Generalizing the Baker-Beynon duality, from `Max` to `Spec`

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What we will talk about

- Preliminaries on the structures involved;

- Baker-Beynon duality and a general approach to 'affine dualities';

- Our results.

ℓ-groups and Riesz spaces

A general approach

Beyond Baker-Beynon duality

The topological side of the duality

Irreducible closed in finite dimension

ℓ-groups and vector lattices

An ℓ-group

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ℓ-groups and Riesz spaces form varieties.

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A proper ℓ-ideal is called maximal if it is maximal wrt inclusion.
A nontrivial ℓ-group/Riesz space A is simple if $\{0\}$ and A are the only ℓ-ideals of A.

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Archimedean $\Rightarrow$ semisimple (when it's finitely generated)

- A/I is simple iff I is maximal.
- A/I is semisimple iff I is intersection of maximal ℓ-ideals.

Piecewise linear functions

A continuous function $f : \mathbb{R}^\kappa \to \mathbb{R}$ is piecewise linear if there exist $g_1, \dots, g_n$ linear homogeneous polynomials such that for each $x \in \mathbb{R}^\kappa$ we have $f(x) = g_i(x)$ for some $i = 1, \dots, n$. Notation:

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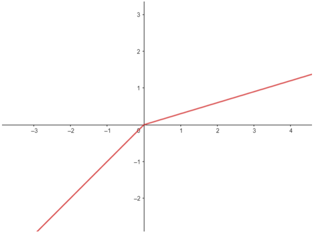

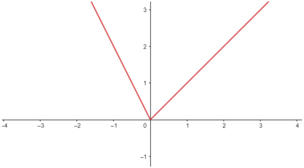

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Theorem (Baker 1968)

- *Every finitely generated Archimedean Riesz space is isomorphic to* $\mathsf{PWL}_\mathbb{R}(C)$ *where* C *is a cone that is closed in* $\mathbb{R}^n$.
- *Every finitely generated Archimedean* ℓ*-group is isomorphic to* $\mathsf{PWL}_\mathbb{Z}(C)$ *where* C *is a cone that is closed in* $\mathbb{R}^n$.

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- $\mathrm{PWL}_\mathbb{R}(\mathbb{R}^\kappa)$ *is isomorphic to the free Riesz space on* κ *generators.*
- $\mathrm{PWL}_\mathbb{Z}(\mathbb{R}^\kappa)$ *is isomorphic to the free* ℓ*-group on* κ *generators.*

Baker-Beynon duality, revisited

Theorem (Beynon 1974)

The category of semisimple Riesz spaces (resp. semisimple ℓ-groups) is dually equivalent to the category of closed cones in some $\mathbb{R}^\kappa$ and piecewise linear maps with real coefficients (resp. with integer coefficients).

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In particular,

$$A \hookrightarrow \prod_{M \in \mathrm{Max}(A)} A/M.$$

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For any $T \subseteq \mathcal{F}_\kappa$ and $S \subseteq A^\kappa$, we define the following operators.

$$\mathbb{V}_A(T) = \{x \in A^\kappa \mid t(x) = 0 \text{ for all } t \in T\}$$
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Basic Galois connection

$$T \subseteq \mathbb{I}_A(S) \quad \text{iff} \quad S \subseteq \mathbb{V}_A(T)$$

From a connection to a duality

The key tool: Algebraic Nullstellensatz

Let J be an ℓ-ideal of $\mathcal{F}_\kappa$. We have $J = \mathbb{I}_A(x)$ for some $x \in A^\kappa$ iff $\mathcal{F}_\kappa / J$ embeds into A.

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$A = \mathbb{R}$ gives the revised Baker-Beynon duality!

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Of course, it won't be possible to find an algebra that embeds all linearly ordered ones. Hence, we impose a bound on the cardinality.

How to get such an algebra?

Given a cardinal α, $\mathcal{F}$ filter in $\mathcal{P}(I)$ is α-regular iff there exists $E \subseteq \mathcal{F}$ of cardinality α such that each $i \in I$ belongs to only finitely many $e \in E$.

