Quantum Monadic Algebras

J. Harding

New Mexico State University
wordpress.nmsu.edu/Hardingj/

jharding@nmsu.edu

BLAST (Chapman) August 2022

Status: This is hopefully the first step in a larger project.

Aim: Renew the perspectives of lattices, geometry and logic into the study of operator algebras, particularly subfactors.

History: From 1930-1970

vN, Stone: modern spectral theorem

Stone: representations of Boolean algebras

B, Menger: finite-dimensional lattice-theoretic projective geometry

B, vN: logic of QM

vN: continuous geometry, rep's of complemented modular lattices

M, vN: rings of operators I, II, III, IV

Frink, Prenowtiz: infinite-dim lattice-theoretic projective geometry

Kaplansky: complete modular ortholattice is a continuous geometry

Gleason: lattice theoretic view of states

Dye: morphisms of vN algebras via projections

Varadarajan: geometric quantum mechanics

various: development of OMLs, relations to vN algebras

From the 1970's — today, this perspective has receded.

Background

Definition B(H) is all bounded operators on a Hilbert space H.

Definition A vN-algebra $\mathcal{M}$ is a $*$-subalgebra of B(H) closed in the WOT.

Definition An element p in $\mathcal{M}$ is a projection if $p = p^2 = p^*$.

Definition $P(\mathcal{M})$ is the projections of $\mathcal{M}$ with $p \leq q$ iff $pq = p = qp$.

Definition $\mathcal{M}$ is a factor if the center of $P(\mathcal{M})$ is $\{0, 1\}$.

Definition An inclusion $\mathcal{N} \leq \mathcal{M}$ of factors is called a subfactor.

Theorem $P(\mathcal{M})$ is a complete orthomodular lattice (OML).

Theorem $\mathcal{M}$ is determined up to Jordan isomorphism by $P(\mathcal{M})$.

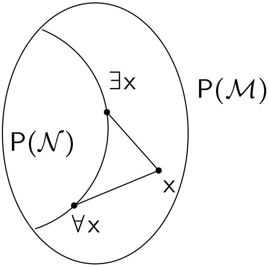
Theorem A factor $\mathcal{M}$ has a unique dimension function $D : P(\mathcal{M}) \to [0, \infty]$.

Factors are given types $I_n, I_\infty, II_1, II_\infty, III$ depending on the range of D which is one of $\{0, 1, \ldots, n\}, \mathbb{N} \cup \{\infty\}, [0, 1], [0, \infty], \{0, \infty\}$.

C^* algebras are often viewed as non-commutative topological spaces, vN algebras are non-commutative measure spaces.

Key observation

For $\mathcal{N} \leq \mathcal{M}$ a subfactor, $P(\mathcal{N})$ is a complete sub-OL of $P(\mathcal{M})$.



$\exists x =$ least in $P(\mathcal{N})$ above x

$\forall x =$ largest in $P(\mathcal{N})$ below x

This is familiar from (classical) logic.

Monadic algebras

Definition A quantifier on a BA B is a map $\exists : B \to B$ where

(Q_1) $\exists 0 = 0$,

(Q_2) $p \leq \exists p$,

(Q_3) $\exists(p \vee q) = \exists p \vee \exists q$,

(Q_4) $\exists\exists p = \exists p$,

(Q_5) $\exists(\exists p)^{\perp} = (\exists p)^{\perp}$.

A monadic algebra $(B, \exists)$ is a BA B with a quantifier $\exists$.

Note: (Q_1) – (Q_5) are equivalent to (Q_1), (Q_2), (Q_6) where

(Q_6) $\exists(p \wedge \exists q) = \exists p \wedge \exists q$.

Quantum monadic algebras

Definition An OL is a bounded lattice L with unary operation $\perp$ where

(O_1) $x \wedge x^\perp = 0$

(O_2) $x \vee x^\perp = 1$

(O_3) $x \leq y \Rightarrow y^\perp \leq x^\perp$

(O_4) $x^{\perp\perp} = x$

It is an OML if it additionally satisfies

(O_5) $x \leq y \Rightarrow x \vee (x^\perp \wedge y) = y$

Monadic OLs are OLs with a quantifier $\exists$ satisfying (Q_1) – (Q_5).

Quantum monadic algebras are monadic OLs that are OMLs.

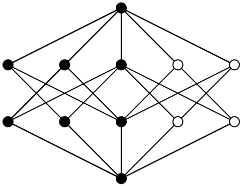
Abbreviation: q-monadic algebras.

Basic examples

Proposition If L is a complete OL and $C \leq L$ is a complete subalgebra, then $\exists x = \bigwedge\{c \in C : x \leq c\}$ is a quantifier and $(L, \exists)$ is a monadic OL.

Note: All complete monadic OLs are obtained in this way.

Example If L is a complete OML and B is a maximal Boolean subalgebra of L (such is called a block), then $B \leq L$ is a complete subalgebra. So each block of a complete OML yields a q-monadic algebra.



Examples of quantum monadic algebras

Example If $\mathcal{N} \leq \mathcal{M}$ then $\mathrm{P}(\mathcal{N}) \leq \mathrm{P}(\mathcal{M})$ yields a q-monadic algebra.

Example A von Neumann algebra $\mathcal{M}$ is specified to Jordan isomorphism by the q-monadic algebra $\mathrm{P}(\mathcal{M}) \leq \mathrm{P}(\mathrm{H})$.

Example A subfactor $\mathcal{N} \leq \mathcal{M}$ gives $\mathrm{P}(\mathcal{N}) \leq \mathrm{P}(\mathcal{M})$ a q-monadic algebra that specifies this subfactor to Jordan isomorphisms.

