrow{x}) \to (H, \xrightarrow{y})$, the frame homomorphism $\mathcal{K}\theta$ which realizes its **W**-reduct drops through q^c_G and q^c_H. That is, there exists a unique frame homomorphism m which makes the diagram commute.

$$
\begin{array}{ccccc}
G & \qquad & \mathcal{K}G & \xrightarrow{q^c_G} & \mathcal{K}_c(G, \xrightarrow{x}) \\
\downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \mathcal{K}\theta} & & \downarrow{\scriptstyle m} \\
H & & \mathcal{K}H & \xrightarrow[q^c_H]{} & \mathcal{K}_c(H, \xrightarrow{y})
\end{array}
$$

The map m satisfies $m(K) = [\theta(K)]_y$ for all $K \in \mathcal{K}_c G$.

We have a functor. Is it adjoint?



Continuous convergence in $\mathcal{R}L$

Classical continuous convegence has an unexpectedly elegant formulation in $\mathcal{R}L$.

Definition

A filter $\mathscr{F}$ on $\mathcal{R}L$ is said to *c-converge to* 0, designated $\mathscr{F} \xrightarrow{c} 0$, if

$$\forall \varepsilon > 0 \left(\bigvee_{\mathscr{F}} \bigwedge_{F} f(0^{\varepsilon}) = \top \right).$$

Continuous convergence in $\mathcal{R}L$

Continuous convergence is very well behaved on $\mathcal{R}L$.

Theorem

- For any frame L, $\xrightarrow{c}$ is an admissible **W**-convergence on $\mathcal{R}L$.
- Every c-Cauchy filter on $\mathcal{R}L$ converges.
- Every **W**-subobject $G \subseteq \mathcal{R}L$ such that $\operatorname{coz} G$ join generates L is c-dense.
- For any frame homomorphism $m\colon L \to M$, the induced **W**-homomorphism $\mathcal{R}m\colon \mathcal{R}L \to \mathcal{R}M = (g \mapsto m \circ g)$ is c-continuous.

Definition

To any frame L, the functor $\mathcal{R}_c$ assigns the **cW**-object $(\mathcal{R}L, \xrightarrow{c})$. To any frame homomorphism $m\colon L \to M$, $\mathcal{R}_c$ assigns the **cW**-homomorphism $\mathcal{R}m$ above.

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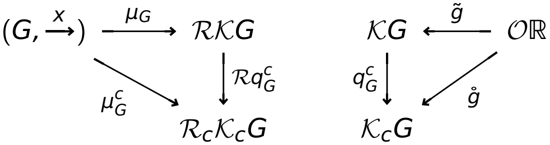
$\mathcal{K}_c \dashv \mathcal{R}_c$

Lemma

For a **cW**-object $(G, \xrightarrow{x})$, let μ_G be the Yosida representation of its **W**-reduct, let $q_G^c \colon \mathcal{K}G \to \mathcal{K}_c G$ be the canonical surjection, and let

$$\mu_G^c \equiv (\mathcal{R}q_G^c) \circ \mu_G = \left(g \mapsto q_G^c \circ \tilde{g} \equiv \mathring{g}\right).$$

Then μ_G^c is continuous, i.e., a **cW**-homomorphism.

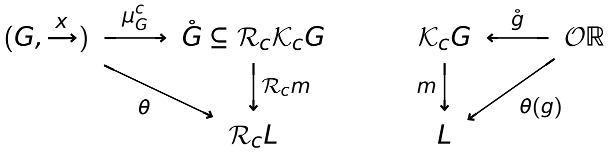


Notation: For $g \in G$, denote $\mu_G^c(g)$ by $\mathring{g}$, and $\left\{\mathring{g} : g \in G\right\}$ by $\mathring{G}$.

$\mathcal{K}_c \dashv \mathcal{R}_c$

Theorem

Let $(G, \xrightarrow{x})$ be a **cW**-object. Then for any frame L and **cW**-homomorphism θ there is a unique frame homomorphism m making the diagram commute.



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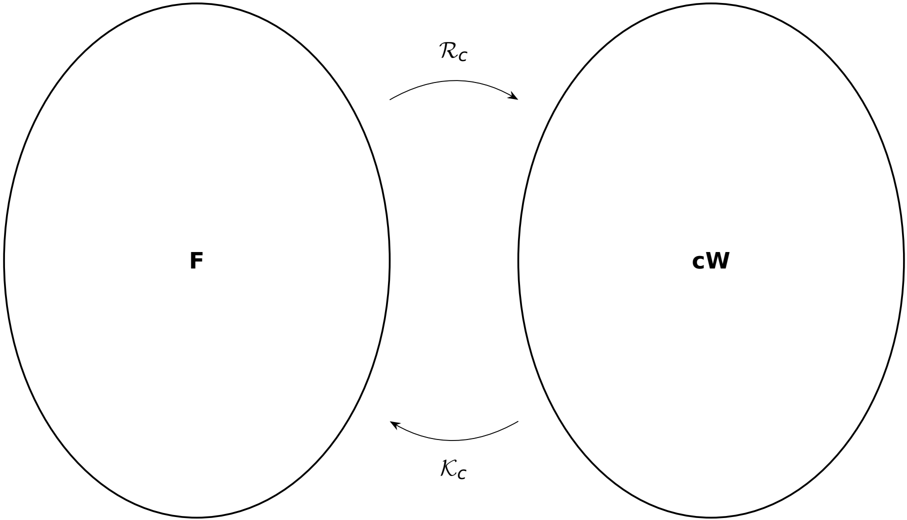


The objects fixed by the adjunction

Every frame is fixed by the adjunction.