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Theorem

Let γ be a cardinal. There exists an ultrapower $\mathcal{U}$ of $\mathbb{R}$ such that every κ-generated linearly ordered ℓ-group/Riesz space with $\kappa \leq \gamma$ embeds into $\mathcal{U}$.

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If $\kappa \leq \gamma$, then every κ-generated ℓ-group/Riesz space is subdirect product of totally ordered ones, that are subalgebras of $\mathcal{U}$!

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Theorem (Carai, L., and Spada)

Let γ be a cardinal. There exists an ultrapower $\mathcal{U}$ of $\mathbb{R}$ such that:

- *The category of κ-generated Riesz spaces for some $\kappa \leq \gamma$ is dually equivalent to the category of subsets of $\mathcal{U}^\kappa$ of type $\mathbb{V}_{\mathcal{U}}(J)$ for some $\kappa \leq \gamma$ and $J \subseteq \mathfrak{F}_\kappa$.*

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- *The category of κ-generated ℓ-groups for some $\kappa \leq \gamma$ is dually equivalent to the category of subsets of $\mathcal{U}^\kappa$ of type $\mathbb{V}_{\mathcal{U}}(J)$ for some $\kappa \leq \gamma$ and $J \subseteq \mathcal{F}_\kappa$.*

Spoiler alert!

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Some remarks:

- All of this can be done with a generic ℓ-group that embeds all ordered ones (up to a cardinality). When A is an ultrapower, $J = \mathbb{I}_\mathcal{U}(x)$ for some $x \in \mathcal{U}^\kappa$ iff J is prime or $J = \mathcal{F}_\kappa$.

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- This family of functors is "compatible" in a sense that I won't make precise here.
- If you add a strong unit everything works. Via Mundici's equivalence, you can work with the equivalent categories of MV-algebras and Riesz MV-algebras, (that are varieties!) and $A = [0, 1]$.

Non standard tools

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If $\mathcal{U} = \prod \mathbb{R} / \mathcal{F}$ for some ultrafilter $\mathcal{F}$, every subset $X \subseteq \mathbb{R}^n$ can be associated with a subset ${}^*X$ of $\mathcal{U}^n$ defined as

$$\left\{x \in \mathcal{U}^n \mid \{i \in I \mid \pi_i(x) \in X\} \in \mathcal{F}\right\}$$

and called the enlargement of X. Similarly, every predicate $P \subseteq \mathbb{R}^n$ and function $f : \mathbb{R}^n \to \mathbb{R}$ can be enlarged to ${}^*P \subseteq \mathcal{U}^n$ and ${}^*f : \mathcal{U}^n \to \mathcal{U}$.

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Transfer principle (Łoś Theorem)

Let φ be a first order formula and ${}^*\varphi$ the formula obtained by replacing every predicate symbol P and every function symbol f with ${}^*P$ and ${}^*f$. Then φ is true in $\mathbb{R}$ iff ${}^*\varphi$ is true in $\mathcal{U}$.

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When $C = \mathbb{V}(J)$ for some J,

- $\mathcal{F}_\kappa \,/\, \mathbb{I}(C) \cong {}^*\mathrm{PWL}_{\mathbb{R}}(C)$ (Riesz spaces).
- $\mathcal{F}_\kappa \,/\, \mathbb{I}(C) \cong {}^*\mathrm{PWL}_{\mathbb{Z}}(C)$ (ℓ-groups).

Up to cardinality γ, both ℓ-groups and Riesz spaces are dual to a category whose objects are subsets of $\mathcal{U}^\kappa$ of type $\mathbb{V}(J)$ for some $\kappa \leq \gamma$ and $J \subseteq \mathcal{F}_\kappa$.

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Hence, for any $S \subseteq \mathcal{U}^\kappa$,

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So we need to consider $\mathcal{U}^\kappa_o := \mathcal{U}^\kappa \setminus \{O\}$ and modify $\mathbb{V}$ accordingly: $\mathbb{V}_o(S) := \mathbb{V}(S) \setminus \{O\}$.