Slogan Subfactors are non-commutative monadic algebras.

Commuting squares

Theorem A subfactor $\mathcal{N} \leq \mathcal{M}$ has a conditional expectation $\mathrm{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$

Note This generalizes conditional expectation from measure theory.

Definition Subfactors $\mathcal{N}, \mathcal{K} \leq \mathcal{M}$ are a commuting square

$$\begin{array}{ccc} \mathcal{N} & \text{———} & \mathcal{M} \\ | & & | \\ \mathcal{N} \cap \mathcal{K} & \text{———} & \mathcal{K} \end{array}$$

if their conditional expectations $\mathrm{E}_{\mathcal{N}}$ and $\mathrm{E}_{\mathcal{K}}$ commute.

Commuting squares are well-known in subfactor theory. They are a non-commutative version of independent σ-algebras.

Theorem $\mathrm{E}_{\mathcal{N}}$ and $\mathrm{E}_{\mathcal{K}}$ commute iff the quantifiers $\exists_{\mathcal{M}}$ and $\exists_{\mathcal{K}}$ commute.

Cylindric algebras

Definition An I-dimensional cylindric algebra $(B, \exists_i, d_{i,j})$ is a BA B with a family $\exists_i$ of unary operations and $d_{i,j}$ of constants where

- (C_1) $\exists_i$ is a quantifier
- (C_2) $\exists_i \exists_j x = \exists_j \exists_i x$
- (C_3) $d_{i,j} = d_{j,i}$ and $d_{i,i} = 1$
- (C_4) if $j \neq i, k$ then $d_{i,k} = \exists_j (d_{i,j} \wedge d_{j,k})$
- (C_5) if $i \neq j$ then $\exists_i (d_{i,j} \wedge x) \wedge \exists_i (d_{i,j} \wedge x^{\perp}) = 0$

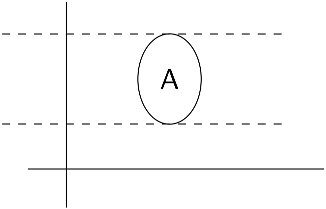
The $\exists_i$ are called cylindrifications and the $d_{i,j}$ are diagonals.

If we remove the d_{ij} we obtain a diagonal-free cylindric algebra.

(C_5) ensures $S_{ij}\, x := \exists_i (d_{ij} \wedge x)$ is a substitution endomorphism.

Cylindric algebras

The name comes from the following "cylindric set algebra".



$B = P(X^2)$

$\exists_1 A =$ the cylinder generated by A

Diagonals are usual diagonal $\subseteq X^2$

Definition Cylindric OLs are the corresponding structures with BAs replaced by OLs and quantum cylindric algebras with OMLs.

The quantum cylindric set algebra

This is closely related to Nik Weaver's quantum logic.

Lemma For $H_1, \ldots, H_n$ Hilbert spaces, each $\mathcal{M}_i \leq B(H_1 \otimes \cdots \otimes H_n)$ is a vN subalgebra where

$$\mathcal{M}_i = \{1 \otimes A : A \in B(\bigotimes_{j \neq i} H_j)\}$$

Diagonals If all H_i are the same, diagonal D_{ij} is projection onto the subspace of the tensor power $H^{\otimes n}$ symmetric in the i^{th}, j^{th} coordinates.

Note This generalizes to infinite tensor products as well.

The quantum cylindric set algebra

Proposition For H_i $(i \in I)$ Hilbert spaces, the quantum cylindric set algebra over $\bigotimes_I H_i$ is a diagonal-free q-cylindric set algebra.

Proposition The quantum cylindric set algebra with diagonals over the tensor power $H^{\otimes I}$ satisfies (C_1) – (C_4) but not (C_5).

Note The issue with (C_5) seems related to difficulties with substitution in Weaver's quantum predicate calculus.

Note Some of the issues with (C_5) are addressed by modifying the axiom to use a Sasaki projection.

Monadic orthoframes

Definition A relational structure $(X, \perp, R)$ is a monadic orthoframe if $\perp$ and R are binary relations on X that satisfy

(M_1) $\perp$ is symmetric and irreflexive

(M_2) R is reflexive and transitive

(M_3) for each $x \in X$, the set $R[\{x\}]^{\perp}$ is closed under R.

Proposition $(X, \neq, R)$ is a monadic orthoframe iff R is an equivalence relation

Definition Set $(X, \perp, R)^{+} = (L, \exists)$ where

1. L is the complete OL of Galois closed subsets of $(X, \perp)$.
2. $\exists A$ is the Galois closure of $R[A]$.

Monadic orthoframes

Theorem Each $(X, \perp, R)^+$ is a monadic OL. Each monadic OL is a subalgebra of such. Each complete monadic OL is isomorphic to such.

Definition $(X, \perp, (R_i)_I)$ is diagonal-free cylindric orthoframe if

(C_1) Each $(X, \perp, R_i)$ is a monadic orthoframe

(C_2) R_i commutes with R_j for each $i, j \in I$

Theorem As above but realizing diagonal-free cylindric OLs as complex algebras of diagonal-free cylindric orthoframes.

Note There are many obstacles to providing similar results for quantum monadic frames.

The contents of this talk will appear in J. Physics A.

A preliminary version is on ArXiv.

There are further logical avenues to pursue, but my main focus is in pushing the view of factors and subfactors from the order-theoretic and geometric interpretation and potential generalizations.

Thank You!