Proposition

Every frame L is canonically isomorphic to $\mathcal{K}_c\mathcal{R}_cL$.

Definition

Two admissible **W**-convergences $\xrightarrow{x}$ and $\xrightarrow{y}$ on a **W**-object G are said to be *equivalent* if $[K]_x = [K]_y$ for all $K \in \mathcal{K}G$. That is, $\xrightarrow{x}$ and $\xrightarrow{y}$ are equivalent if $\mathcal{K}_c(G, \xrightarrow{x}) = \mathcal{K}_c(G, \xrightarrow{y})$.

Definition

Let $(G, \xrightarrow{x})$ be a **cW**-object. Denote by $\xrightarrow{cx}$ the **W**-convergence on G which is inherited from the restriction of $\xrightarrow{c}$ on $\mathcal{R}_c\mathcal{K}_c(G, \xrightarrow{x})$ to $\mathring{G}$. That is, $\mathscr{F} \xrightarrow{cx} 0$ if $\mathring{\mathscr{F}} \xrightarrow{c} 0$, i.e., if $\bigvee_{\mathscr{F}} \bigwedge_F \mathring{f}(0^\varepsilon) = \top$ for all $\varepsilon > 0$.

Lemma

On any **cW**-object $(G, \xrightarrow{x})$, the coarsest admissible **W**-convergence equivalent to $\xrightarrow{x}$ is $\xrightarrow{cx}$.

Complete **cW**-objects

Definition

We say that an admissible **W**-convergence $\xrightarrow{x}$ on a **W**-object G is *coarse* if it is coarser than any equivalent admissible **W**-convergence on G. That is, $\xrightarrow{x}$ is coarse if it coincides with $\xrightarrow{cx}$. A **cW**-object $(G, \xrightarrow{x})$ is said to be *complete* if $\xrightarrow{x}$ is coarse and G is x-Cauchy complete, i.e., every x-Cauchy filter converges. We denote by **ccW** the full subcategory of **cW** comprised of the complete objects.

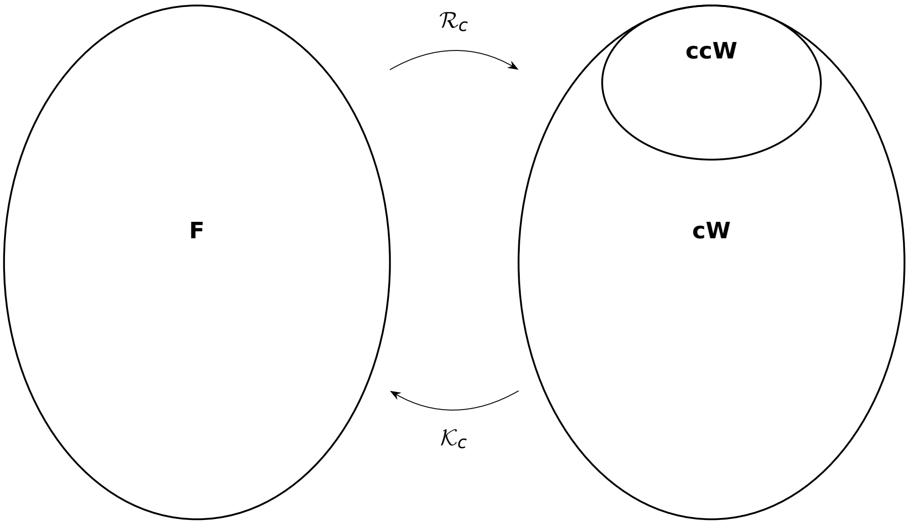
Proposition

A **cW**-object is complete if and only if it is isomorphic to $(\mathcal{R}L, \xrightarrow{c})$ for some frame L.

Theorem

1. **ccW** is bireflective in **cW**, and a reflector for the object $(G, \xrightarrow{x})$ is its embedding $\mu_G^c \colon (G, \xrightarrow{x}) \to \mathcal{R}_c\mathcal{K}_c(G, \xrightarrow{x})$.
2. The restrictions of the functors $\mathcal{R}_c$ and $\mathcal{K}_c$ provide an equivalence between the categories **F** and **ccW**.

The objects fixed by the adjunction



A quick application

Proposition

The following are equivalent for a $\mathbf{W}$-object G.

1. G is isomorphic to $\mathcal{R}L$ for some frame L.
2. There is an admissible $\mathbf{W}$-convergence $\xrightarrow{x}$ for which G is cx-Cauchy complete, i.e., $(G, \xrightarrow{cx})$ is a $\mathbf{ccW}$-object.
3. G is ca-Cauchy complete.

Thank you very much.