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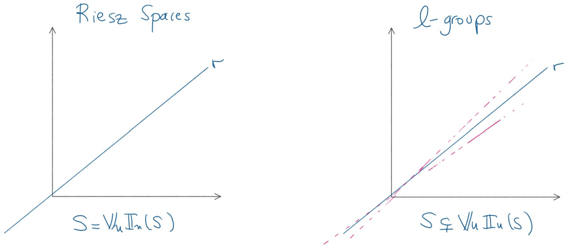

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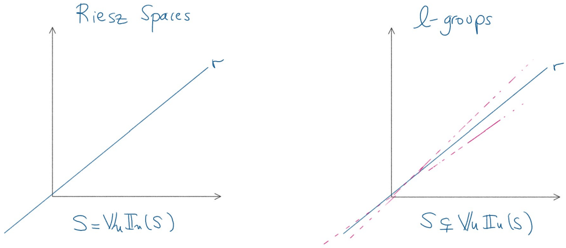

The Zariski topology on $\mathcal{U}_o^\kappa$ is not even T_0. Indeed, $t(x) = 0$ implies $t(x + x) = t(x) + t(x)$. Whence x and $2x$ cannot be separated by an open set.

The topology on the quotient

Therefore, we will consider the T_0-reflection of $\mathcal{U}_o^\kappa$. This is equivalently obtained by taking a quotient over the relation

$$x \sim y \text{ if and only if } \mathbb{V}\,\mathbb{I}(x) = \mathbb{V}\,\mathbb{I}(y).$$

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This is because all closed subsets of $\mathcal{U}_o^\kappa$ are saturated w.r.t. the relation $\sim$.

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The compact opens of $\mathcal{U}_o^\kappa$ are exactly the complements of the basic closed sets $\mathbb{V}(t)$.

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The other direction is a consequence of the fact that finitely generated ℓ-ideals are principal in both cases.

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Indeed, notice that being irreducible means to be join-prime in the lattice of closed sets. The latter is order-dual to the lattice of ℓ-ideals, in which prime ℓ-ideals (they are $\mathbb{I}(x)$!) are exactly the meet-prime elements.

The Zariski topology is generalized spectral

Proposition

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Theorem

The map $\mathtt{e}\colon \mathcal{U}_o^\kappa/\!\sim\; \to \mathtt{Spec}(\mathcal{F}_\kappa)$ *that sends* $x/\!\sim\; \mapsto \mathbb{I}(x)$ *is a homeomorphism. Moreover,* $\mathtt{Spec}(\mathcal{F}_\kappa)$ *embeds into* $\mathcal{U}_o^\kappa$ *as a dense subset.*

The duality is induced by $\mathtt{Spec}$

Corollary

For any κ-generated ℓ-groups A there exists an embedding of $\mathtt{Spec}(A)$ into $\mathcal{U}_o^\kappa$ such that $A \cong {}^*\mathrm{PWL}_{\mathbb{Z}}(\mathtt{Spec}(A))$.

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The Corollary requires the Axiom of Choice! How to get rid of it?

Indices and irreducible closed

Orthogonal decomposition theorem (Goze 1995)

Any $x \in \mathcal{U}_o^n$ can be written in a unique way as

$$x = \alpha_1 v_1 + \cdots + \alpha_k v_k$$

where

1. $v_1, \ldots, v_k$ are orthonormal vectors of $\mathbb{R}^n$,
2. $0 < \alpha_1, \ldots, \alpha_k \in \mathcal{U}$, and
3. α_{i+1}/α_i is infinitesimal for every $i < k$.

Cones and indices

Thus, each $x \in \mathcal{U}_o^n$ gets associated with a sequence $\mathsf{v} = (v_1, \dots, v_k)$ of orthonormal vectors, which we call index of x, $\iota(x)$.

