Natural deduction in quantum logic

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Orthomodular logic

Propositional variables: $X_1, X_2, \ldots$

Connectives: $\wedge$, $\neg$

Valuations: $\mathit{Form} \xrightarrow{v} \mathcal{L}$, where $\mathcal{L}$ is an orthomodular lattice

Valid entailment:

$$P \vdash Q \quad \Longleftrightarrow \quad v(P) \leq v(Q) \text{ for all valuations } v$$

We can define the validity of $\mathcal{S} \vdash Q$, where $\mathcal{S}$ is any set of formulas.

We shouldn't!

The sequent calculus *NOM*

$$\frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q} \qquad \frac{\Gamma \vdash P \wedge Q}{\Gamma \vdash P} \qquad \frac{\Gamma \vdash P \wedge Q}{\Gamma \vdash Q}$$

$$\frac{\Gamma, P \vdash Q}{\Gamma \vdash P \rightarrow Q} \qquad \frac{\Gamma \vdash P \rightarrow Q}{\Gamma, P \vdash Q}$$

$$\frac{\Gamma, P \vdash Q \quad \Gamma, \neg P \vdash Q}{\Gamma \vdash Q} \qquad \frac{\Gamma \vdash \neg P}{\Gamma, P \vdash Q}$$

$$\frac{}{\Gamma, P \vdash P} \qquad \frac{\Gamma \vdash P \quad \Gamma, P \vdash Q}{\Gamma \vdash Q} \qquad \frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma, P \vdash Q}$$

$$\frac{\Gamma, P, Q \vdash P \quad \Gamma, P, Q \vdash R \quad \Gamma, Q, P \vdash Q}{\Gamma, Q, P \vdash R}$$

Soundness and completeness

Theorem

For all formulas P and Q of orthomodular logic, t.f.a.e.:

1. $P \vdash Q$ *is derivable in NOM,*
2. $P \vdash Q$ *is valid in orthomodular logic.*

(Here, $\rightarrow$ potentially occurs in the derivation but not in $P \vdash Q$.)

1. Why is *NOM* sound and complete?

2. How is *NOM* related to quantum theory?

3. Is *NOM* really a natural deduction system?

Sasaki adjunction

In any orthomodular lattice:

$$(A \vee \neg B) \wedge B \leq C \quad \Longleftrightarrow \quad A \leq \neg B \vee (B \wedge C)$$

- $A \mathbin{\&} B \;:=\; (A \vee \neg B) \wedge B$ is a kind of conjunction.
- $B \to C \;:=\; \neg B \vee (B \wedge C)$ is a kind of implication.

All understood by 1970.

Extended orthomodular logic

Propositional variables: $X_1, X_2, \dots$

Connectives: $\wedge$, $\neg$, $\rightarrow$

Valuations: $\mathit{Form} \xrightarrow{v} \mathcal{L}$, where $\mathcal{L}$ is an orthomodular lattice

Valid entailment:

$$P_1, \dots, P_n \vdash Q \quad \iff \quad v(P_1) \,\&\, \cdots \,\&\, v(P_n) \leq v(Q) \quad \text{for all } v$$

(By convention, & associates to the left.)

Proof of completeness, step 1

Use $P \vee Q$ to abbreviate $\neg(\neg P \wedge \neg Q)$.

We can derive $\dfrac{\Gamma \vdash P}{\Gamma \vdash P \vee Q}$ and $\dfrac{\Gamma \vdash Q}{\Gamma \vdash P \vee Q}$.

We can also derive $\dfrac{\Gamma \vdash P \rightarrow Q}{\Gamma \vdash \neg P \vee (P \wedge Q)}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, P \vdash P} \quad \dfrac{\Gamma \vdash P \rightarrow Q}{\Gamma, P \vdash Q}}{\Gamma, P \vdash P \wedge Q}}{\Gamma, P \vdash \neg P \vee (P \wedge Q)} \qquad \dfrac{\dfrac{}{\Gamma, \neg P \vdash \neg P}}{\Gamma, \neg P \vdash \neg P \vee (P \wedge Q)}}{\Gamma \vdash \neg P \vee (P \wedge Q)}$$

Proof of completeness, step 2

Proposition

If $\Gamma \vdash Q$ is derivable in *NOM*, then $P, \Gamma \vdash Q$ is derivable in *NOM*.

We can also derive $\dfrac{\Gamma, P, Q \vdash P}{\Gamma, P, \neg Q \vdash P}$.

Assume $P \vdash Q$ is derivable. Then $Q, P \vdash Q$ is derivable.

$$\cfrac{\cfrac{\cfrac{\overline{Q, P \vdash Q}}{Q, \neg P \vdash Q}}{Q \vdash \neg P \to Q}}{Q \vdash \neg\neg P \vee (\neg P \wedge Q)}$$

Therefore, the Lindenbaum algebra is orthomodular

(if it is orthocomplemented).

Quantum logic

Write $\mathcal{C}(\mathcal{H})$ for the orthomodular lattice of closed subspaces of $\mathcal{H}$.

Valuations: $\mathit{Form} \xrightarrow{v} \mathcal{C}(\mathcal{H})$

Proposition

For all $A_1, \ldots, A_n, B \in \mathcal{C}(\mathcal{H})$, the following are equivalent:

1. $A_1 \mathbin{\&} \cdots \mathbin{\&} A_n \leq B$,
2. $[A_n] \cdots [A_1]\mathcal{H} \subseteq B$,
3. After measuring $A_1, \ldots, A_n$ to be true,
   a measurement of B will find it to be true.

Decidability

NOM is sound and complete for orthomodular logic.

NOM is sound but **not** complete for quantum logic.

Question. Is there a recursive extension of *NOM* that is sound and complete for quantum logic?