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The vector $w = ((1,0))$ is the truncation of v snd

$$\mathsf{Hcone}(w) = \{(x,0) \in \mathcal{U}_o^2 \mid x \in \mathcal{U}\} \subseteq \mathsf{Hcone}(v).$$

Theorem (Carai, L., Spada)

In the Zariski topology of $\mathcal{U}_o^n$ *relative to Riesz spaces each irreducible closed of* $\mathcal{U}_o^n$ *is* $\mathsf{Hcone}(\mathsf{v})$ *for some index* v*. More precisely,*

$$\mathbb{V}\,\mathbb{I}(x) = \mathsf{Hcone}(\iota(x)).$$

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If $w \in \mathbb{R}^n$, $\langle w \rangle$ is the rational envelop of w:

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Using a sort of Gram-Schmidt process, we can associate to each index v a unique $\mathbb{Z}$-reduced index $\mathsf{red}(\mathsf{v})$.

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$$\bigcup\{\mathsf{Hcone}(\mathsf{w}) \mid \mathsf{red}(\mathsf{w}) = \mathsf{v}\}.$$

for some $\mathbb{Z}$-reduced index v.

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$\mathcal{U}_o^\kappa/\sim$ is homeomorphic to $\mathtt{Spec}(\mathcal{F}_\kappa)$. Moreover, $\mathtt{Spec}(\mathcal{F}_\kappa)$ can be canonically embedded into $\mathcal{U}_o^\kappa$ as a dense subset.

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where $(v_1, \ldots, v_k)$ is the reduced index of all points in $\mathcal{U}_o^n$ such that $\mathbb{I}(x) = P$.

Indices and cones

If $\mathbf{v}$ is an index, we say that a closed cone $C \subseteq \mathbb{R}^n$ is a $\mathbf{v}$-cone if there exist real numbers $r_2, \ldots, r_k > 0$ such that C is generated by $\{v_1,\ v_1 + r_2 v_2,\ \ldots,\ v_1 + r_2 v_2 + \cdots + r_k v_k\}$.

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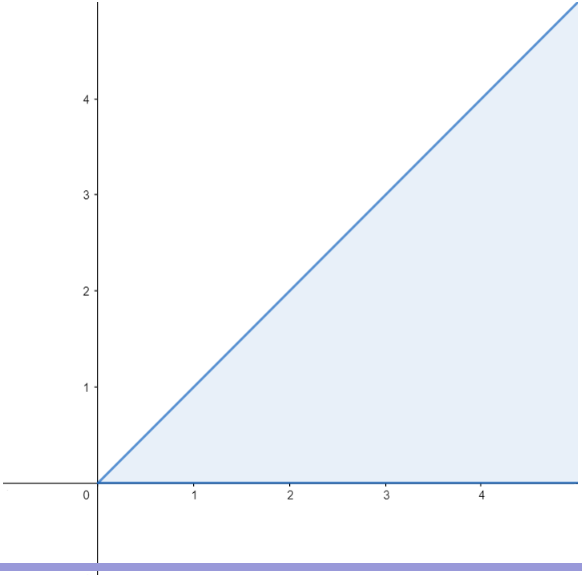
Proposition

$\mathsf{Hcone}(\mathbf{v})$ is the intersection of the enlargements of all the $\mathbf{v}$-cones.

$v = ((1,0),(0,1))$.



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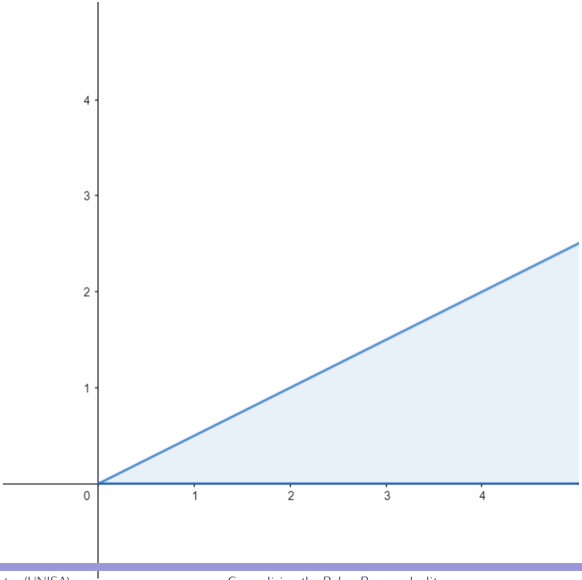

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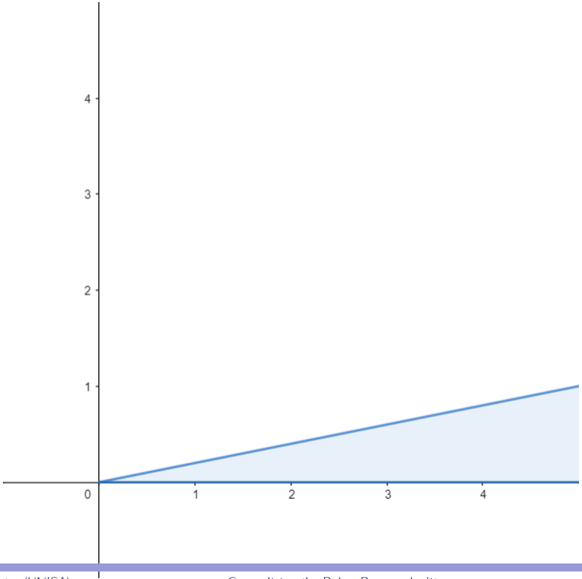

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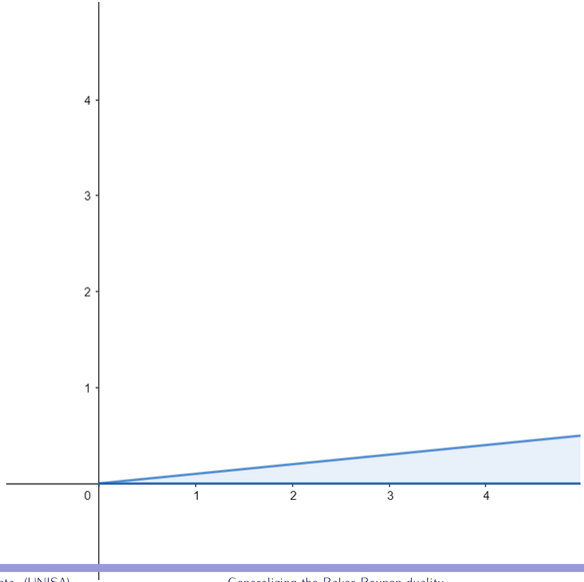

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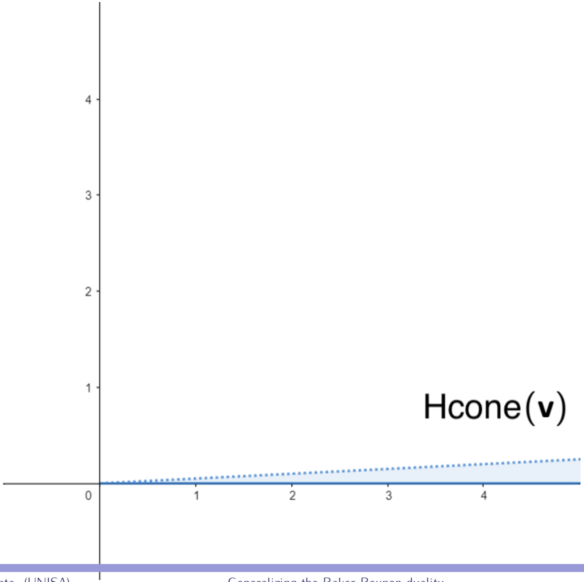

Primes and indices

Theorem (Carai, L., Spada)

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As a corollary, we obtain the description of prime ℓ-ideals in finitely generated Riesz spaces and ℓ-groups due to Panti.

Theorem (Panti 1999)

Each prime ℓ*-ideal of* $\mathcal{F}_n$ *is of the form*
$\{f \in \mathrm{PWL}(\mathbb{R}^n) \mid f$ *vanishes on a* v*-cone*$\}$ *for a uniquely determined (reduced) index* v.

Thank you!