This question is **not** settled by recent work on synchronous games!

(At least it doesn't appear to be...)

What is natural deduction?

Natural deduction is characterized by **subproofs**.

Characteristics:

1. Formulas are asserted on the basis of assumptions.
2. Rules are valid after any sequence of assumptions.
3. $P \to Q$ is proved by assuming P and asserting Q.

A sequent calculus may be a system of natural deduction (Gentzen 1936).

NOM is a system of natural deduction

1. Formulas are asserted on the basis of assumptions.

$$\underbrace{P_1, \dots, P_n}_{\text{assumptions}} \quad \vdash \quad \underbrace{Q}_{\text{assertion}}$$

2. Rules are valid after any sequence of assumptions.

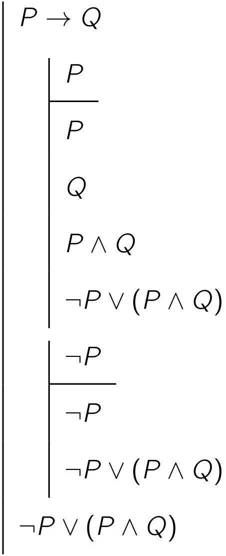
$$\frac{\Gamma, P \vdash Q \qquad \Gamma, \neg P \vdash Q}{\Gamma \vdash Q} \qquad \leftsquigarrow \qquad \Gamma \text{ is } \textbf{any} \text{ sequence of formulas.}$$

3. $P \to Q$ is proved by assuming P and asserting Q.

$$\frac{\Gamma, P \vdash Q}{\Gamma \vdash P \to Q}$$

NOM can be presented in the notation of Fitch

$$\frac{\Gamma \vdash P \to Q}{\Gamma \vdash \neg P \vee (P \wedge Q)}$$



$P \to Q$	$\Gamma \vdash P \to Q$
P	
P	$\Gamma, P \vdash P$
Q	$\Gamma, P \vdash Q$
$P \wedge Q$	$\Gamma, P \vdash P \wedge Q$
$\neg P \vee (P \wedge Q)$	$\Gamma, P \vdash \neg P \vee (P \wedge Q)$
$\neg P$	
$\neg P$	$\Gamma, \neg P \vdash \neg P$
$\neg P \vee (P \wedge Q)$	$\Gamma, \neg P \vdash \neg P \wedge (P \vee Q)$
$\neg P \vee (P \wedge Q)$	$\Gamma \vdash \neg P \vee (P \wedge Q)$

Rules requiring a further assumption can be eliminated

$$\frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q} \qquad \frac{\Gamma \vdash P \wedge Q}{\Gamma \vdash P} \qquad \frac{\Gamma \vdash P \wedge Q}{\Gamma \vdash Q}$$

$$\frac{\Gamma, P \vdash Q}{\Gamma \vdash P \to Q} \qquad \frac{\Gamma \vdash P \quad \Gamma \vdash P \to Q}{\Gamma \vdash Q}$$

$$\frac{\Gamma, P \vdash Q \quad \Gamma, \neg P \vdash Q}{\Gamma \vdash Q} \qquad \frac{\Gamma \vdash P \quad \Gamma \vdash \neg P}{\Gamma \vdash Q}$$

$$\frac{\Gamma \vdash P \wedge Q}{\Gamma, P \vdash P \wedge Q} \qquad \frac{\Gamma \vdash P \to Q}{\Gamma, P \vdash P \to Q} \qquad \frac{\Gamma \vdash \neg P}{\Gamma, P \vdash \neg P}$$

$$\frac{}{\Gamma, P \vdash P} \qquad \frac{\Gamma, P \vdash Q \to P \quad \Gamma, P \vdash Q \to R \quad \Gamma, Q \vdash P \to Q}{\Gamma \vdash Q \to (P \to R)}$$

Quantum predicate logic

Formulas: No variable can occur more than once in any atomic formula.

Structures: $v = \begin{cases} \mathit{Form}() \xrightarrow{v} \{\top, \bot\} \\ \mathit{Form}(x_1) \xrightarrow{v} \mathcal{C}(\mathcal{H}) \\ \mathit{Form}(x_1, x_2) \xrightarrow{v} \mathcal{C}(\mathcal{H} \otimes \mathcal{H}) \\ \vdots \end{cases}$

For $P \in \mathit{Form}(x_1, \ldots, x_n)$:

$$v((\forall x_n)P) = \sup\{A \in \mathcal{C}(\underbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{n-1}) : A \otimes \mathcal{H} \leq v(P)\}$$

This is Weaver's semantics for the universal quantifier.

Additional rules

$$\frac{\Gamma \vdash P[x/y]}{\Gamma \vdash (\forall x)P} \qquad \frac{\Gamma \vdash (\forall x)P}{\Gamma \vdash P[x/t]} \qquad \frac{\Gamma, P, Q \vdash R}{\Gamma, Q, P \vdash R}$$

(* y is not free in $\Gamma \vdash (\forall x)P$ in the first rule.)

(** P and t have no free variables in common in the second rule.)

(*** P and Q have no free variables in common in the third rule.)

This extension of *NOM* is sound for quantum predicate logic.

Existential quantifier

Use $(\exists x)P$ to abbreviates $\neg(\forall x)\neg P$.

We can derive:

$$\frac{\Gamma \vdash P[x/t]}{\Gamma \vdash (\exists x)P} \qquad \frac{\Gamma \vdash (\exists x)P \quad \Gamma, P[x/y] \vdash Q \quad \Gamma, Q, P[x/y] \vdash Q}{\Gamma \vdash Q}$$

(* P and t have no free variables in common in the first rule.)
(** y must not appear freely in $\Gamma \vdash (\exists x)P$ or in Q.)

"The creation of y such that P must be compatible with Q."